simple symm. r.w.
Galton-Watson
⋮

Markov chain : $X_0, X_1, X_2, \dots \quad (X_n)_{n\in\mathbb{N}}$

$$P(\underbrace{X_{n+1}}_{}\overset{=x_{n+1}}{\in} A \mid X_0 = x_0, \dots, X_{n-1} = x_{n-1}, X_n = x_n) = $$

$$= P(X_{n+1} \in A \mid X_n = x_n) \quad \text{``} = x_{n+1}\text{''}$$

Summarizing

$$\lim_{n\to\infty} \mathbf{P}\left(\max_{k\le n} S_k \le \sqrt{n}\sigma x\right) = 2\Phi(x) - 1.$$

3.4 Markov property

Assume that we have a SBM (W_t) and we know everything up to time s. Conditioned on that information, what is the distribution of W_t, $t > s$?

Formally, $(W_t, \mathcal{F}_t)$ is a SBM, and we are interested in the conditional probabilities

$$\mathbf{P}(W_t \in A|\mathcal{F}_s).$$

Since $W_t = W_s + W_t - W_s$, where W_s is $\mathcal{F}_s$-measurable and $W_t - W_s$ is independent of $\mathcal{F}_s$, we obtain

Markov prop. starting here

$$\mathbf{P}(W_t \in A|\mathcal{F}_s) = \mathbf{P}(W_t \in A|W_s) = \mathbf{P}_{W_s}(W_{t-s} \in A),$$

where $\mathbf{P}_x(W_u \in A) = \mathbf{P}(W_u \in A|W_0 = x)$, that is under $\mathbf{P}_x$ W is a SBM starting at x. That is knowing the whole past up to s gives no more information than knowing only W_s. This is the Markov property.

To make the previous argument formal we need the following.

Exercise 23. Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, $\mathcal{G} \subset \mathcal{A}$ a sub-σ-algebra, X, Y random variables such that X is independent of $\mathcal{G}$ and Y is $\mathcal{G}$-measurable. Then

$$\mathbf{P}(X + Y \in A|\mathcal{G}) = \mathbf{P}(X + Y \in A|Y) \quad \mathbf{P} - \text{a.s.}$$

and

$$\mathbf{P}(X + Y \in A|Y = y) = \mathbf{P}(X + y \in A) \quad \mathbf{P}Y^{-1} - \text{a.s.}$$

For the latter note that for some $\sigma(Y)/\mathcal{B}(\mathbb{R})$-measurable h

$$\mathbf{P}(X + Y \in A|Y) = h(Y).$$

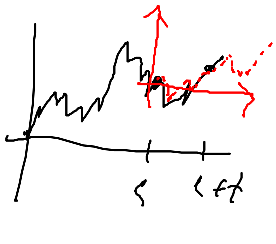
So the latter statement claims that $h(y) = \mathbf{P}(X + y \in A)$ a.s. with respect to the induced measure $\mathbf{P}Y^{-1}$.

A (d-dimensional) adapted process (X_t) is *Markov process with initial distribution* μ if

(i) $\mathbf{P}(X_0 \in A) = \mu(A)$; $\quad \mu$ prob meas.

(ii) $\mathbf{P}(X_{t+s} \in A|\mathcal{F}_s) = \mathbf{P}(X_{t+s} \in A|X_s)$, for all A and $t, s > 0$.

$\cup$ information up to s



$P_x \leftarrow$ standing

Sometimes it is more convenient to work with various initial distributions. A *Markov family* is an adapted process (X_t) together with a family of probability measures $(\mathbf{P}_x)$ such that

(i) $\mathbf{P}_x(X_0 = x) = 1$;

(ii) $\mathbf{P}_x(X_{t+s} \in A | \mathcal{F}_s) = \mathbf{P}_x(X_{t+s} \in A | X_s)$;

(iii) $\mathbf{P}_x(X_{t+s} \in A | X_s = y) = \mathbf{P}_y(X_t \in A)$ $\mathbf{P}_x X_s^{-1}$-a.s.

For a given $\omega \in \Omega$ denote $X_{s+\cdot}$ the function X_{s+t}, that is we shift the path by s. The property in the definition of Markov process easily extends to path.

$(X_{s+\cdot})$ process

$X'_t = X_{s+t}$.

Proposition 9. *For a Markov family for any $F \in \mathcal{B}(\mathbb{R}^{[0,\infty)})$*

(i) $\mathbf{P}_x(X_{s+\cdot} \in F | \mathcal{F}_s) = \mathbf{P}_x(X_{s+\cdot} \in F | X_s)$;

(ii) $\mathbf{P}_x(X_{s+\cdot} \in F | X_s = y) = \mathbf{P}_y(X_\cdot \in F)$ $\mathbf{P}_x X_s^{-1}$*-a.s.*

The proof goes by the usual technical machinery. The sets F satisfying the above properties forms a λ-system and it contains the finite dimensional cylinders.

