It is enough to show that on this subsequence the second and third terms in decomposition (5) tends to 0. For the second term

$$\begin{aligned}
\left| \int_A (X_\tau - X_{\tau\wedge n_k}) \mathrm{d}\mathbf{P} \right| &= \left| \int_{A\cap\{\tau>n_k\}} (X_\tau - X_{\tau\wedge n_k}) \mathrm{d}\mathbf{P} \right| \\
&\le \int_{A\cap\{\tau>n_k\}} (|X_\tau| + |X_{n_k}|) \mathrm{d}\mathbf{P} \\
&\le \int_{\{\tau>n_k\}} |X_\tau| \, \mathrm{d}\mathbf{P} + \int_{\{\tau>n_k\}} |X_{n_k}| \, \mathrm{d}\mathbf{P}.
\end{aligned}$$

Similarly, for the third term

$$\begin{aligned}
\left| \int_A (X_\sigma - X_{\sigma\wedge n_k}) \mathrm{d}\mathbf{P} \right| &= \left| \int_{A\cap\{\sigma>n_k\}} (X_\sigma - X_{n_k}) \mathrm{d}\mathbf{P} \right| \\
&\le \int_{\{\sigma>n_k\}} |X_\sigma| \, \mathrm{d}\mathbf{P} + \int_{\{\tau>n_k\}} |X_{n_k}| \, \mathrm{d}\mathbf{P}.
\end{aligned}$$

Using (2) both upper bounds tend to 0. □

Corollary 2. *Assume that (X_n) is (super-, sub-) martingale, τ is a stopping time, $\mathbf{E}(|X_\tau|) < \infty$ and (3) holds. Then*

(i) $\mathbf{E}(X_\tau|\mathcal{F}_1) \le X_1$ *and* $\mathbf{E}(X_\tau) \le \mathbf{E}(X_1)$ *for supermartingales;*

(ii) $\mathbf{E}(X_\tau|\mathcal{F}_1) \ge X_1$ *and* $\mathbf{E}(X_\tau) \ge \mathbf{E}(X_1)$ *for submartingales;*

(iii) $\mathbf{E}(X_\tau|\mathcal{F}_1) = X_1$ *and* $\mathbf{E}(X_\tau) = \mathbf{E}(X_1)$ *for martingales.*

Some conditions are needed for the optional stopping to hold.

Example 2 (Simple symmetric random walk)**.** Let $\xi, \xi_1, \xi_2, \ldots$ are iid random variables with $\mathbf{P}(\xi = \pm 1) = 1/2$. Let $S_0 = 1$ and $S_n = S_{n-1} + \xi_n$. Then (S_n) is martingale. Let $\tau = \min\{n : S_n = 0\}$. Then τ is a stopping time and the martingale $(S_{\tau\wedge n})_n$ tends to 0 a.s. The optional stopping does not hold as $S_\tau \equiv 0$ a.s., while $S_0 = 1$. Clearly, condition (3) does not hold.

Theorem 5 (Wald identity)**.** *Let $X, X_1, X_2, \ldots$ be iid random variables with $\mathbf{E}X = \mu \in \mathbb{R}$, and let τ be a stopping time with $\mathbf{E}(\tau) < \infty$. Put $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$. Then $\mathbf{E}(S_\tau) = \mu\mathbf{E}(\tau)$.*

$$E(S_n) = E\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} E X_i = n\mu.$$

$\{\tau \le k-1\}^c = \{\tau \ge k\} \in \mathcal{F}_{k-1}$

$\sum_{i=1}^{\tau} X_i$ $\quad \in \mathcal{F}_{k-1} = \sigma(X_1, \dots, X_{k-1})$

