countable set. Similarly,

$$\left\{\omega \in \mathbb{R}^{[0,\infty)} : \sup_{0\le t\le 1} \omega_t \le x\right\}, \quad x \in \mathbb{R},$$

is not $\mathcal{B}^{[0,\infty)}$-measurable, so we cannot define $\sup_{t\in[0,1]} \widetilde{W}_t$.

Thus the setup in Kolmogorov's consistency theorem cannot deal with continuous processes. We need a different approach.

Recall that Y is a *modification* of X if $X_t = Y_t$ a.s. for any fix t, i.e. $\mathbf{P}(X_t = Y_t) = 1$ for each $t \ge 0$.

Theorem 12 (Kolmogorov continuity theorem)**.** *Let $(X_t)_{t\in[0,T]}$ be a stochastic process on $(\Omega, \mathcal{A}, \mathbf{P})$, such that for some positive constants α, β, C*

$$\mathbf{E}|X_t - X_s|^\alpha \le C|t-s|^{1+\beta}, \quad 0 \le s, t \le T. \quad \leftarrow$$

increment is small

Then X has a continuous modification $\widetilde{X}$ which is Hölder continuous with exponent γ for every $\gamma \in (0, \beta/\alpha)$, that is for some $h(\omega)$ a.s. positive random variable and $\delta > 0$

$|f(t)-f(s)| \le |t-s|^\gamma C$

$$\mathbf{P}\left(\omega : \sup_{0<t-s<h(\omega)} \frac{\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)}{|t-s|^\gamma} \le \delta\right) = 1.$$

Proof. We can assume that $T = 1$. By Chebyshev

$\boxed{X \ge 0,\ P(X > c) \le \frac{E(X)}{c}}$

Markov

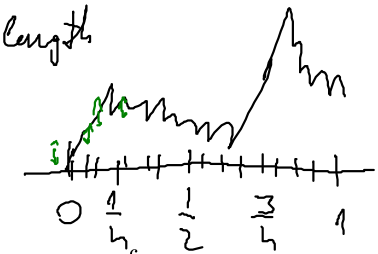
$$P(|X_t - X_s|^\alpha > \varepsilon^\alpha) = \mathbf{P}(|X_t - X_s| > \varepsilon) \le \varepsilon^{-\alpha}\mathbf{E}|X_t - X_s|^\alpha \overset{\text{assumption}}{\le} C\varepsilon^{-\alpha}|t-s|^{1+\beta},$$

in particular $X_t \to X_s$ in probability as $t \to s$. Fix $\gamma \in (0, \beta/\alpha)$. Then

$$\begin{aligned}
&\mathbf{P}\left(\max_{1\le k\le 2^n} |X_{k2^{-n}} - X_{(k-1)2^{-n}}| > 2^{-\gamma n}\right) \\
&\le 2^n \mathbf{P}\left(|X_{k2^{-n}} - X_{(k-1)2^{-n}}| > 2^{-\gamma n}\right) \\
&\le 2^n C 2^{\alpha\gamma n} 2^{-n(1+\beta)} \\
&= C 2^{-n\underbrace{(\beta - \alpha\gamma)}_{>0}}.
\end{aligned}$$

$\le \underbrace{\sum_{k=1}^{2^n} P(|X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| > 2^{-\gamma n})}$

$\frac{1}{2^n}$ length



By the first Borel–Cantelli lemma with probability 1 only finitely many of the events

$$\le C\, 2^{-\gamma n\cdot(-\alpha)} (2^{-n})^{1+\beta} \quad \left[\max_{1\le k\le 2^n} |X_{k2^{-n}} - X_{(k-1)2^{-n}}| > 2^{-\gamma n}\right]$$

s fix $\quad t \to s \quad X_t \xrightarrow{P} X_s$

variables

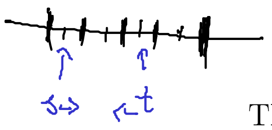
$X_t(\omega)$ $\qquad X_t \qquad X(t,\omega)$

function

occur. That is, there is a set Ω_0 with $\mathbf{P}(\Omega_0) = 1$, and a threshold $n_0(\omega)$ (depending on ω!) such that for $\omega \in \Omega_0$

$$\max_{1\le k\le 2^n} |X_{k2^{-n}} - X_{(k-1)2^{-n}}| \le 2^{-\gamma n}, \quad n \ge n_0(\omega).$$