Markov property states that the process restarts at fixed times t. Sometimes we need to restart the process at stopping times τ. This property is the *strong Markov property.*

A (d-dimensional) adapted process (X_t) is *strong Markov process with initial distribution* μ if

$\mathcal{F}_\tau$ pre-τ σ-algebra

(i) $\mathbf{P}(X_0 \in A) = \mu(A)$;

random

(ii) $\mathbf{P}(X_{\tau+t} \in A | \mathcal{F}_\tau) = \mathbf{P}(X_t \in A | X_\tau)$, for all A and stopping time τ.

Similarly, a *strong Markov family* is an adapted process (X_t) together with a family of probability measures $\mathbf{P}_x$ such that

(i) $\mathbf{P}_x(X_0 = x) = 1$;

(ii) $\mathbf{P}_x(X_{\tau+t} \in A | \mathcal{F}_\tau) = \mathbf{P}_x(X_{\tau+t} \in A | X_\tau)$ for all A and stopping time τ;

(iii) $\mathbf{P}_x(X_{\tau+t} \in A | X_\tau = y) = \mathbf{P}_y(X_t \in A)$ $\mathbf{P}_x X_\tau^{-1}$-a.s. for all A and stopping time τ;

Proposition 10. *For a strong Markov family for any $F \in \mathcal{B}((\mathbb{R})^{[0,\infty)})$*

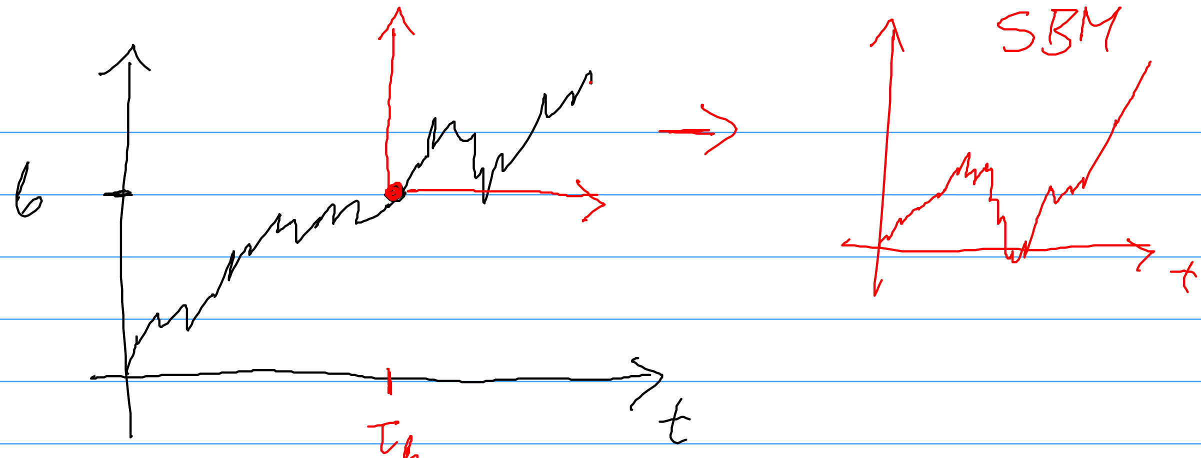
(i) $\mathbf{P}_x(X_{\tau+\cdot} \in F | \mathcal{F}_\tau) = \mathbf{P}_x(X_{\tau+\cdot} \in F | X_\tau)$;

(ii) $\mathbf{P}_x(X_{\tau+\cdot} \in F | X_\tau = x) = \mathbf{P}_x(X_\cdot \in F)$ $\mathbf{P}_x X_\tau^{-1}$*-a.s.*

We proved that SBM is Markov. In fact, it is strong Markov.

{thm:SBM-strong}

Theorem 20. *SBM is a strong Markov process.*



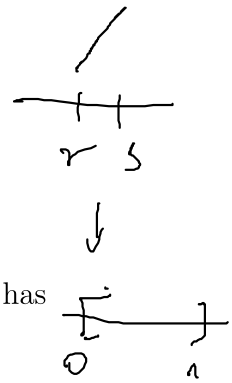
← random!!