Proof. First assume $X \geq 0$. We have

$E(X) = \int_0^\infty P(X > x)\,dx = \sum_{k=1}^\infty P(X \geq k)$

$$
\begin{aligned}
\mathbf{E}(S_\tau) &= \mathbf{E}\left(\sum_{k=1}^{\infty} \mathbf{I}(\tau \geq k) X_k\right) \\
&= \sum_{k=1}^{\infty} \mathbf{E}(\mathbf{I}(\tau \geq k) X_k) \\
&\overset{?}{=} \sum_{k=1}^{\infty} \mathbf{E}\mathbf{I}(\tau \geq k)\mathbf{E}(X_k) \\
&= \mu \sum_{k=1}^{\infty} \mathbf{P}(\tau \geq k) \\
&= \mu \mathbf{E}(\tau).
\end{aligned}
$$

$E\left[\overbrace{I(\tau \geq k)} \cdot X_k\right] = E(I(\tau \geq k)) \cdot E(X_k)$

independence

To see the general case consider the decomposition $S_\tau = S_\tau^{(+)} - S_\tau^{(-)}$ where

$$S_\tau^{(+)} = \sum_{k=1}^{\infty} X_k^+ \mathbf{I}(\tau \geq k)$$

$a^+ = \max(0, a)$

and

$$S_\tau^{(-)} = \sum_{k=1}^{\infty} X_k^- \mathbf{I}(\tau \geq k).$$

$a^- = \max(0, -a)$

$a = a^+ - a^-$ $\square$

As a simple application of the optional stopping problem we consider the gambler's ruin problem. There is an elementary but longer way to derive these formulas.

Example 3 (Gambler's ruin)**.** Let $X, X_1, X_2, \dots$ be iid random variables such that $\mathbf{P}(X = 1) = p = 1 - \mathbf{P}(X = -1)$, $0 < p < 1$, and put $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$. Fix $a, b \in \mathbb{N}$ and let

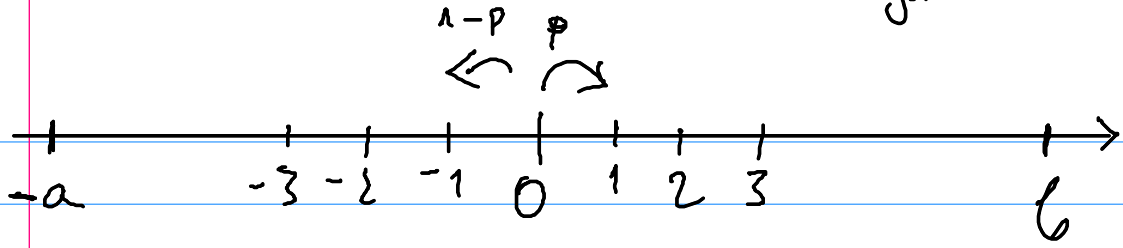
$$\tau = \tau_{a,b}(p) = \overset{\text{min}}{\inf}\{n : S_n \overset{= b}{\geq} b \text{ or } S_n \leq -a\},$$

with the convention $\inf \emptyset = \infty$. Let $(\mathcal{F}_n)$ be the natural filtration, i.e. $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, $n \in \mathbb{N}$.