Fix $\omega \in \Omega_0$. Put $D_n = \{k2^{-n} : k = 0, 1, \ldots, 2^n\}$, and $D = \cup_{n=0}^{\infty} D_n$. Then for $n \ge n_0(\omega)$ and $m > n$ induction gives that



$$|X_t(\omega) - X_s(\omega)| \le 2 \sum_{j=n+1}^{m} 2^{-\gamma j}, \quad t, s \in D_m, \ |t - s| \le 2^{-n}.$$

$\frac{k}{2^m}, \frac{\ell}{2^m}$

This implies that $(X_t(\omega))_{t\in D}$ is uniformly continuous in $t \in D$. Indeed, for any $t, s \in D$ with $0 < t - s < h(\omega) = 2^{-n_0(\omega)}$ there is an $n \ge n_0(\omega)$ such that $2^{-n-1} \le t - s < 2^{-n}$, thus

$2\cdot 2^{-\gamma(n+1)}\cdot\sum_{j=0}^{\infty} 2^{-\gamma j}$

$$|X_t(\omega) - X_s(\omega)| \le 2 \sum_{j=n+1}^{\infty} 2^{-\gamma j} = 2^{-\gamma(n+1)} \frac{2}{1 - 2^{-\gamma}} \le |t - s|^{\gamma} \frac{2}{1 - 2^{-\gamma}}.$$

uniform continuity!

Informally, we proved that (X_t) behaves regularly on D. We define $\widetilde{X}$. If $\omega \notin \Omega_0$ let $\widetilde{X}(\omega) = 0$, (or anything). If $\omega \in \Omega_0$ and $t \in D$ let $\widetilde{X}_t(\omega) = X_t(\omega)$, while if $t \notin D$ choose a sequence $s_n \in D$ such that $s_n \to t$ and let

D is dense

$$\widetilde{X}_t(\omega) = \lim_{n\to\infty} X_{s_n}(\omega).$$

$\widetilde{X}$ cont. Is it a modification?

By the uniform continuity and the Cauchy criteria the limit on the right-hand side exist.

The a.s. uniqueness of the stochastic limit together with the stochastic continuity of X implies that $\widetilde{X}$ is a modification of X. □

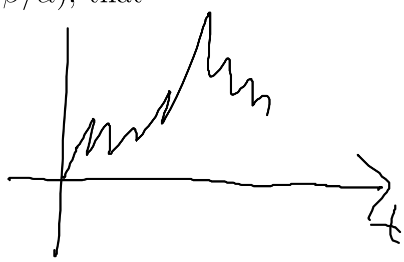
Exercise 15 (Random fields)**.** A *random field* is a collection of random variables indexed by a partially ordered set. Let $(X_t)_{t\in[0,T]^d}$ be a random field satisfying

$$\mathbf{E}|X_t - X_s|^{\alpha} \le C\|t - s\|^{d+\beta},$$

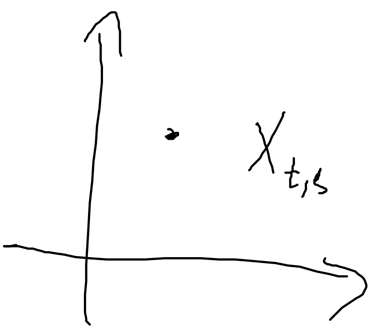
$\|\cdot\|$ usual euclidean norm

$t\in[0,T]$

for some positive constants. Show that there exists a continuous modification $\widetilde{X}$ which is Hölder continuous with exponent γ for every $\gamma \in (0, \beta/\alpha)$, that is for some $h(\omega)$ a.s. positive random variable and $\delta > 0$



$$\mathbf{P}\left(\omega : \sup_{0<\|t-s\|<h(\omega)} \frac{\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)}{\|t - s\|^{\gamma}} \le \delta\right) = 1.$$