$W_t(\omega)$

$W_t \quad W(t, \omega)$

3.5 Path properties

Theorem 21. *Almost surely the sample path of a SBM is not monotone in any interval.*



Proof. Let

$$A = \{\omega : W(\cdot, \omega) \text{ is monotone on some interval}\}.$$

countable union

Clearly

$$A = \cup_{r,s\in\mathbf{Q}} \{\omega : W(\cdot, \omega) \text{ is monotone on } [r, s]\}.$$

Since this is a countable union it is enough to prove that each event has probability zero. To ease notation choose $r = 0$, $s = 1$, and put

$$B = \{\omega : W(\cdot, \omega) \text{ is nondecreasing on } [0, 1]\}.$$

We have $N(0, \frac{1}{n}) \sim W(\frac{i+1}{n}) - W(\frac{i}{n}) \geq 0$

$$B = \cap_{n=1}^{\infty} \{\omega : W((i+1)/n, \omega) \geq W(i/n, \omega),\ i = 0, 1, \ldots, n-1\} =: \cap_{n=1}^{\infty} B_n.$$

By the independent increment property

$$\mathbf{P}(B_n) = \prod_{i=0}^{n-1} \mathbf{P}(W((i+1)/n) \geq W(i/n)) = 2^{-n},$$

which implies that $\mathbf{P}(B) = 0$ as claimed. $\square$

For any interval $[a, b]$ let $\Pi_n = \{a = t_0 < t_1 < \ldots < t_n = b\}$ a partition with mesh

$$\|\Pi_n\| = \max\{t_i - t_{i-1} : i = 1, 2, \ldots, n\}.$$

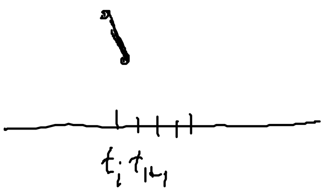
$0 \leq a < b < \infty$

We determine the quadratic variation of the Wiener process.

Theorem 22. *Let* $\Pi_n = \{a = t_0 < t_1 < \ldots < t_n = b\}$, $n = 1, 2, \ldots$, *a sequence of partitions of* $[a, b]$ *such that* $\|\Pi_n\| \to 0$. *Then*

$$\sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})^2 \xrightarrow{L^2} b - a.$$

random



$$X_n \xrightarrow{L^2} X \quad (n \to \infty)$$

$$X_n \in L^2, \; E\left[(X_n - X)^2\right] \longrightarrow 0.$$

- $\|\Pi_n\| \to 0$ $\qquad t_i = \frac{i}{n}$

$C^1 \ni f$ smooth $\quad [0,1] \quad 0 = t_0 < t_1 < \ldots < t_n = 1$

$$\sum_{i=0}^{n-1} \underbrace{\left(f(t_{i+1}) - f(t_i)\right)}_{f'(t_i)\cdot(t_{i+1}-t_i)}{}^2 \leq C \cdot \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2$$

$$f(t+h) - f(t) \underset{h \text{ small}}{\sim} f'(t)\, h$$

$$\leq C \cdot \underbrace{\sum (t_{i+1} - t_i)}_{1} \cdot \underbrace{\max(t_{i+1} - t_i)}_{= \|\Pi_n\|} \longrightarrow 0.$$

-

nyugodtan: $\quad W(t_{i+1}) - W(t_i) \sim (t_{i+1} - t_i)^{1/2}$

$$\sum_i (W_{t_i} - W_{t_{i-1}})^2 \xrightarrow{L^2} 1$$

Proof. Assume that $[a, b] = [0, 1]$. We have to show that

$$\mathbf{E}\left[\left(\sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})^2 - 1\right)^2\right] \longrightarrow 0.$$

Using $1 = \sum_{i=1}^{n}(t_i - t_{i-1})$ we have

independent Gaussian r.v.

$$\begin{aligned}
&\mathbf{E}\left(\sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})^2 - 1\right)^2 = \\
&\sum_{i,j=1}^{n} \mathbf{E}\left(\left[(W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right]\left[(W_{t_j} - W_{t_{j-1}})^2 - (t_j - t_{j-1})\right]\right).
\end{aligned} \tag{6}$$ {eq:Wquad-1}

If $i \neq j$ then $W_{t_i} - W_{t_{i-1}}$ and $W_{t_j} - W_{t_{j-1}}$ are independent. Therefore

$$\mathbf{E}\left[(W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right] = 0,$$

so the mixed products in (6) are 0. Using that $W_t - W_s \sim N(0, t - s)$ we obtain

$$\begin{aligned}
\mathbf{E}\left(\sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})^2 - 1\right)^2 &= \sum_{i=1}^{n}\mathbf{E}\left[(W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right]^2 \\
&= \sum_{i=1}^{n}(t_i - t_{i-1})^2\mathbf{E}\left[\left(\frac{W_{t_i} - W_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}\right)^2 - 1\right]^2 \\
&= \mathbf{E}(Z^2 - 1)^2\sum_{i=1}^{n}(t_i - t_{i-1})^2,
\end{aligned}$$

where $Z \sim N(0, 1)$. Since

$$\sum_{i=1}^{n}(t_i - t_{i-1})^2 \leq \|\Pi_n\| \sum_{i=1}^{n}(t_i - t_{i-1}) = \|\Pi_n\| \to 0,$$

the proof is ready. □

Under some extra conditions a.s. convergence hold. Recall that in general neither L^2 convergence nor a.s. convergence implies the other. Moreover, L^2 convergence implies a.s. convergence on a subsequence. However, if $\sum_{n=1}^{\infty}\|\Pi_n\| < \infty$ then the Borel–Cantelli lemma implies a.s. convergence.