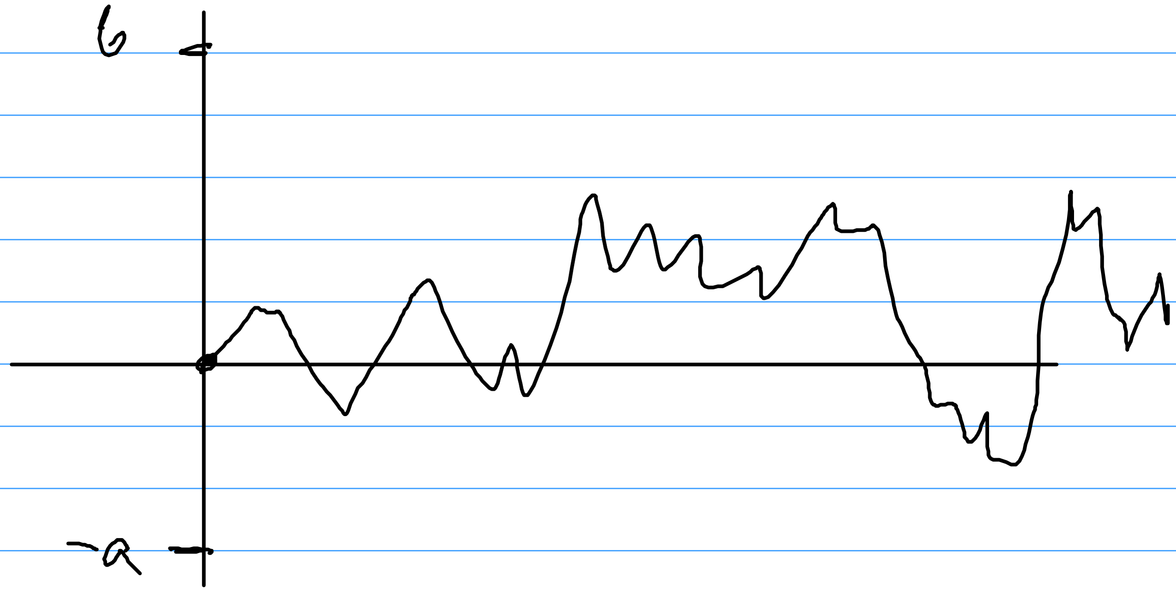
It is easy to show that $\overbrace{\mathbf{P}(\tau < \infty) = 1}^{\text{HW}}$, and τ is a stopping time. Furthermore, $|S_\tau| \leq \max(a, b)$, in particular $\mathbf{E}|S_\tau| < \infty$ and

$$\liminf_{n \to \infty} \int_{\{\tau > n\}} |S_n| \, d\mathbf{P} \leq \liminf_{n \to \infty} \max(a, b) \mathbf{P}(\tau > n) = 0.$$

We can apply optional stopping



Gambler: starts with a ducats stops if he loses all his money or he wins b ducats



$\boxed{p = \frac{1}{2}}$

- If $\boxed{p = \frac{1}{2}}$ then $E(X) = 0 \Rightarrow (S_n)$ is a martingale

$$E(S_\tau) = E(S_0) = 0$$

$$\parallel$$

$$-a \cdot P(S_\tau = -a) + b \cdot \underbrace{\left[1 - P(S_\tau = -a)\right]}_{P(S_\tau = b)}$$

$$-a\, P(S_\tau = -a) - b\, P(S_\tau = -a) + b = 0$$

$$\boxed{\mathbf{P}(S_\tau = -a) = \frac{b}{a+b}}$$

First assume that $p = 1/2$. Then $\mathbf{E}X = 0$ and (S_n) is a martingale. Therefore, by the optional stopping theorem

$$\begin{aligned} 0 = \mathbf{E}S_0 = \mathbf{E}S_\tau &= -a\mathbf{P}(S_\tau = -a) + b\mathbf{P}(S_\tau = b) \\ &= -a(1 - \mathbf{P}(S_\tau = b)) + b\mathbf{P}(S_\tau = b). \end{aligned}$$

Thus

$$\mathbf{P}(S_\tau = b) = \frac{a}{a+b} \quad \text{and} \quad \mathbf{P}(S_\tau = -a) = \frac{b}{a+b}.$$

Using that $(S_n^2 - n)$ is a martingale, we can determine $\mathbf{E}\tau$. Since

opt. stopping $$0 = \mathbf{E}(S_0^2 - 0) = \mathbf{E}(S_\tau^2 - \tau)$$

$\mathcal{F}_n = \sigma(X_1, \dots, X_n)$

we obtain

$$\mathbf{E}\tau = \mathbf{E}(S_\tau^2) = a^2\mathbf{P}(S_\tau = -a) + b^2\mathbf{P}(S_\tau = b) = a^2\frac{b}{a+b} + b^2\frac{a}{a+b} = ab.$$

