2.3 Inequalities

Theorem 8 (Doob's maximal inequality)**.** *Let* (X_t) *be a right-continuous submartingale.* deterministic

(i) For any $0 < S < T < \infty$, $x > 0$

$$x\mathbf{P}(\sup_{S \le t \le T} X_t \ge x) \le \mathbf{E}X_T^+.$$

(ii) If (X_t) *is nonnegative and* $p > 1$ *then*

$$\mathbf{E}\left[\left(\sup_{S \le t \le T} X_t\right)^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbf{E}[X_T^p].$$

Proof. (i): Let F_n be as above. Then $(X_t, \mathcal{F}_t)_{t \in F_n}$ is a discrete time martingale. Therefore, by Doob's maximal inequality

$\mathbb{Q} \cap [S, T] = \{r_1, r_2, \dots\}$

$$y\mathbf{P}\left(\overset{\max}{\sup_{t \in F_n}} X_t > y\right) \le \mathbf{E}X_T^+.$$

$F_n = \{r_1, \dots, r_n\} \cup \{S, T\}$

Right-continuity implies

$$\left\{\sup_{S \le t \le T} X_t > y\right\} = \cup_{n=1}^\infty \left\{\sup_{t \in F_n} X_t > y\right\},$$

and the union is increasing. Letting $n \to \infty$

$y \to x-$

$$y\mathbf{P}\left(\sup_{S \le t \le T} X_t > y\right) \le \mathbf{E}X_T^+.$$

Letting $y \uparrow x$ the result follows.

Part (ii) follows as in the discrete time case. □

Exercise 9. Let N be a Poisson process with intensity $\lambda > 0$. Show that for any $c > 0$

$$\limsup_{t \to \infty} \mathbf{P}\left(\sup_{0 \le s \le t} (N_s - \lambda s) \ge c\sqrt{\lambda t}\right) \le \frac{1}{c\sqrt{2\pi}},$$

and

$$\limsup_{t \to \infty} \mathbf{P}\left(\inf_{0 \le s \le t} (N_s - \lambda s) \le -c\sqrt{\lambda t}\right) \le \frac{1}{c\sqrt{2\pi}}.$$

Show that for any $0 < S < T < \infty$

$$\mathbf{E} \sup_{S \le t \le T} \left(\frac{N_t}{t} - \lambda\right)^2 \le \frac{4T\lambda}{S^2}.$$

$(N_t - \lambda t)_{t \ge 0}$ is a mtg.

$$P\left(\sup_{0\leq s\leq t}(N_s-\lambda s)\geq c\sqrt{\lambda t}\right)\leq \frac{1}{c\sqrt{\lambda t}}\cdot E(N_t-\lambda t)^+$$

Need: $\lim_{t\to\infty}\frac{1}{\sqrt{\lambda t}}E(N_t-\lambda t)^+ = \frac{1}{\sqrt{2\pi}}$

$\boxed{\lambda=1}$

$$E\left[(N_t-t)^+\right]=\sum_{k=\lceil t\rceil}^{\infty}\frac{t^k}{k!}\cdot e^{-t}\cdot(k-t)=$$

$N_t\sim \text{Poisson}(t)$

upper integer part $\lceil 3.5\rceil = 4$

Recall: $X\sim\text{Poisson}(\lambda)$

$$P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda}.$$

$k=0,1,2,\dots$

$$=\sum_{k=\lceil t\rceil}^{\infty}\frac{t^k}{(k-1)!}e^{-t}-\sum_{k=\lceil t\rceil}^{\infty}\frac{t^{k+1}}{k!}e^{-t}$$

$$=\frac{t^{\lceil t\rceil}}{(\lceil t\rceil-1)!}e^{-t}\sim\frac{\sqrt{t}}{\sqrt{2\pi}}$$