$$\left(\sum_i \left[(W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right]\right)^2 =$$

$$= \sum_{1 \leq i,j \leq n} \left((W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right) \cdot \left((W_{t_j} - W_{t_{j-1}})^2 - (t_j - t_{j-1})\right)$$

$i \neq j$

$i < j$

$$E\left[\left((W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right)\left((W_{t_j} - W_{t_{j-1}})^2 - (t_j - t_{j-1})\right)\right]$$

independence

$$\overset{\downarrow}{=} \underbrace{E\left[\left((W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right)\right]}_{=0} \underbrace{E\left[(W_{t_j} - W_{t_{j-1}})^2 - (t_j - t_{j-1})\right]}_{=0}$$

$$W_{t_i} - W_{t_{i-1}} \sim N(0, t_i - t_{i-1})$$

$$E() = \sum_{i=1}^n E\left[\left((W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})\right)^2\right] + \underbrace{\text{mixed products}}_{=0}$$

$$E\left[\left(\underbrace{(W_{t_i} - W_{t_{i-1}})^2}_{(t_i - t_{i-1}) \cdot Z^2} - (t_i - t_{i-1})\right)^2\right] = (t_i - t_{i-1})^2 \cdot \underbrace{E\left[(Z^2 - 1)^2\right]}_{< \infty}$$

$$Z \sim N(0,1)$$

$$\sum \ldots = \sum_{i=1}^n (t_i - t_{i-1})^2 \cdot E\left[(Z^2 - 1)^2\right] \longrightarrow 0.$$

//

$$\sum_{i=1}^{n}(W_{t_i}-W_{t_{i-1}})^2 \xrightarrow[\|\Pi\|_n \to 0]{L^2} b-a \qquad \Big| \qquad L^2 \not\Rightarrow \text{a.s.} \quad \not\Leftarrow$$

L^2 conv. $\Rightarrow$ a.s. conv on a subseq.

previous proof + Borel-Cantelli

HW

Exercise 24. Let $\Pi_n = \{a = t_0 < t_1 < \ldots < t_n = b\}$, $n = 1, 2, \ldots$, a sequence of partitions of $[a, b]$ such that $\sum_{n=1}^{\infty} \|\Pi_n\| < \infty$. Then a.s.

$$\sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})^2 \longrightarrow b - a.$$

Corollary 6. *Let* (Π_n) *be a sequence of partitions of the interval* $[a, b]$ *such that* $\sum_{n=1}^{\infty} \|\Pi_n\| < \infty$. *Then* $\underbrace{\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|}_{\text{total variation}} \to \infty$ *a.s.*

Proof. Clearly,

$$b-a \leftarrow \sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})^2 \le \underbrace{\sup_{1\le i\le n} |W_{t_i} - W_{t_{i-1}}|}_{\to 0 \text{ by uniform cont.}} \sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|.$$

The left-hand side converges to $b - a$ a.s. on a subsequence. On the right-hand side the first factor goes to 0 a.s. by the continuity of the Wiener process. (Recall that continuous function is uniformly continuous on compacts.) Therefore the second term necessarily tends to infinity. □

We proved that the sample path of W are Hölder continuous with exponent $< 1/2$, and that the sample path are not of bounded variation. These results suggest that the trajectories are quite irregular. In fact, they are a.s. nowhere differentiable.

Theorem 23 (Paley, Wiener, Zygmund (1933)). *Almost surely the path* $W(\cdot, \omega)$ *is nowhere differentiable.*

$W(t) \qquad W(t,\omega)$

Proof. For $n, k \in \mathbb{N}$ consider

$$X_{nk} = \max\left\{ \left|W\left(k2^{-n}\right) - W\left((k-1)2^{-n}\right)\right|, \left|W\left((k+1)2^{-n}\right) - W\left(k2^{-n}\right)\right|, \right.$$
$$\left. \left|W\left((k+2)2^{-n}\right) - W\left((k+1)2^{-n}\right)\right| \right\}.$$

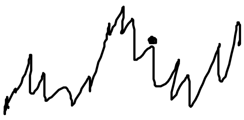
Using the independent increment property and the scale invariance

σ-subadd.

