$N : \Omega \to \mathbb{N} = \{0, 1, 2, \dots\}$ stopping time

if $\{N \leq n\} \in \mathcal{F}_n$

$(\Omega, \mathcal{F}, P)$ prob. space $(\mathcal{F}_n)$ filtration

$\{\emptyset, \Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$ (X_n) adapted (to $(\mathcal{F}_n)$) if

1 Discrete time martingales

X_n is $\mathcal{F}_n$-meas.

1.1 Definition, properties

1.2 Martingale convergence theorem

$E[X_{n+1} | \mathcal{F}_n] = X_n$ a.s.

mtg.

1.3 Doob's decomposition and the martingale Borel–Cantelli lemma

1.4 Doob's maximal inequality

$\leq$ supermtg

Our first optional stopping theorem is the following.

{thm:opt-1}

$\geq$ submtg.

Theorem 1. *Let $(X_n)_n$ be a submartingale and let N be a bounded stopping time, i.e. $N \leq k$ a.s. for some $k \in \mathbb{N}$. Then*

$E[X_{n+1} | \mathcal{F}_n] \geq X_n$

$$\mathbf{E}X_0 \leq \mathbf{E}X_N \leq \mathbf{E}X_k.$$

Proof. We proved that the stopped process $(X_{n \wedge N})_n$ is submartingale, thus

$E[X_{n+1}] \geq E[X_n]$

$$\mathbf{E}X_0 = \mathbf{E}X_{N \wedge 0} \overset{\text{submtg}}{\leq} \mathbf{E}X_{N \wedge k} = \mathbf{E}X_N.$$

For the other direction, put $K_n = \mathbf{I}(N < n) = \mathbf{I}(N \leq n-1)$. Then K_n is $\mathcal{F}_{n-1}$-measurable, so $(K_n)_n$ is predictable. Therefore $(K \cdot X)_n$ is submartingale, where

$\Downarrow$

$\{N \leq n-1\} \in \mathcal{F}_{n-1}$

$E(X_0) \leq E(X_n) \leq E(X_k)$

$$(K \cdot X)_n = \sum_{i=1}^{n} \underbrace{\mathbf{I}(N \leq i-1)}_{K_i}(X_i - X_{i-1}) = X_n - X_{N \wedge n}.$$

for n fix deterministic

That is

$$\mathbf{E}X_k - \mathbf{E}X_N = \mathbf{E}(K \cdot X)_k \geq \mathbf{E}(K \cdot X)_0 = 0.$$

□

An easy consequence is Doob's maximal inequality.

Theorem 2 (Doob's maximal inequality)**.** *Let $(X_k, \mathcal{F}_k)_k$ be a submartingale, and put*

$$M_n = \max_{1 \leq k \leq n} X_k.$$

Then for any $x > 0$

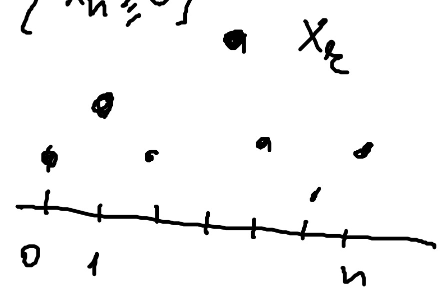
$$x\mathbf{P}(M_n \geq x) \leq \int_{\{M_n \geq x\}} X_n \mathrm{d}\mathbf{P} \leq \mathbf{E}X_n^+, = \int_{\{X_n \geq 0\}} X_n \, dP$$

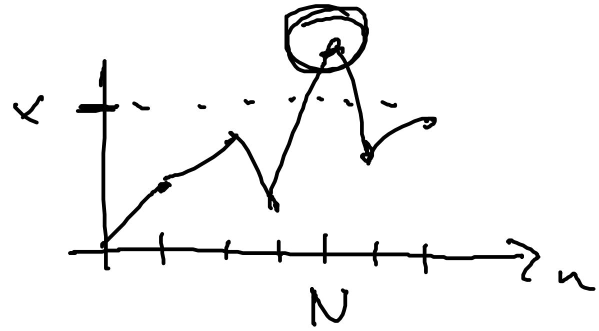
where $a^+ = \max\{a, 0\}$.

$\Uparrow$

positive part

?