$W_t - W_s \sim N(0, t-s)$ $\sim Z \sim N(0,1)$

$$E|W_t - W_s|^\alpha = E\left|\frac{W_t - W_s}{\sqrt{t-s}}\right|^\alpha \cdot (t-s)^{\frac{\alpha}{2}} = E|Z|^\alpha \cdot (t-s)^{\frac{\alpha}{2}}$$

Exercise 16. Show that if $W_t - W_s \sim N(0, t-s)$ then for any $n > 0$ $\alpha = 2n$

$$\mathbf{E}|W_t - W_s|^{2n} = C_n |t-s|^n,$$

$C_n = E(|Z|^{2n})$

where $C_n = \mathbf{E}|Z|^n$, $Z \sim N(0,1)$.

Corollary 4. *Wiener process exists.*

Proof. We need only the continuity part. The condition of Kolmogorov continuity theorem holds with $\alpha = 2n$ and $\beta = n - 1$ for any $n > 1$. Thus there exists a continuous modification on $[0, N]$, for any $N \in \mathbb{N}$. Necessarily, X^{N_1} and X^{N_2} agrees a.s. for any fix $t \in [0, N_1 \wedge N_2]$, which allows us to extend the process to $[0, \infty)$. □

$[0, N]$ $\Downarrow$ $[0, \infty)$

$\gamma = 1$

In fact, we proved that the Wiener process is locally γ-Hölder continuous for any $\gamma < 1/2$.

$\gamma = \frac{1}{2}$

Exercise 17. Let (N_t) be a Poisson process with intensity 1. Compute the order $\mathbf{E}|N_t - N_s|^\alpha$ for $t - s$ small. (Thus the condition in the continuity theorem holds for $\beta = 0$. Well, of course, Poisson processes are not continuous.)

More generally, we obtain a result on continuity of Gaussian processes.

Theorem 13. *Let (X_t) be a Gaussian process with continuous mean function m, and covariance function r. If there exist positive constants δ, C such that for all s, t*

$$r(t,t) - 2r(s,t) + r(s,s) \le C|t-s|^\delta,$$

then (X_t) has a continuous modification which is locally γ-Hölder continuous for any $\gamma \in (0, \delta/2)$.

$X_t - m(t) = \tilde{X}_t$

Proof. Subtracting the mean function we may and do assume that $m(t) \equiv 0$. Simply

$$\mathbf{Var}(X_t - X_s) = r(t,t) - 2r(s,t) + r(s,s) = \sigma^2(s,t),$$

therefore

$$\mathbf{E}|X_t - X_s|^\alpha = \mathbf{E}|Z|^\alpha \sigma(s,t)^\alpha,$$

with $Z \sim N(0,1)$. Thus

$$\mathbf{E}|X_t - X_s|^\alpha \le C|t-s|^{\delta\alpha/2},$$

which implies that the condition of the continuity theorem holds with $\alpha > 0$, $\beta = \delta\alpha/2 - 1$. Letting $\alpha \to \infty$ the result follows. □

$m(t) = E(X_t)$ $\quad r(s,t) = Cov(X_s, X_t) = E((X_s - m(s))(X_t - m(t)))$

For SBM: $m(t) \equiv 0$, $r(s,t) = \min(s,t)$.

$$\mathbb{E}|W_t - W_s|^\alpha = C \cdot |t-s|^{\frac{\alpha}{2}} = C \cdot |t-s|^{1+\overbrace{\frac{\alpha}{2}-1}^{=\beta}}$$

Cond. holds α, $\beta = \frac{\alpha}{2} - 1$.

$$\gamma \in \left(0, \frac{\frac{\alpha}{2}-1}{\alpha}\right) = \left(0, \frac{1}{2} - \frac{1}{\alpha}\right)$$

$\alpha \to \infty$

$\Rightarrow$ path is Hölder cont with any exponent $< \frac{1}{2}$.