$$\mathbf{P}(X_{nk} \le \varepsilon) = (\mathbf{P}(|W(1/2^n)| \le \varepsilon))^3 \le \left(2 \cdot 2^{n/2}\varepsilon\right)^3.$$

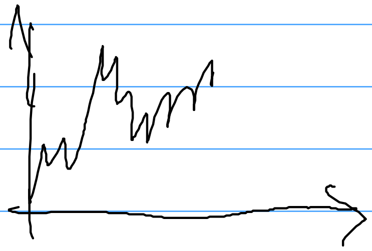
Putting $Y_n = \min_{1 \le k \le n2^n} X_{nk}$ we obtained $P(Y_n \le \varepsilon) = P\left(\bigcup\{X_{nk} \le \varepsilon\}\right) \le$

$$\boxed{\mathbf{P}(Y_n \le \varepsilon) \le \sum_{k=1}^{n2^n} \mathbf{P}(X_{nk} \le \varepsilon) < n\, 2^n \left(2 \cdot 2^{n/2}\, \varepsilon\right)^3}$$



$$P\left(\left|W\left(\tfrac{1}{2^n}\right)\right| \le \varepsilon\right) = P\left(\tfrac{1}{2^{n/2}}|Z| \le \varepsilon\right) = P\left(|Z| \le \varepsilon \cdot 2^{n/2}\right) = \int_{-\varepsilon 2^{n/2}}^{\varepsilon 2^{n/2}} \varphi(y)\,dy \le 2\varepsilon 2^{n/2}$$

for any [a,b] $\sum_i^t |W_{t_i} - W_{t_{i-1}}| \to \infty$



not of bounded variation

Lebesgue-Stieltjes integrals:

$$\mu \longleftrightarrow F$$

$\int \dots dF$ F monoton nondecreasing

$\int \dots d(F-G)$ F, G monotone nondec.

H is of bounded variation $\boxed{H = F - G}$

We can define LS integrals with respect to functions of bounded variation.

$\Rightarrow$ Cannot define $\int \dots dW_t$ pathwise !!

depends on w

Introduce the event

$$A = \{\omega : W(\cdot\,,\omega) \text{ is } \underline{\text{somewhere}} \text{ differentiable}\}.$$

If $\omega \in A$ then there exist $t = \underline{t(\omega)}$ such that $W'(t,\omega) = D(\omega) \in \mathbb{R}$. Thus

definition of derivative

$$\lim_{s\to t}\left|\frac{W(s,\omega)-W(t,\omega)}{s-t}\right| = |D(\omega)| < \infty\,.$$

Therefore there exists $\delta(\underline{\omega}) = \delta(\omega,t) > 0$ such that for $|s-t| < \delta(\omega)$

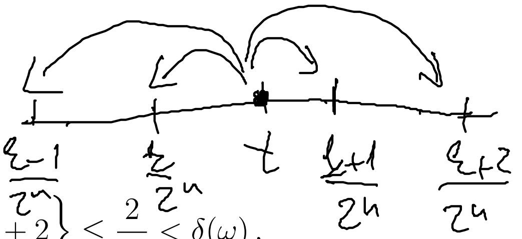
$$|W(s,\omega)-W(t,\omega)| \le (|D(\omega)|+1)|s-t|\,.$$

Let $n_0(\omega) = n_0(\omega,t)$ so large that

$$2^{-n_0(\omega)} < \frac{\delta(\omega)}{2}\,,\quad \underline{n_0(\omega) \ge \max\{4(|D(\omega)|+1), t+1\}}.$$

Fix $n \ge n_0(\omega)$ and let

$$\frac{k(\omega)}{2^n} \le t < \frac{k(\omega)+1}{2^n}\,.$$



Then

$$\max\left\{\left|t-\frac{j}{2^n}\right| \;:\; j = k(\omega)-1,\, k(\omega),\, k(\omega)+1,\, k(\omega)+2\right\} \le \frac{2}{2^n} < \underline{\delta(\omega)}\,,$$

thus

$|W(\frac{j}{2^n}) - W(\frac{j-1}{2^n})|$ ⊂ def. of X

triangle ineq.

$$X_{n,k(\omega)}(\omega) \le \max\left\{\left|W\left(\frac{j}{2^n},\omega\right)-W(t,\omega)\right| + \left|W\left(\frac{j-1}{2^n},\omega\right)-W(t,\omega)\right|\right\}$$
$$\le 2\big(|D(\omega)|+1\big)\,\frac{2}{2^n} = 4\Big(|D(\omega)|+1\Big)\,\frac{1}{2^n} \le \frac{n}{2^n}\,,$$

where the max is taken on the set $j \in \{k(\omega), k(\omega)+1, k(\omega)+2\}$.