The case $p \neq 1/2$ is different. Introduce

$$Z_n = s^{S_n} = \prod_{k=1}^{n} s^{X_k}$$

$\mathbf{E}[Z_n \mid \mathcal{F}_{n-1}] = s^{S_{n-1}}\cdot$

with respect to the natural filtration

with $s = (1-p)/p = 1/r$. Then (Z_n) is a martingale and

$s^{S_{n-1}+X_n}$ $\cdot\, \mathbf{E}[s^{X_n} \mid \mathcal{F}_{n-1}]$

$$Z_\tau = s^b\mathbf{I}(S_n = b) + s^{-a}\mathbf{I}(S_n = -a) \leq s^b + s^{-a},$$

thus $\mathbf{E}Z_\tau < \infty$ and

$$\liminf_{n\to\infty} \int_{\{\tau > n\}} |Z_n| \,\mathrm{d}\mathbf{P} \leq (s^b + s^{-a}) \liminf_{n\to\infty} \mathbf{P}\{\tau > n\} = 0.$$

$= Z_{n-1} \cdot \mathbf{E}(s^X)$

Again, by the optional sampling theorem

HW $\mathbf{E}(s^X) = 1$

$\mathbf{E}(Z_0) = \mathbf{E}(Z_\tau)$

$$\begin{aligned} &s^{-a}\mathbf{P}(S_\tau = -a) + s^b\,(1 - \mathbf{P}(S_\tau = -a)) \\ &= s^{-a}\mathbf{P}(S_\tau = -a) + s^b\mathbf{P}(S_\tau = b) \\ &= \mathbf{E}(s^{S_\tau}) = \mathbf{E}(Z_\tau) \\ &= \mathbf{E}(Z_1) = \mathbf{E}(s^X) = 1. \end{aligned}$$

Rearranging we obtain

$$\mathbf{P}(S_\tau = -a) = \frac{1 - s^b}{s^{-a} - s^b}\,\frac{r^b}{r^b} = \frac{r^b - 1}{r^{a+b} - 1} = \frac{1 - r^b}{1 - r^{a+b}}.$$

$E(X) = P(X=1) - P(X=-1)$

$E(X) \quad = p - (1-p) = 2p-1$

To obtain $\mathbf{E}\tau$, using the Wald identity

$$\mathbf{E}S_\tau = (2p-1)\mathbf{E}\tau,$$

from which

$$\mathbf{E}\tau = \frac{1}{2p-1}\mathbf{E}S_\tau = \frac{1}{2p-1}\left[-a\mathbf{P}(S_\tau = -a) + b\mathbf{P}(S_\tau = b)\right].$$

Exercise 4. Show that $\tau < \infty$ a.s.

2 Continuous time martingales

$0 \qquad T$

2.1 Definitions and simple properties

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability spance and $(\mathcal{F}_t)_{t\geq 0}$ a *filtration*, i.e. an increasing sequence of σ-algebras. The time horizon is either finite or infinite, $t \in [0, T]$ or $t \in [0, \infty)$.

$\mathcal{F}_t \supset \mathcal{F}_s$, $t > s$

In what follows we always assume that the filtration satisfies the *usual propreties*:

(i) $\mathcal{F}_0$ contains the $\mathbf{P}$-null sets;

$\cap \mathcal{F}_t \supset$ in general

(ii) $(\mathcal{F}_t)_t$ is right-continuous, i.e. $\cap_{s>t}\mathcal{F}_s =: \mathcal{F}_{t+} = \mathcal{F}_t$.

Let (X_t) and (Y_t) be stochastic processes. The process Y is a *modification* of X if $X_t = Y_t$ a.s. for any fix t, i.e. $\mathbf{P}(X_t = Y_t) = 1$ for each $t \geq 0$. The processes X and Y are *indistuinguishable* if their sample path are the same almost surely, i.e.

$$\mathbf{P}(X_t = Y_t,\ t \geq 0) = 1.$$

intersection of σ-alg. is a σ-alg.