Proof of Thm. 13.

We need $\mathbb{E}|X_t - X_s|^\alpha < ?$

$$X_t - X_s \sim N(0, ?) \quad ? = r(t,t) + r(s,s) - 2r(s,t)$$

$\mathbb{E}(X_t) = 0 \quad \mathbb{E}(X_s) = 0$

$$Var(X_t - X_s) = Cov(X_t - X_s, X_t - X_s) =$$

we know $r(s,t) = E(X_s X_t)$.

$$= \mathrm{Cov}(X_t, X_t) - 2\mathrm{Cov}(X_t, X_s) + \mathrm{Cov}(X_s, X_s)$$

$$= r(t,t) - 2r(s,t) + r(s,s) = \sigma^2(s,t)$$

$$E(|X_t - X_s|^\alpha) = E\left(\left|\frac{X_t - X_s}{\sigma(s,t)}\right|^\alpha\right) \sigma(s,t)^\alpha$$

$$= E(|Z|^\alpha) \cdot \sigma(s,t)^\alpha \underset{\text{const.}}{\leq} C \cdot |t-s|^{\delta \frac{\alpha}{2}}$$

$$\alpha = \alpha, \quad \beta = \frac{\delta\alpha}{2} - 1$$

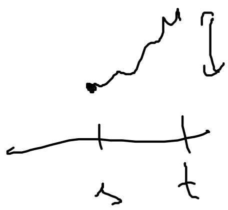
$$\gamma \in \left(0, \frac{\beta}{\alpha}\right) \qquad \alpha \to \infty$$

$$\frac{\delta}{2} - \frac{1}{\alpha}$$

$$X \sim N(0, \sigma^2) \Rightarrow \frac{X}{\sigma} \sim N(0,1)$$

Exercise 18 (Fractional Brownian motion)**.** Fractional Brownian motion with Hurst index $H \in (0,1)$ is a Gaussian process $(B(t))$ with mean function $m(t) \equiv 0$ and covariance function

$$r(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$



Note that $H = 1/2$ corresponds to the usual Brownian motion.

(i) Show that it is self-similar, i.e. $B(at) \sim a^H B(t)$.

(ii) Show that it has stationary increments: $B(t) - B(s) \sim B(t-s)$.

(iii) Prove that a continuous modification exists, which is γ-Hölder for any $\gamma < H$. (That is H is the 'roughness parameter': for small H the process strongly oscillates, while for H close to 1 the paths are almost smooth.)

(iv) Are the increments independent?

Exercise 19. Let $(X_t)_{t\in[0,1]}$ be a continuous Gaussian process with mean 0 and covariance function $r(s,t)$. Show that $Y = \int_0^1 X_t \mathrm{d}t \sim N(0,\sigma^2)$, where

$$\sigma^2 = \int_0^1 \int_0^1 r(s,t)\, \mathrm{d}s\, \mathrm{d}t\,.$$

random bound.

Show that $Y_t = \int_0^t X_s \mathrm{d}s$ is a Gaussian process. Determine its covariance function.

A version of the continuity theorem is the following.