$n!\sim\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n$ Stirling formula

$$E \sup_{S \leq t \leq T} \left(\frac{N_t}{t} - \lambda\right)^2 = E \sup_{S \leq t \leq T} \frac{(N_t - \lambda t)^2}{t^2} \leq$$

$$\leq S^{-2} \cdot E \sup_{S \leq t \leq T} (N_t - \lambda t)^2 \underset{p=2}{\leq} S^{-2} \left(\frac{2}{2-1}\right)^2 \cdot E\left[(N_T - \lambda T)^2\right]$$

$$= \frac{4 \cdot \lambda T}{S^2}.$$

Corollary 3. *Let N be a Poisson process with intensity $\lambda > 0$. Then*

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda \quad a.s.$$

Proof. By Chebyshev's inequality

λt

$$\mathbf{P}\left(\left|t^{-1}N_t - \lambda\right| > \varepsilon\right) \le \frac{\mathbf{Var}(N_t)}{t^2\varepsilon^2} = \frac{\lambda}{\varepsilon^2 t}.$$

$|N_t - \lambda t| > \varepsilon t$

By the first Borel–Cantelli-lemma almost surely

$$\boxed{\lim_{n\to\infty} \frac{N_{2^n}}{2^n} = \lambda.}$$

$P\left(\left|\frac{N_{2^n}}{2^n} - \lambda\right| > \varepsilon\right) \le c \cdot 2^{-n}$

So on a subsequence we are done. In between we have

$$\mathbf{P}\left(\sup_{2^n \le t \le 2^{n+1}} \left|t^{-1}N_t - \lambda\right| > \varepsilon\right) \le \frac{\mathbf{E}\left(\sup_{2^n\le t\le 2^{n+1}} |t^{-1}N_t - \lambda|\right)^2}{\varepsilon^2}$$

$\sigma = 2^n \quad \tau = 2^n + 1$

$$\le \frac{4\,2^{n+1}\lambda}{2^{2n}\varepsilon^2} = 2^{-n}\frac{8\lambda}{\varepsilon^2}.$$

Applying Borel–Cantelli again, we are done. □

2.4 Optional stopping

Let $(X_t, \mathcal{F}_t)_{t\in[0,\infty)}$ be a right-continuous submartingale. It has a *last element* X_∞, if X_∞ is measureable with respect to the σ-algebra $\mathcal{F}_\infty = \sigma\left(\cup_{t\ge 0}\mathcal{F}_t\right)$, $\mathbf{E}|X_\infty| < \infty$ and for all $t \ge 0$ $\mathbf{E}[X_\infty|\mathcal{F}_t] \ge X_t$ a.s.

If we work on the finite time horizon $[0, T]$, $T < \infty$, then the submartingale $(X_t)_{t\in[0,T]}$ has a last element X_T (by definition!).

Theorem 9 (Optional stopping)**.** *Let $(X_t, \mathcal{F}_t)_{t\ge 0}$ be a right-continuous submartingale with last element X_∞. Let $\sigma \le \tau$ be stopping times. Then*

$$\mathbf{E}[X_\tau|\mathcal{F}_\sigma] \ge X_\sigma \quad a.s.$$

Proof. Assume that τ is bounded, i.e. $\tau \le K$. Let

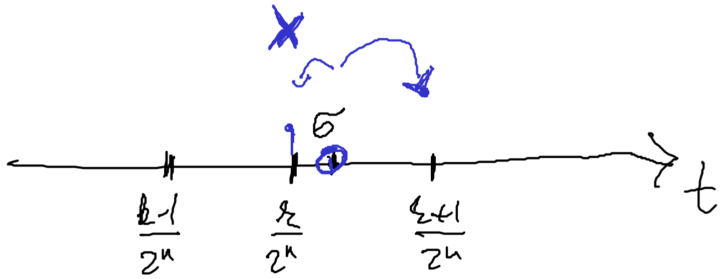
n large

$$\sigma_n(\omega) = k/2^n, \quad \text{if } \sigma(\omega) \in [(k-1)/2^n, k/2^n),$$

$\left\{ \frac{k}{2^n},\ k \in \mathbb{N} \right\}$

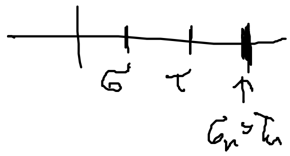
$\mathcal{F}_\sigma$: events prior to σ.