They have the *same finite dimensional distributions* if for all $0 \leq t_1 < t_2 < \ldots < t_n < \infty$ and $A \in \mathcal{B}(\mathbb{R}^n)$

$$\mathbf{P}\left((X_{t_1}, \ldots, X_{t_n}) \in A\right) = \mathbf{P}\left((Y_{t_1}, \ldots, Y_{t_n}) \in A\right).$$

Example 4. Let U be uniform$(0,1)$, and $X_t \equiv 0$, $t \in [0,1]$, and $Y_t = \mathbf{I}(U = t)$. Then Y is a modification of X, but they are not indistuinguishable, since

$$\mathbf{P}(X_t = Y_t,\ t \geq 0) = 0.$$

X

The process $(X_t)_t$ is *adapted to the filtration* $(\mathcal{F}_t)_t$, if X_t is $\mathcal{F}_t$-measureable for each $t \geq 0$. The process $(X_t, \mathcal{F}_t)_t$ is a *martingale* if

Y

(i) $(X_t)_t$ is adapted to $(\mathcal{F}_t)_t$;

(ii) $\mathbf{E}|X_t| < \infty$ for all $t \geq 0$;

(iii) $\mathbf{E}[X_t|\mathcal{F}_s] = X_s$ a.s. for all $t \geq s$.

It is sub- or supermartingale if (i) and (ii) holds, and (iii) holds with $\geq$ or $\leq$ instead of $=$.

A random variable $\tau : \Omega \to [0, \infty)$ is a *stopping time* if $\{\tau \leq t\} \in \mathcal{F}_t$. The σ-algebra of the events prior to τ, or pre-τ-σ-algebra is

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Exercise 5. Show that $\mathcal{F}_\tau$ is indeed a σ-algebra.

The next result is obvious, but very useful.

Proposition 2. *Let $(X_t, \mathcal{F}_t)$ be a (sub-, super-) martingale. Then for any sequence $0 \leq t_0 < t_1 < \ldots < t_N < \infty$ the process $(X_{t_n}, \mathcal{F}_{t_n})_{n=0}^N$ is a discrete time martingale.*

Lemma 3. *Let σ, τ be stopping times.*

(i) τ is $\mathcal{F}_\tau$-measureable.

(ii) If $\tau \equiv t$ then $\mathcal{F}_\tau = \mathcal{F}_t$.

(iii) $\sigma \wedge \tau = \min(\sigma, \tau)$ and $\sigma \vee \tau = \max(\sigma, \tau)$ are stopping times.

(iv) If $\sigma \leq \tau$, then $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$.

(v) If $(X_t)_t$ is right-continuous and adapted then X_τ is $\mathcal{F}_\tau$-measurable.

Exercise 6. Prove the lemma.

Remark 1. In continuous time the technical detailes are trickier.

The process $(X_t)_t$ is adapted to $(\mathcal{F}_t)_t$, if X_t is $\mathcal{F}_t$-measurable, and it is *progressively measureable* with respect to $(\mathcal{F}_t)_t$, if for all $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$

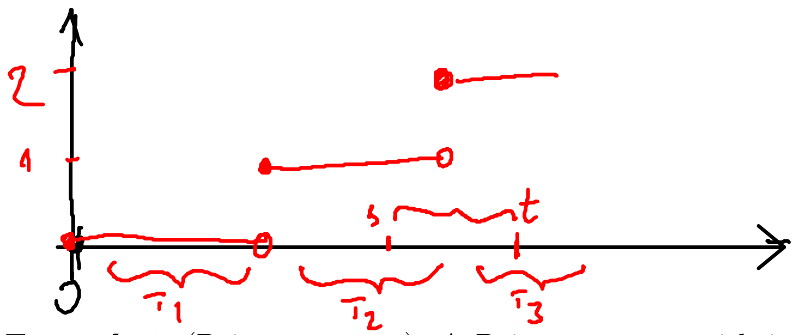
$[0,t] \times \Omega$

$$\{(s, \omega) : \; s \leq t, \; X_s(\omega) \in A\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t,$$

where $\mathcal{B}$ stands for the Borel sets, and $\otimes$ is the product σ-algebra. In what follows we always need progressive measureability, adaptedness is not enough.