{thm:Kol-cont2}

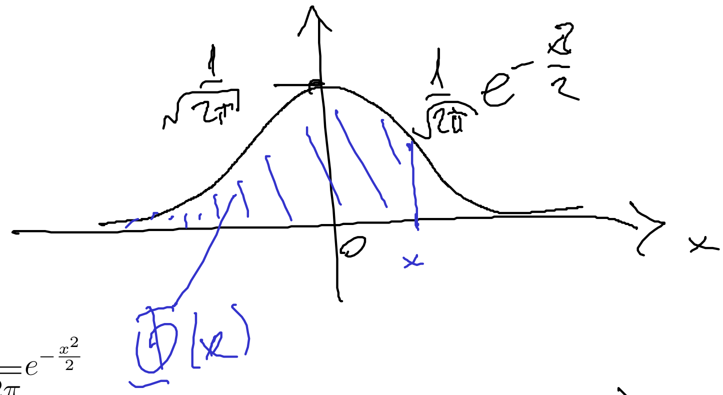
Theorem 14. *Let $T \subset \mathbb{R}$ finite or infinite interval, and $(X_t)_{t\in T}$ a stochastic process such that for $\delta > 0$ small enough*

$$\mathbf{P}\left(|X_t - X_s| \geq g(\delta)\right) \leq h(\delta) \quad \textit{whenever } |s-t| < \delta\,,\ s,t \in T,$$

where g and h are continuous function such that

$$\sum_{n=1}^{\infty} g\left(2^{-n}\right) < \infty\,, \quad \sum_{n=1}^{\infty} 2^n h\left(2^{-n}\right) < \infty\,,$$

Then X has a continuous modification.



Recall that

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

is the standard normal density function, and

$$\Phi(x) = \int_{-\infty}^{x} \varphi(y) \mathrm{d}y \qquad \Phi(x) = P(Z \leq x)$$

is the standard normal distribution function.

{lemma:Phi-bound}

Lemma 4. *For any $x > 0$*

$$\left(\frac{1}{x} - \frac{1}{x^3}\right) \varphi(x) \leq \overbrace{1 - \Phi(x)}^{P(Z > x)} \leq \frac{1}{x} \varphi(x)$$

and

$$\lim_{x \to \infty} \frac{1 - \Phi(x)}{\frac{1}{x} \varphi(x)} = 1.$$

Proof. The first follows from integrating the inequality

$$\left(1 - \frac{3}{y^4}\right) \varphi(y) \leq \varphi(y) \leq \left(1 + \frac{1}{y^2}\right) \varphi(y),$$

on (x, ∞). The second is immediate from the first. □

Using Theorem 14 we obtain a better criteria for continuity.

Corollary 5. *Let $T \subset \mathbb{R}$ be a finite or infinite interval and let $(X_t)_{t \in T}$ be a Gaussian process with continuous mean function m, and covariance function r such that for δ small enough*

$$\sup_{|s-t| \leq \delta} \left(r(t,t) - 2r(s,t) + r(s,s)\right) \leq C \left(-\log \delta\right)^{-3(1+\alpha)}$$

for some $C > 0$, $\alpha > 0$. Then (X_t) has a continuous modification.

3.2 The space $C[0, \infty)$

As SBM is continuous, its natural space is the space of continuous functions. Instead of a collection of random variables a stochastic process (W_t) can be understood as a random element of a function space.

Recall that ρ is a metric if on S

$$1-\Phi(x) = 1-\int_{-\infty}^{x} \varphi(y)\,dy$$

$$\frac{d}{dx}\left[1-\Phi(x)\right] = -\varphi(x).$$

$$\left(\frac{\varphi(x)}{x}\right)' = \varphi'(x)\cdot\frac{1}{x} + \varphi(x)\cdot\left(-\frac{1}{x^2}\right)$$

$$= (-x)\varphi(x)\cdot\frac{1}{x} + \varphi(x)\left(-\frac{1}{x^2}\right)$$

$$= -\varphi(x)\left(1+\frac{1}{x^2}\right)$$

+ differentiate the lower bound

+ integrate out everything on (x,∞).

Why is normal distribution important?

CLT : central limit theorem

Lévy CLT: $\xi, \xi_1, \xi_2, \dots$ iid
independent, identically distrib.