Proof. The second inequality is obvious.

Let $N = \min\{\min\{k : X_k \geq x, k = 1, 2, \ldots, n\}, n\}$. Then N is a bounded stopping time. Since $X_N \geq x$ on $\{M_n \geq x\}$

$N \leq n$

$$\int_{\{M_n \geq x\}} x\,dP = x\mathbf{P}\{M_n \geq x\} \leq \int_{\{M_n \geq x\}} X_N \mathrm{d}\mathbf{P}.$$

By Theorem 1 $\mathbf{E}X_N \leq \mathbf{E}X_n$, and $X_N = X_n$ on the event $\{M_n < x\}$, thus

$$E(X_N) = \int_{\{M_n \geq x\}} X_N\,dP + \int_{\{M_n<x\}} X_N \mathrm{d}\mathbf{P} \leq \int_{\{M_n<x\}} X_n \mathrm{d}\mathbf{P} + \int_{\{M_n \geq x\}} X_n\,dP = E(X_n)$$

proving the statement. $\square$

$$\Rightarrow \int_{\{M_n \geq x\}} X_N\,dP \leq \int_{\{M_n \geq x\}} X_n\,dP$$

We obtain a new proof for Kolmogorov's maximal inequlity.

Example 1 (Kolmogorov's maximal inequality)**.** Let $\xi, \xi_1, \ldots$ be independent random variables with $\mathbf{E}\xi_i = 0$, and $\mathbf{E}\xi_i^2 = \sigma_i^2 < \infty$. Then $X_n = \sum_{i=1}^n \xi_i$ is a martingale with respect to the natural filtration. Therefore $(X_n^2)_n$ is a submartingale and

natural $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$

$$\mathbf{P}\left(\max_{1\leq k\leq n} |X_k| \geq x\right) = \mathbf{P}\left(\max_{1\leq k\leq n} X_k^2 \geq x^2\right) \leq x^{-2}\mathbf{E}X_n^2 = x^{-2}\sum_{i=1}^n \sigma_i^2.$$

y convex

$$P(|X_n| \geq x) \leq \frac{E(X_n^2)}{x^2}$$

Markov (Chebishev)

For an infinite sequence we obtain the following.

Corollary 1. *If* $(X_k, \mathcal{F}_k)$ *is a submartingale and* $x > 0$*, then*

$$\mathbf{P}(\sup_n X_n \geq x) \leq \frac{1}{x}\sup_n \mathbf{E}X_n^+.$$

Proof. Follows from the previous result combined with the monotone convergence theorem. $\square$

Exercise 1. Prove the corollary.

For the L^p version we need a lemma.

{lemma:max-ineq}

Lemma 1. *Let X, Y be nonnegative random variables such that*

$$\mathbf{P}(X \geq x) \leq \frac{1}{x} \int_{\{X \geq x\}} Y \mathrm{d}\mathbf{P}, \quad x > 0.$$

Then for any $p > 1$

$$\mathbf{E}(X^p) \leq \left(\frac{p}{p-1} \right)^p \mathbf{E}(Y^p)$$

Doob's max

$X = M_n$

$Y = X_n$.

Proof. Note the for a nonnegative random variable X

$$\mathbf{E}X^p = \int_0^\infty p x^{p-1} \underbrace{[1 - F(x)]}_{P(X > x)} \mathrm{d}x,$$

where $F(x) = \mathbf{P}(X \leq x)$ is the distribution function of X. Indeed,

$p > 0$

$$\mathbf{E}X^p = \int_\Omega X^p \mathrm{d}\mathbf{P} = \int_\Omega \int_0^\infty \mathbf{I}(x < X(\omega)) p x^{p-1} \mathrm{d}x \mathrm{d}\mathbf{P}(\omega)$$
$$= \int_0^\infty p x^{p-1} [1 - F(x)] \mathrm{d}x.$$