The next statement says that the situation is not too bad.

Proposition 3. *If $(X_t)_t$ is right continuous and adatped, then it is progressively measureable.*



$\tau_1, \tau_2, \tau_3, \dots$ iid $\mathrm{Exp}(\lambda)$

Example 5 (Poisson process)**.** A Poisson process with intensity $\lambda > 0$ is an adapted integer valued RCLL (right continuous with left limits) process $N = (N_t, \mathcal{F}_t)_{t\geq 0}$ such that

(i) N has independent increments, that is $N_t - N_s$ is independent of $\mathcal{F}_s$ for any $s < t$,

(ii) $N_0 = 0$ a.s.,

(iii) $N_t - N_s \sim \text{Poisson}(\lambda(t-s))$.

Exercise 7. Show that $(N_t - \lambda t)$ is martingale.

Proposition 4. *Let (X_t) be a martingale, and φ a convex function such that $\mathbf{E}|\varphi(X_t)| < \infty$ for all $t \geq 0$. Then $(\varphi(X_t))$ is submartingale.*

Furthermore if (X_t) is a submartingale and φ nondecreasing and convex that $\mathbf{E}|\varphi(X_t)| < \infty$ for all $t \geq 0$, then $(\varphi(X_t))$ is a submartingale.

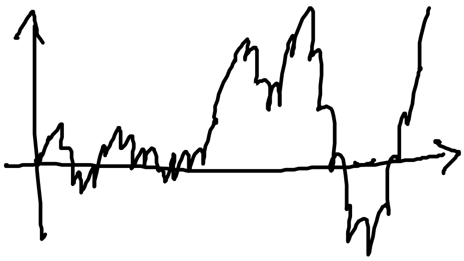
Example 6 (Wiener process)**.** The Wiener process or standard Brownian motion is an adapted process $W = (W_t, \mathcal{F}_t)_{t\geq 0}$ such that

(i) W has independent increments, that is $W_t - W_s$ is independent of $\mathcal{F}_s$ for any $s < t$,

(ii) $W_0 = 0$ a.s.,

(iii) $W_t - W_s \sim \mathrm{N}(0, t-s)$, ← Gaussian

(iv) W_t has continuous sample path.



Exercise 8. Show that (W_t) and $(W_t^2 - t)$ are martingales.

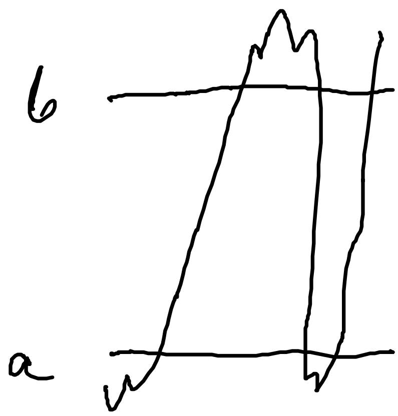
2.2 Martingale convergence theorem

Consider an adapted stochastic process $(X_t)_{t\geq 0}$. Fix $a < b$, and a finite set $F \subset [0, \infty)$. Let U_F denote the number of *upcrossings* of the interval $[a, b]$ by the restricted process $(X_t)_{t\in F}$. Formally, let $\tau_0 = 0$, and

$$\tau_{2k-1} = \min\{t \in F: \ t \geq \tau_{2k-2}, X_t < a\},$$
$$\tau_{2k} = \min\{t \in F: \ t \geq \tau_{2k-1}, X_t > b\}.$$

The number of upcrossings on F is

$$U_F(a, b) = U_F = \max\{k : \tau_{2k} < \infty\}.$$



$(N_t) \sim$ Poisson process $\lambda > 0$

- ind. inc.
- $N_t - N_s \sim \text{Poisson}(\lambda(t-s))$
- $N_0 = 0$ a.s

$$\mathcal{F}_t = \sigma(N_s : s \leq t) \quad \text{natural filtration}$$

$N_t - \lambda t$ is mtg.