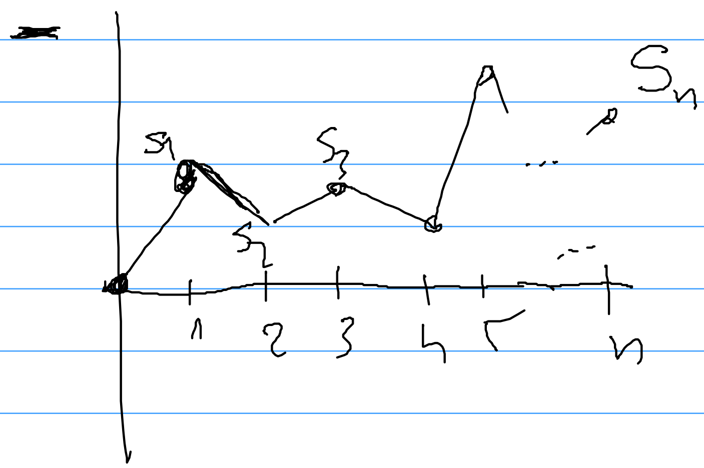
$E\xi^2 < \infty$

$$\frac{\sum_{i=1}^{n} \xi_i - nE\xi}{\sqrt{n} \cdot \sqrt{Var(\xi)}} \xrightarrow{\mathcal{D}} Z \sim N(0,1)$$

by def.

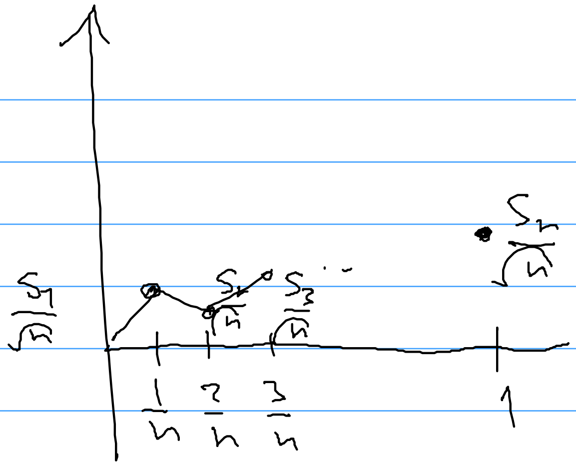
$$P\left(\frac{\sum_{i=1}^{n} \xi_i - nE(\xi)}{\sqrt{n\, Var(\xi)}} \leq x \right) \longrightarrow \Phi(x)$$

$F(x) = P(X \leq x)$
right cont.



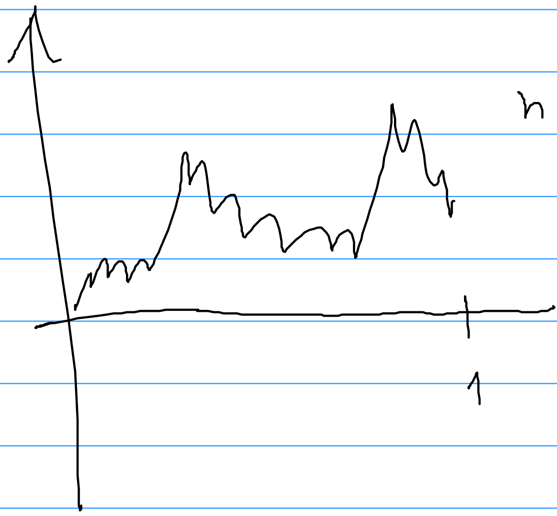
$S_n = \xi_1 + \dots + \xi_n$

$E\xi = 0 \quad E\xi^2 = 1.$



$\rightarrow$ random function

invariance principle of Donsker



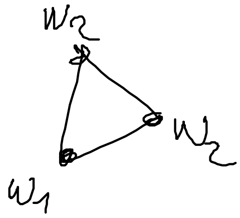
n is large

$\Big\downarrow \mathcal{D}$?