Since $k(\omega) \le n\,2^n$, we obtained

$$Y_n(\omega) = \min_{1\le k\le n2^n} X_{nk}(\omega) \le n/2^n.$$

Thus $\underline{\omega \in A}$ implies $\omega \in A_n = \{\omega : Y_n(\omega) \le n/2^n\}$ for all $n \ge n_0(\omega)$ so

$$\omega \in \liminf_{n\to\infty} A_n = \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m$$
$$= \{\omega : \omega \in A_k \text{ except finitely many } k\}.$$

$$\mathbf{P}(Y_n \leq \varepsilon) \leq n \cdot 2^n \cdot \left(2 \cdot 2^{n/2} \cdot \varepsilon\right)^3 = n 2^n \left(2 \frac{n}{2^n \varepsilon}\right)^3$$

$$\varepsilon = \frac{n}{2^n} \qquad = 8 \cdot n^4 \cdot 2^{-\frac{n}{2}}$$

That is $A \subset B := \liminf_{n\to\infty} A_n$. Using the Fatou lemma

$$\begin{aligned}\mathbf{P}(B) &\leq \liminf_{n\to\infty} \mathbf{P}(A_n) \leq \liminf_{n\to\infty} \mathbf{P}\left(Y_n \leq \frac{n}{2^n}\right) \\ &\leq \liminf_{n\to\infty} n\, 2^n \left(2 \cdot 2^{n/2}\, \frac{n}{2^n}\right)^3 = 0.\end{aligned}$$

So $A \subset B$ and $\mathbf{P}(B) = 0$ as claimed. □

Note that we don't claim that $A \in \mathcal{A}$. Now we see the usefulness of the *usual conditions*. The usual conditions include that $\mathcal{F}_0$ contains the null-sets of $\mathcal{A}$.

Let

$$Z(\omega) = \{t : W(t,\omega) = 0\}$$

denote the set of zeros. Let λ be the Lebesgue measure. By Fubini

$$\begin{aligned}\mathbf{E}\lambda(Z) &= \int_\Omega \lambda(Z(\omega))\mathbf{P}(\mathrm{d}\omega) \\ &= \int_\Omega \int_{\mathbb{R}} \mathbf{I}(W(t,\omega) = 0)\, \mathrm{d}t \mathbf{P}(\mathrm{d}\omega) \\ &= \int_{\mathbb{R}} \mathbf{P}(W(t,\omega) = 0) \mathrm{d}t = 0.\end{aligned}$$

Since $\lambda(Z) \geq 0$ this implies $\lambda(Z) = 0$ a.s.

Theorem of iterated logarithm

Theorem 24 (Khinchin, 1933). *For almost every* ω

$$\boxed{\limsup_{t\downarrow 0} \frac{W_t(\omega)}{\sqrt{2t \log\log 1/t}} = 1} \quad \text{and} \quad \liminf_{t\downarrow 0} \frac{W_t(\omega)}{\sqrt{2t \log\log 1/t}} = -1,$$

and

$$\limsup_{t\to\infty} \frac{W_t(\omega)}{\sqrt{2t \log\log t}} = 1 \quad \text{and} \quad \liminf_{t\to\infty} \frac{W_t(\omega)}{\sqrt{2t \log\log t}} = -1.$$

Proof. By symmetry it is enough to prove the limsup results, and since $(tW_{1/t})_t$ is SBM it is enough to prove at 0.

Let

$$X_t = \exp\left\{\lambda W_t - \frac{\lambda^2}{2} t\right\}. \qquad \forall\, W \qquad \lambda > 0 \text{ anything}$$