$s < t$:

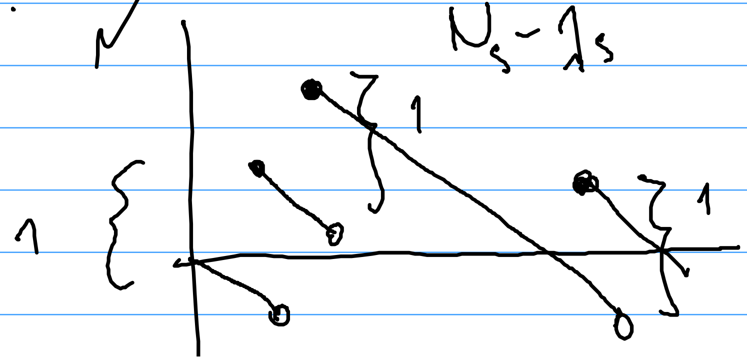
$$\mathbb{E}[N_t - \lambda t \mid \mathcal{F}_s] = \mathbb{E}[N_s + N_t - N_s - \lambda t \mid \mathcal{F}_s]$$

$$= N_s + \mathbb{E}[N_t - N_s - \lambda t \mid \mathcal{F}_s] =$$

indep.

$$= N_s + \mathbb{E}(N_t - N_s - \lambda t) = N_s + \lambda(t-s) - \lambda t$$

$$= N_s - \lambda s. \checkmark$$



(W_t):

W_t is mtg.

$t>s$

$$E[W_t \mid \mathcal{F}_s] = E[\overbrace{W_s}^{\text{meas. } \mathcal{F}_s} + \overbrace{W_t - W_s}^{\text{indep. of } \mathcal{F}_s} \mid \mathcal{F}_s]$$

$$= W_s + E(\underbrace{W_t - W_s}_{\sim N(0,t-s)}) = W_s \quad \checkmark$$

$\Rightarrow (W_t)$ mtg.

$\Rightarrow (W_t^2)$ submtg.

$(W_t^2 - t)$ is mtg.

$$E[W_t^2 - t \mid \mathcal{F}_s] = E[\overbrace{W_s^2}^{\mathcal{F}_s \text{ meas.}} + W_t^2 - W_s^2 - t \mid \mathcal{F}_s]$$

$$= W_s^2 - t + E[\underbrace{W_t^2 - W_s^2}_{(W_t - W_s)(W_t + W_s)} \mid \mathcal{F}_s]$$

$$= W_s^2 - t + E\Big[\underbrace{W_s}_{\text{meas.}} \cdot (W_t - W_s) + W_t \cdot (W_t - W_s) \,\Big|\, \mathcal{F}_s\Big]$$

$$= W_s^2 - t + W_s \cdot E\big[W_t - W_s \,\big|\, \mathcal{F}_s\big] +$$

$$+ E\Big[(W_s + W_t - W_s)(W_t - W_s) \,\Big|\, \mathcal{F}_s\Big]$$

$$= W_s^2 - t + W_s \cdot E(W_t - W_s) +$$

$$+ W_s \cdot E(W_t - W_s) + E\big((W_t - W_s)^2\big)$$

$$= W_s^2 - t + 0 + 0 + t - s = W_s^2 - s$$

$$\Rightarrow \quad (W_t^2 - t) \quad \text{is a mtg.}$$

We can extend the definition of infinite sets $I \subset [0, \infty)$ as

$$U_I = \sup\{U_F : \ F \subset I, \ F \text{ finite}\}.$$

We have the upcrossing inequality.