HW σ_n, τ_n stopping time



and define τ_n similarly. Then σ_n and τ_n are stopping times, and $\boxed{\sigma_n \leq \tau_n}$. We can apply the optional stopping theorem for the sumbartingale $(X_{k/2^n}, \mathcal{F}_{k/2^n})$, and stopping times σ_n, τ_n. Then

$$\mathbf{E}[X_{\tau_n} | \mathcal{F}_{\sigma_n}] \geq X_{\sigma_n},$$

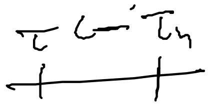


that is for $A \in \mathcal{F}_{\sigma_n}$

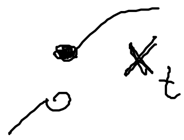
def. of cond. exp :
$$\int_A X_{\tau_n} \mathrm{d}\mathbf{P} \geq \int_A X_{\sigma_n} \mathrm{d}\mathbf{P}.$$

Since $\sigma_n \geq \sigma$ for each n, $\mathcal{F}_{\sigma_n} \supset \mathcal{F}_\sigma$. Therefore, for $A \in \mathcal{F}_\sigma$

$$\int_A X_{\tau_n} \mathrm{d}\mathbf{P} \geq \int_A X_{\sigma_n} \mathrm{d}\mathbf{P}.$$



By the right-continuity $X_{\tau_n} \to X_\tau$ and $X_{\sigma_n} \to X_\sigma$ a.s. This combined with the uniform integrability implies



$$\int_A X_\tau \mathrm{d}\mathbf{P} \geq \int_A X_\sigma \mathrm{d}\mathbf{P},$$

proving the result. $\Rightarrow E[X_\tau | \mathcal{F}_\sigma] \geq X_\sigma$. □

Exercise 10. Prove that σ_n, τ_n are indeed stopping times.

2.5 Doob-Meyer decomposition

The Doob-Meyer decomposition is the continuous time analogue of the Doob's decomposition of submartingales. While the latter is basically trivial, the Doob-Meyer decomposition is highly nontrivial, and needs further assumptions.

Recall that a class $\mathcal{D}$ of random variables are *uniformly integrable*, if for any $\varepsilon > 0$ there exists $K > 0$ such that for all $X \in \mathcal{D}$

$$\int_{|X|>K} |X| \mathrm{d}\mathbf{P} < \varepsilon.$$

Put

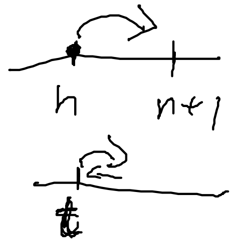
$$\mathcal{S}_a = \{\tau : \tau \text{ stopping time }, \tau \leq a\}.$$

The adapted process (X_t) belongs to the class DL is for any $a > 0$ the class $\{X_\tau\}_{\tau \in \mathcal{S}_a}$ of random variables is uniformly integrable.

(X_n) submtg. $(X_n, \mathcal{F}_n)$

$X_n = \underset{\text{mtg.}}{M_n} + \underbrace{A_n}_{\text{predictable}}$ nondecreasing $A_0 = 0$

predictable



Theorem 10 (Doob-Meyer decomposition)**.** *Let $(X_t, \mathcal{F}_t)_t$ be a right-continuous submartingale in DL. Then there exist (M_t) and (A_t) such that (M_t) is a martingale, (A_t) is an adapted nondecreasing right-continuous process with $A_0 \equiv 0$, and*

$$X_t = \underbrace{M_t}_{\text{fair}} + \underbrace{A_t}_{\text{a.s. nondecreasing}}, \quad t \geq 0.$$

Furthermore, the decomposition is unique.

Example 7. If (N_t) is a Poisson process with intensity $\lambda > 0$, then it is a submartingale. Its Doob-Meyer decomposition is

$$N_t = \overbrace{(N_t - \lambda t)}^{\text{mtg}} + \overbrace{\lambda t}^{\text{deterministic}}.$$

If (W_t) is a standard Brownian motion, then (W_t^2) is a submartingale and its Doob-Meyer decomposition is

$$W_t^2 = \underbrace{(W_t^2 - t)}_{\text{mtg.}} + \overset{\text{det.}}{t}.$$

3 Wiener process

3.1 First properties and existence

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. Then $W = (W_t, \mathcal{F}_t)_{t \geq 0}$ is a *Wiener process* or *standard Brownian motion* if

(W1) $W_0 = 0$ a.s.,

(W2) W has independent increments, that is $W_t - W_s$ is independent of $\mathcal{F}_s$ for any $s < t$,