SBM

$C[0,\infty) = \{ f : f : [0,\infty) \to \mathbb{R},\ f \text{ is continuous} \}$

ρ is metric — if and only if



(i) $\rho \geq 0$, $\rho(\omega_1, \omega_2) = 0$ iff $\omega_1 = \omega_2$;

(ii) symmetric;

(iii) the triangle inequality holds, i.e.

$$\rho(\omega_1, \omega_2) \leq \rho(\omega_1, \omega_3) + \rho(\omega_2, \omega_3).$$

Then (S, ρ) is a metric space.

The sequence (x_n) is *Cauchy* if for each $\varepsilon > 0$ there exist $n_0(\varepsilon)$ such that $\rho(x_m, x_n) \leq \varepsilon$ for all $m, n \geq n_0$. The space (S, ρ) is *complete* if every Cauchy sequence converges. A set $A \subset S$ is dense, if for any $x \in S$ there exists a sequence $(x_n) \subset A$ such that $x_n \to x$. The space (S, ρ) is *separable* if there exists a countable dense subset.

Let $C[0, \infty)$ denote the space of continuous real functions on $[0, \infty)$ with metric

any norm on $[0,n] \to [0,\infty)$

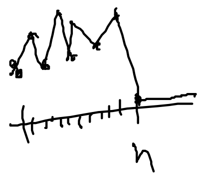
$$\rho(\omega_1, \omega_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{t \in [0,n]} \left(|\omega_1(t) - \omega_2(t)| \wedge 1 \right).$$

min

Proposition 7. *ρ is a metric, and $(C[0, \infty), \rho)$ is a complete separable metric space.*

Proof. It is clear that ρ is a metric. Fix a Cauchy sequence (x_n). For any fix $N \in \mathbb{N}$ the limit $\lim_{n\to\infty} x_n(t) = x_\infty(t)$ exists for $t \in [0, N]$, and it is continuous. Thus x_∞ exists and continuous.

To find a countable dense subset consider functions which are 0 for $t \geq n$, and it is rational at k/n for $k = 0, 1, \ldots, n^2 - 1$. $\square$



If (S, ρ) is a metric space we can define open sets. The σ-algebra generated by open sets is the Borel-σ-algebra $\mathcal{B}(S)$. With this $\boxed{(S, \mathcal{B}(S))}$ is a measurable space.

$\mathbb{R}^d$

If $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space and $(S, \mathcal{B}(S))$ is a measurable space then a measurable $X : \Omega \to S$ is a *random variable / random element* in S. It induces a probability measure $\mathbf{P} \circ X^{-1}$ on S as

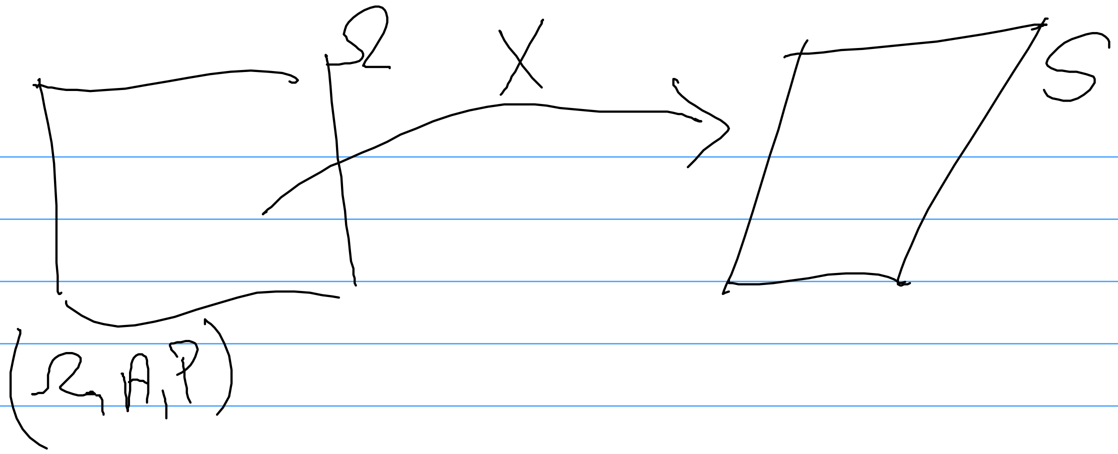
$$\mathbf{P} \circ X^{-1}(B) = \mathbf{P}(X \in B) = \mathbf{P}(\{\omega : X(\omega) \in B\}).$$