This is a martingale, therefore by the maximal inequality

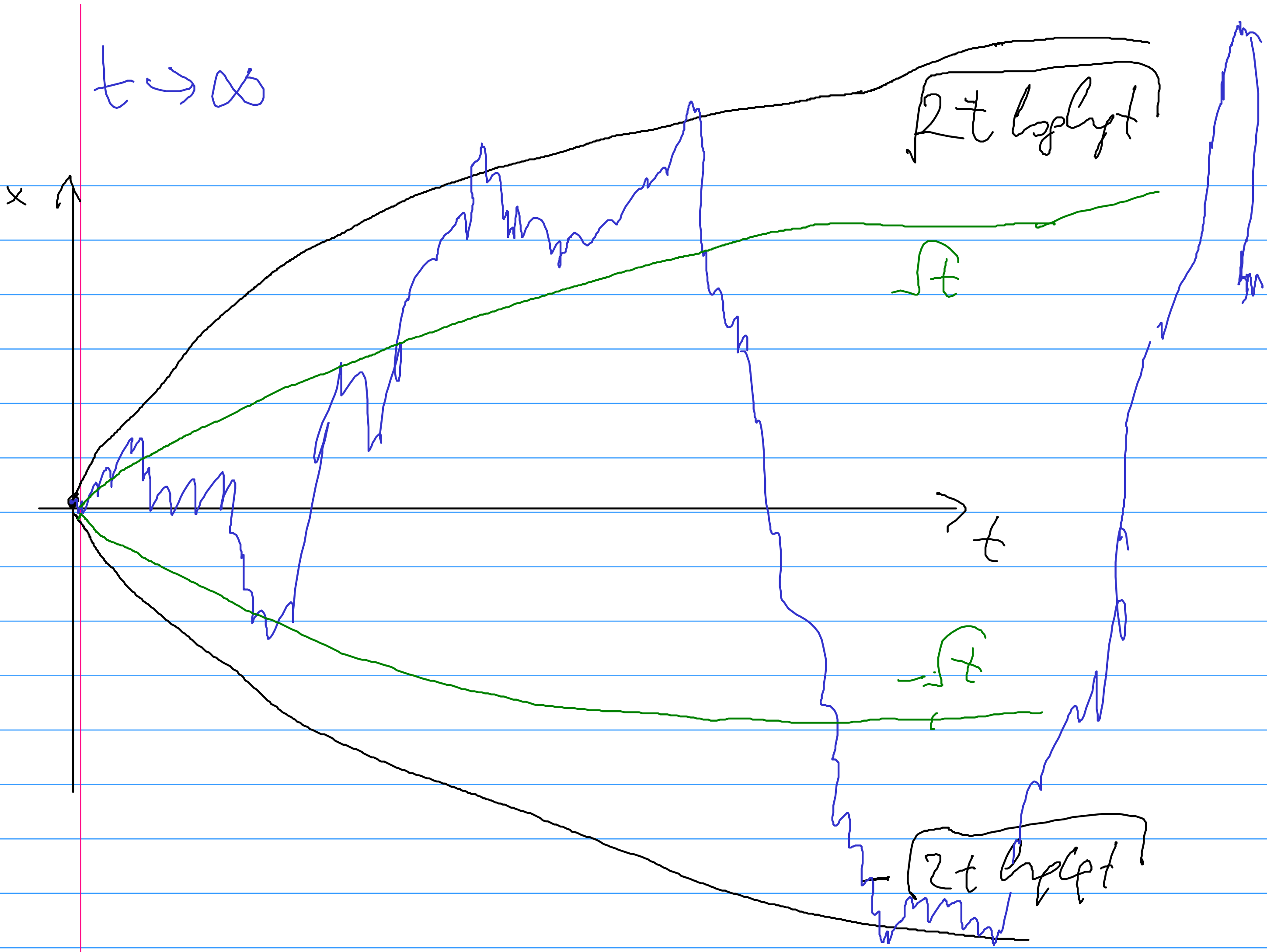
$$\mathbf{P}\left(\max_{s\in[0,t]} \left(W_s - \frac{\lambda}{2} s\right) \geq \beta\right) = \mathbf{P}\left(\max_{s\in[0,t]} X_s \geq e^{\lambda\beta}\right) \leq \frac{E(X_t^+)}{e^{\lambda\beta}} = e^{-\lambda\beta}. \qquad E(X_t^+) = 1$$

Strong LLN.

$$\overline{\lim_{t\to\infty}} \frac{W_t}{\sqrt{2t \cdot \log\log t}} = 1 \qquad \underline{\lim} \ldots = -1$$

$$\lim_{t\to\infty} \frac{W_t}{t} = 0 \text{ a.s.}$$

$t \to \infty$



$$\lambda\beta = (1+\delta)\cancel{\theta^{-n}} \cdot \cancel{2\theta^{n}} \log\log \theta^{-n} \cdot \frac{1}{\cancel{2}}$$

Put $h(t) = \sqrt{2t \log\log(1/t)}$. Fix $\theta, \delta \in (0,1)$. Choose $\lambda = (1+\delta)\theta^{-n}h(\theta^n)$, $\beta = h(\theta^n)/2$, and $t = \theta^n$. Then

$$\mathbf{P}\left(\max_{s\in[0,t]}\left(W_s - \frac{\lambda}{2}s\right) \geq \beta\right) \leq e^{-\lambda\beta} = (n \log 1/\theta)^{-(1+\delta)}. \quad = \text{const} \cdot n^{-(1+\delta)}$$

This is summable, therefore by the (fix n) Borel–Cantelli lemma there exists $N(\omega)$, and $\Omega_{\delta,\theta}$ with $\mathbf{P}(\Omega_{\delta,\theta}) = 1$ such that

$$\max_{s\in[0,\theta^n]}\left(W_s - \frac{1+\delta}{2}s\theta^{-n}h(\theta^n)\right) \leq \frac{1}{2}h(\theta^n) \quad \text{for } n \geq N(\omega).$$