(W3) $W_t - W_s \sim \mathrm{N}(0, t - s)$,

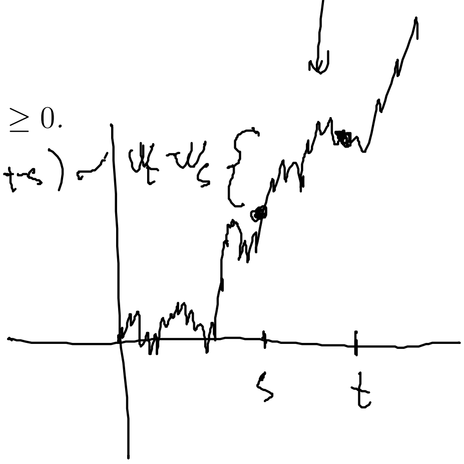
(W4) W_t has continuous sample path.

HW

Exercise 11. Show that (W2) and (W3) with $s = 0$ (i.e. $W_t \sim \mathrm{N}(0, t)$) implies (W3).

Proposition 5. *(i)* $\mathbf{E}(W_t) = 0$ *for all* t.

(ii) $\mathbf{Cov}(W_s, W_t) = \mathbf{E}(W_s W_t) = \min(s, t) =: s \wedge t$, $s, t \geq 0$.



(W3'): $W_t \sim N(0,t)$.

(W2) + (W3') $\Rightarrow$ (W3)

$$E(W_t) = 0 \quad \Leftarrow \quad W_t \sim N(0,t).\ \checkmark$$

$\underline{s<t}$

$$\mathrm{Cov}(W_s, W_t) = E(W_s W_t) - \overbrace{E(W_s)}^{=0} \cdot \overbrace{E(W_t)}^{=0}$$

$$\left(= E\left[(W_s - E(W_s))(W_t - E(W_t))\right] \right)$$

$$= E(W_s \cdot W_t) = E\left(W_s \cdot (W_s + W_t - W_s)\right)$$

$$= E(W_s^2) + E\left(W_s (W_t - W_s)\right)$$

indep.

$$= \underbrace{\mathrm{Var}(W_s)}_{=s} + 0 = s.\ \checkmark$$

(iii) For any $k \in \mathbb{N}$ and $0 \le t_1 < \cdots < t_k$, the random vector $(W_{t_1}, \dots, W_{t_k})$ has a multivariate normal distribution with mean 0 and covariance

$$\Sigma = \Sigma_{t_1,\dots,t_k} = \begin{pmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_k \end{pmatrix}.$$

Proof. Part (i) and (ii) are trivial. For part (iii) note that by the independent increment property the components of

$$X = (W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_k} - W_{t_{k-1}})^\top$$

are independent normal random variables. Therefore X is a multivariate normal. Since

$$(W_{t_1}, \dots, W_{t_k})^\top = AX,$$

the statement follows from the fact that a linear tranformation of a multivariate normal is normal with covariance matrix $A\mathbf{Cov}(X)A^\top$. □

Let (X_t) be a stochastic process with finite second moment. Then $m(t) = \mathbf{E}X_t$ is the mean value and $r(s,t) = \mathbf{Cov}(X_s, X_t) = \mathbf{E}([X_s - m(s)][X_t - m(t)])$, is the covariance function.

Clearly r is symmetric, and nonnegative definite, i.e.

$[0, \infty)$

$$\sum_{j=1}^{k} \sum_{\ell=1}^{k} c_j c_\ell \, r(t_j, t_\ell) \ge 0\,, \quad k \in \mathbb{N}\,,\ t_1, \dots, t_k \in T\,,\ c_1, \dots, c_k \in \mathbb{R}\,.$$

Definition 1. The stochastic process (X_t) is a Gaussian process with mean function $m(t)$ and covariance function $r(t,s)$ if for any $k \in \mathbb{N}$ and $t_1, \dots, t_k$ the random vector $(X_{t_1}, \dots, X_{t_k})$ has multivariate normal distribution with mean $(m(t_1), \dots, m(t_k))$ and covariance $(r(t_j, t_\ell))_{j,\ell=1}^k$.

A simple, but not very interesting example to a Gaussian process is $X_t = a(t)Z + b(t)$, where $Z \sim N(0,1)$.