Let (P_n) be a sequence of probability measure on $(S, \mathcal{B}(S))$ and P another measure on it. Then P_n *converges weakly to* P, $P_n \xrightarrow{w} P$, if

$$\boxed{\lim_{n\to\infty} \int_S f(s) \mathrm{d}P_n(s) = \int_S f(s) \mathrm{d}P(s)}$$

σ-alg. $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$(\Omega, \mathcal{A}, P)$



We can define a measure on S:

$$\mu(A) = P(X \in A) = P(\underbrace{\{\omega : X(\omega) \in A\}}_{\in \mathcal{A}})$$

$A \in \mathcal{B}(S)$ $\quad P X^{-1}(A)$

Special case: $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$: Lebesgue-Stieltjes measure

$X : \Omega \to \mathbb{R} \longrightarrow F(x) = P(X \leq x)$

$\downarrow$

induces a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$X_t(\omega)$

Earlier: stoch. proc.: a lot of random variables
$t \in [0, \infty)$

Now: stoch. proc: random element in $C[0, \infty)$

$(\Omega, \mathcal{A}, P)$ random var. $X : \Omega \to \mathbb{R}$ + measurability

$X^{-1}(\mathcal{B}(\mathbb{R})) \subseteq \mathcal{A}$.

random element $X : \Omega \to S$ $(S, \mathcal{B}(S))$ measurable space

for every continuous real function f. Note that the limit measure is necessarily a probability measure.

Let X_n and X be random elements in S, defined possibly on different probability spaces. The sequence (X_n) *converges in distribution to* X if the corresponding induced measures converge weakly. Equivalently,

$X^{-1}(\mathcal{B}(S)) \subseteq \mathcal{A}$.

$$\mathbf{E}f(X_n) \to \mathbf{E}f(X)$$

for all continuous and bounded f.

Assume that $X_n \to X$ in distribution. For any $0 \le t_1 < \ldots < t_k$ consider the projection $\pi_{t_1,\ldots,t_k} : C[0,\infty) \to \mathbb{R}^d$

$$\pi_{t_1,\ldots,t_k}(\omega) = (\omega(t_1), \ldots, \omega(t_k)) .$$

This is clearly continuous. For a continuous bounded function $f : \mathbb{R}^d \to \mathbb{R}$ the composite function $f(\pi_{t_1,\ldots,t_k})$ is bounded and continuous. Therefore, by the definition of convergence in distribution

$$\mathbf{E}f(\pi_{t_1,\ldots,t_k}(X_n)) \to \mathbf{E}f(\pi_{t_1,\ldots,t_k}(X))$$

that is

$$\mathbf{E}f(X_n(t_1), \ldots, X_n(t_k)) \to \mathbf{E}f(X(t_1), \ldots, X(t_k)).$$

That is, for any $0 \le t_1 < \ldots < t_k$

$$(X_n(t_1), \ldots, X_n(t_k)) \xrightarrow{\mathcal{D}} (X(t_1), \ldots, X(t_k)).$$

This means that the finite dimensional distributions converge.

We proved the following.

Proposition 8. *If (X_n) converges in distribution X then all finite dimensional distributions converge.*

The converse is not true in general.

Example 8. Let

$$X_n(t) = nt\mathbf{I}_{[0,(2n)^{-1}]}(t) + (1 - nt)\mathbf{I}_{((2n)^{-1},n^{-1}]}(t), \quad t \ge 0.$$

Then all finite dimensional distributions converge to the corresponding finite dimensional distributions of $X \equiv 0$. However, convergence as a process does not hold.