Thus for $t \in (\theta^{n+1}, \theta^n]$

$$W_t(\omega) \leq \max_{s\in[0,\theta^n]} W_s(\omega) \leq (1+\delta/2)\, h(\theta^n) \leq (1+\delta/2)\, \theta^{-1/2}\, h(t).$$

Therefore for $n \geq N(\omega)$

$$\sup_{t\in(\theta^{n+1},\theta^n]} \frac{W_t(\omega)}{h(t)} \leq (1+\delta/2)\, \theta^{-1/2},$$

$\left[h(\theta^n) \leq \theta^{-\frac{1}{2}} h(t) \right.$ if $t \in (\theta^{n+1}, \theta^n]$

which implies as $n \to \infty$

$$\limsup_{t\downarrow 0} \frac{W_t(\omega)}{h(t)} \leq (1+\delta/2)\, \theta^{-1/2}.$$

Letting $\delta \downarrow 0$ and $\theta \uparrow 1$ through rationals we obtain

$$\boxed{\limsup_{t\downarrow 0} \frac{W_t(\omega)}{h(t)} \leq 1.} \qquad (7) \quad \texttt{\{eq:loglog-1\}}$$

For the opposite direction we need the second Borel–Cantelli lemma, which requires independence. Fix $\theta \in (0,1)$ and let

$$A_n = \{W_{\theta^n} - W_{\theta^{n+1}} \geq \sqrt{1-\theta}\, h(\theta^n)\}.$$

$$x = \frac{\sqrt{1-\theta}\, h(\theta^n)}{\sqrt{\theta^n - \theta^{n+1}}}$$

Putting $x_n = \sqrt{2\log n + 2\log\log 1/\theta}$

$$\mathbf{P}(A_n) = \mathbf{P}\left(\frac{W_{\theta^n} - W_{\theta^{n+1}}}{\sqrt{\theta^n - \theta^{n+1}}} \geq x\right) \geq Cx^{-1}e^{-\frac{x^2}{2}} \geq C'\frac{1}{n\sqrt{\log n}},$$

$\| \theta$

$Z \sim N(0,1) \quad P(Z > \sqrt{2\log n}) \leq$

$1 - \Phi(\sqrt{2\log n}) \sim \frac{1}{\sqrt{2\log n}} \cdot e^{-\log n} \cdot \frac{1}{\sqrt{2\pi}}$

$$\sum \frac{1}{n\sqrt{\log n}} = \infty$$

large sometimes

$$\underbrace{W_{\theta^n} - W_{\theta^{n+1}}}_{\text{small}} \geq \sqrt{1-\theta}\, h(\theta^n)$$

where we use Lemma 4. The lower bound is a divergent series in n, therefore the event A_n occur infinitely often. On the other hand by (7) (for $-W_t$)

always

$$-W_{\theta^{n+1}} \leq 2h(\theta^{n+1}) \leq 4\theta^{1/2} h(\theta^n)$$

easy def.

for all $n \geq N(\omega)$. Therefore whenever A_n occur

$$\frac{W_{\theta^n}(\omega)}{h(\theta^n)} \geq \sqrt{1-\theta} - 4\sqrt{\theta}.$$

W_{θ^n} something large

Letting $n \to \infty$ we have

$$\limsup_{t \downarrow 0} \frac{W_t}{h(t)} \geq \sqrt{1-\theta} - 4\sqrt{\theta},$$

and the result follows by letting $\theta \downarrow 0$. □

Exercise 25. Show that if W is SBM then for any λ

$$X_t = \exp\left\{\lambda W_t - \frac{\lambda^2}{2} t\right\}$$

is a martingale.

4 Stochastic integral

Here we define the integration with respect to the Brownian motion. Note that SBM is not of bounded variation, therefore we cannot define the integral pathwise. This is the major difficulty in the theory.

4.1 Integration of simple processes

In what follows we work on $[0, T]$, for $T < \infty$. Let $(W_t, \mathcal{F}_t)$ be SBM.

The process (X_t) is a *simple process*, if

$$X_t(\omega) = \xi_0(\omega)\mathbf{I}_{\{0\}}(t) + \sum_{i=1}^{n-1} \xi_i(\omega)\mathbf{I}_{(t_i, t_{i+1}]}(t),$$

where $0 = t_0 < t_1 < \ldots < t_n = T$ is a partition of $[0, T]$, and ξ_i is $\mathcal{F}_{t_i}$-measurable.

That is $(X_t(\omega))$ is a step function for each $\omega \in \Omega$, where the step sizes are random. Note that ξ_i is measurable with respect to the σ-algebra corresponding to the left end point of the interval.