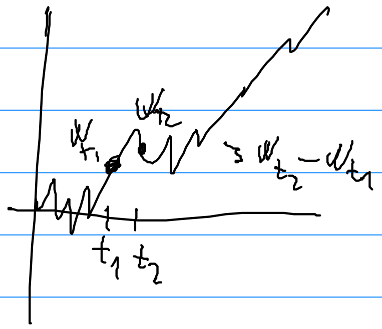
We proved that the Wiener process (W_t) is a Gaussian process with mean $m(t) \equiv 0$ and covariance function $r(s,t) = \min(s,t)$. This could be the definition of the Wiener process.

Proposition 6. *Let (W_t) be a continuous Gaussian process with mean 0 and covariance function $r(s,t) = \min(s,t)$. Then (W_t) is a Wiener process.*

$$0 \leq t_1 < t_2 < \dots < t_\ell$$

$$\begin{pmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ W_{t_\ell} \end{pmatrix} = \overset{A}{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & & \\ & & & 1 & \\ & & & & 1 \\ 1 & \dots & & & 1 \end{pmatrix}} \overbrace{\begin{pmatrix} W_{t_1} \\ W_{t_2} - W_{t_1} \\ W_{t_3} - W_{t_2} \\ \vdots \\ W_{t_\ell} - W_{t_{\ell-1}} \end{pmatrix}}^{\underline{X}}$$

difficult?



$$\begin{pmatrix} W_{t_1} \\ W_{t_2} - W_{t_1} \\ W_{t_3} - W_{t_2} \\ \vdots \\ W_{t_\ell} - W_{t_{\ell-1}} \end{pmatrix}$$

multivariate normal

mean $= \underline{0} = (0, \dots, 0)$

covariance matrix:

$$\begin{pmatrix} t_1 & & & & 0 \\ 0 & t_2 - t_1 & & & \\ 0 & 0 & t_3 - t_2 & & \\ & & & \ddots & \\ 0 & & & 0 & t_\ell - t_{\ell-1} \end{pmatrix}$$

$$\begin{pmatrix} w_{t_1} \\ \vdots \\ w_{t_\ell} \end{pmatrix} = A \cdot \underline{X} \qquad \underline{X} \sim \mathcal{N}_\ell(\underline{0}, \Sigma_X)$$

$$A\underline{X} \sim \mathcal{N}_\ell\left(0,\ A\Sigma_X A^T\right)$$

$$\begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} t_1 & & & 0 \\ & t_2 - t_1 & & \\ & & \ddots & \\ 0 & & & t_\ell - t_{\ell-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ & 1 & & 1 \\ & & \ddots & \vdots \\ 0 & & & 1 \end{pmatrix}$$

HW

$$= \begin{pmatrix} t_1 & \cdots & & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & \ddots & \\ t_1 & t_2 & & t_\ell \end{pmatrix}$$

Proof of Prop. 6.

(W0) : $W_0 = 0$ a.s.

(W1) : independent incr.

(W2) : $W_t - W_s \sim N(0, t-s)$

(W3) cont. ✓ assumed

(W0) : $Cov(W_0, W_0) = Var\, W_0 = 0 \Rightarrow W_0 = 0$ a.s. ✓

(W2) : $W_t - W_s$: normal ✓

$E(W_t - W_s) = 0$.

$Var(W_t - W_s) = Cov(W_t - W_s, W_t - W_s) =$

$= Cov(W_t, W_t) + Cov(W_s, W_s) - 2\, Cov(W_s, W_t)$

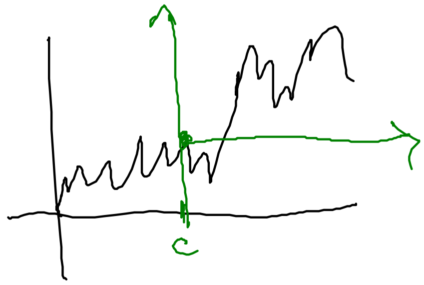
$= r(t,t) + r(s,s) - 2r(s,t)$

$= t + s - 2s = t - s$. ✓

(W3) : multivariate normal + $Cov = 0 \Rightarrow$ independence

HW

i)



Exercise 12. Prove the statement.