Theorem 6 (Upcrossing inequality). *Let (X_t) be a right-continuous submartingale. For any $a < b$ and $0 \leq S \leq T < \infty$*

$$(b - a)\mathbf{E}U_{[S,T]} \leq \mathbf{E}(X_T - a)^+ - \mathbf{E}(X_S - a)^+.$$

Proof. Consider an enumeration of the countable set $\mathbf{Q} \cap [S, T]$ as

$$\mathbf{Q} \cap [S, T] = \{q_1, q_2, \ldots\},$$

and let $F_n = \{q_1, \ldots, q_n\} \cup \{S, T\}$. Then $(X_t, \mathcal{F}_t)_{t \in F_n}$ is a discrete time sub martingale, therefore, by the upcrossing inequality

$$(b - a)\mathbf{E}U_{F_n} \leq \mathbf{E}(X_T - a)^+ - \mathbf{E}(X_S - a)^+$$

Since F_n is increasing, U_{F_n} is increasing, and by the right-continuity of (X_t)

$$\lim_{n \to \infty} U_{F_n} = U_{[S,T]} \quad \text{a.s.}$$

In particular, $U_{[S,T]}$ is measurable, and by the monotone convergence theorem the result follows. $\square$

Theorem 7 (Martingale convergence theorem). *Let (X_t) be a right-continuous submartingale such that*

$$\sup_{t \geq 0} \mathbf{E}(X_t^+) < \infty.$$

Then $\lim_{t \to \infty} X_t = X$ exists a.s. and $\mathbf{E}|X| < \infty$.

Proof. By the upcrossing inequality and the monotone convergence theorem for any $a < b$

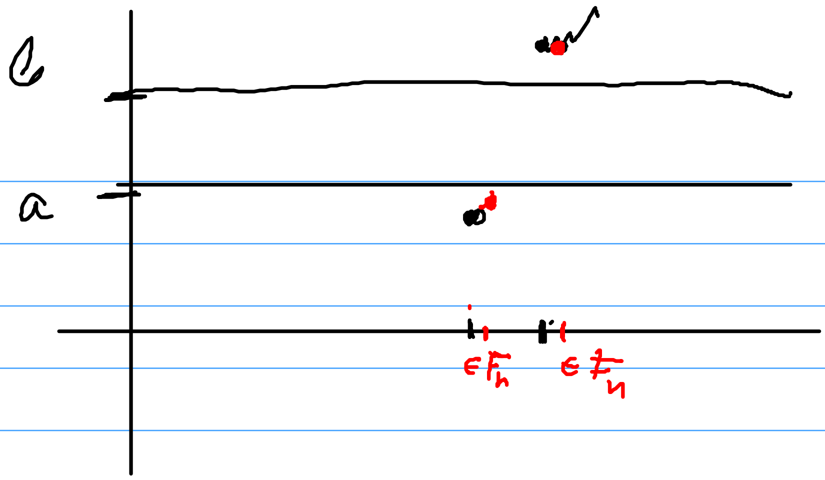
$$\mathbf{E}U_{[0,\infty)}(a, b) \leq \frac{\sup_{t \geq 0} \mathbf{E}X_t^+ + |a|}{b - a}.$$

Therefore, for any $a < b$ the upcrossings $U_{[0,\infty)}(a, b)$ are a.s. finite. Thus almost surely the uppcrossings are finite for all $a < b$ rationals, implying the existence of the limit.

The integrability of the limit follows from Fatou's lemma. $\square$

$A_{a,b} = \{ U_{[0,\infty)}(a,b) = \infty \}$ $\quad P(A_{a,b}) = 0$

$\bigcup_{a,b \in \mathbb{Q}} A_{a,b} = A \quad P(A) = 0.$



I