HW

Exercise 13. Let $(W(t))$ be SBM. Show that
(i) $W_1(t) = W(c+t) - W(c)$, $t \geq 0$;
(ii) $W_2(t) = \sqrt{c}\, W(t/c)$, $t \geq 0$;
(iii) $W_3(t) = tW(1/t)$
are SBM.

$aZ \sim N(0, a^2)$
$Z \sim N(0,1)$

Kolmogorov's consistency theorem yields the the existence of Gaussian processes.

$\mathbb{T} = [0, \infty)$

Theorem 11. *Let $\mathbb{T} \subset \mathbb{R}$, and let $m(t)$ be an arbitrary function and $r(s,t)$ a nonnegative definite function. Then there exists a Gaussian process $(X_t)_{t\in\mathbb{T}}$ with mean function m and covariance function r.*

Therefore, *apart from continuity*, we have a Wiener process. That is, we have a probability space $(\mathbb{R}^{[0,\infty)}, \mathcal{B}^{[0,\infty)}, \mathbf{P})$ and a stochastic process $(\widetilde{W}_t(\omega) = \omega_t)_{t\geq 0}$, which satisfies (W1)–(W3).

Let $C = C[0,\infty)$ be the space of continuous function on $[0,\infty)$. We have to show that $\mathbf{P}(\widetilde{W} \in C) = 1$. *The problem is that C does not belong to the product σ-algebra $\mathcal{B}^{[0,\infty)}$.* Indeed, it can be shown that

$$\mathcal{B}^{[0,\infty)} = \cup\{\pi_K^{-1}(\mathcal{B}^K) : K \subset [0,\infty), K \text{ countable}\}.$$

Therefore, if $C \in \mathcal{B}^{[0,\infty)}$, then $C = \pi_K^{-1}(\mathcal{B}^K)$ for some countable $K \subset [0,\infty)$. But continuity cannot be determined from the values of the function on a countable set. Similarly,

$$\left\{\omega \in \mathbb{R}^{[0,\infty)} : \sup_{0\leq t\leq 1} \omega_t \leq x\right\}, \quad x \in \mathbb{R},$$

is not $\mathcal{B}^{[0,\infty)}$-measurable, so we cannot define $\sup_{t\in[0,1]} \widetilde{W}_t$.

Thus the setup in Kolmogorov's consistency theorem cannot deal with continuous processes. We need a different approach.

Recall that Y is a *modification* of X if $X_t = Y_t$ a.s. for any fix t, i.e. $\mathbf{P}(X_t = Y_t) = 1$ for each $t \geq 0$.

Theorem 12 (Kolmogorov continuity theorem)**.** *Let $(X_t)_{t\in[0,T]}$ be a stochastic process on $(\Omega, \mathcal{A}, \mathbf{P})$, such that for some positive constants α, β, C*

$$\mathbf{E}|X_t - X_s|^\alpha \leq C|t-s|^{1+\beta}, \quad 0 \leq s,t \leq T.$$

Kolmogorov consistency thm:

$(W_{t_1}, W_{t_2}, \dots, W_{t_n})$ finite dimensional distributions

$\sim N_n(m_n, \Sigma_n)$

$n \in \mathbb{N}, \ t_1, \dots, t_n \in [0, \infty)$

$\Sigma_n = \left(r(t_i, t_j) \right)_{i,j=1}^n$ $\quad m_n = (m(t_1), \dots, m(t_n))^T$

We want a process $(W_t)_t$ $\quad t \in [0, \infty)$

We can define/determine the finite dimensional distributions.

Under what conditions does (W_t) exist?

family of finite dimensional dist. $(\mu_{t_1, \dots, t_n})$ | process $(W_t)_{t \in [0, \infty)}$

Kolmogorov: – permutation invariant

consistent
(konsistent)

$$\mu_{1,2}(A_1, A_2) = \mu_{2,1}(A_2, A_1)$$

$$\overset{\shortparallel}{P}(W_1 \in A_1, W_2 \in A_2) \qquad \overset{\shortparallel}{P}(W_2 \in A_2, W_1 \in A_1)$$

– $P(W_1 \in A_1, W_2 \in \mathbb{R}) = P(W_1 \in A_1)$

compatibility (kompatibilität)