$\int_\Omega I(x < X(\omega)) dP(\omega)$

$= P(X > x) = 1 - F(x)$

The result follows using Hölder's inequality as

$$\begin{aligned}
\mathbf{E}X^p &= \int_0^\infty p x^{p-1} [1 - F(x)] \mathrm{d}x \\
&\leq \int_0^\infty p x^{p-1} \frac{1}{x} \int_{\{X \geq x\}} Y(\omega) \mathrm{d}\mathbf{P}(\omega) \, \mathrm{d}x \\
&= \int_0^\infty \int_\Omega p x^{p-2} \mathbf{I}(X(\omega) \geq x) Y(\omega) \mathrm{d}\mathbf{P}(\omega) \mathrm{d}x \\
&= \int_\Omega Y(\omega) \left(\int_0^{X(\omega)} p x^{p-2} \mathrm{d}x \right) \mathrm{d}\mathbf{P}(\omega) \\
&= \int_\Omega Y X^{p-1} \frac{p}{p-1} \mathrm{d}\mathbf{P} \\
&\leq \frac{p}{p-1} \left(\mathbf{E}Y^p \right)^{1/p} \left(\mathbf{E}X^{(p-1)q} \right)^{1/q} \\
&= \frac{p}{p-1} \left(\mathbf{E}Y^p \right)^{1/p} \left(\mathbf{E}X^p \right)^{1/q},
\end{aligned}$$

$p > 1$

$\int Y \cdot X^{p-1} dP$

$\leq \left(\int Y^p dP \right)^{1/p} \cdot \left(\int (X^{p-1})^q dP \right)^{1/q}$

$\frac{1}{p} + \frac{1}{q} = 1$

$p + q = pq$

$(p-1)q =$

$= -q + pq = p$.

where p and q are conjugate exponents, i.e. $1/p + 1/q = 1$. □

Hölder's inequality: $\int fg \, d\mu \leq \left(\int f^p d\mu \right)^{1/p} \cdot \left(\int g^q d\mu \right)^{1/q}$

$$E X^p \leq \frac{q}{p-1} \cdot (E Y^p)^{1/p} \cdot (E X^p)^{\frac{1}{q}}$$

$$(E X^p)^{1-\frac{1}{q} = \frac{1}{p}} \leq \frac{q}{p-1} (E Y^p)^{1/p}$$

$$E(X^p) \leq \left(\frac{p}{p-1}\right)^p \cdot E(Y^p) \quad \checkmark$$

Theorem 3 (L^p maximal inequality)**.** *(i) Let $(X_k)_{k=1}^n$ be a nonnegative submartingale and $p \in (1, \infty)$. Then*

$$\mathbf{E} \max\{X_1^p, \ldots, X_n^p\} \le \left(\frac{p}{p-1}\right)^p \mathbf{E} X_n^p.$$

(ii) Let $(X_k)_{k=1}^\infty$ be a nonnegative submartingale and $p \in (1, \infty)$. Then

$$\mathbf{E} \sup_{n \in \mathbb{N}} X_n^p \le \left(\frac{p}{p-1}\right)^p \sup_{n \in \mathbb{N}} \mathbf{E} X_n^p.$$

Proof. Statement (i) follows from Doob's maximal inequality and Lemma 1.

(ii) follows from (i) and the monotone convegence theorem as

$$\begin{aligned}
\mathbf{E} \sup_n X_n^p &= \lim_{n\to\infty} \mathbf{E} \max_{1 \le k \le n} X_k^p \quad \le \left(\frac{p}{p-1}\right)^q E(X_n^p) \\
&\le \liminf_{n\to\infty} \left(\frac{p}{p-1}\right)^p \mathbf{E} X_n^p \\
&\le \left(\frac{p}{p-1}\right)^p \sup_n \mathbf{E} X_n^p.
\end{aligned}$$

□

1.5 Optional stopping theorem

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability measure and $(\mathcal{F}_n)_n$ a filtration on it. Recall that a random variable $\tau : \Omega \to \mathbb{N}$ is *stopping time*, if $\{\tau \le n\} \in \mathcal{F}_n$ for each $n \in \mathbb{N}$.

We already used the following simple observation.

Proposition 1. *The following are equivalent.*

(i) τ *is stopping time;*

(ii) $\{\tau > n\} \in \mathcal{F}_n$ *for each* $n \in \mathbb{N}$*;*

(iii) $\{\tau = n\} \in \mathcal{F}_n$ *for each* $n \in \mathbb{N}$.

Exercise 2. Prove this result.

Let τ be a stopping time. The *σ-algebra of the events prior to τ*, or short *pre-τ-sigma algebra* is defined as

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : \; A \cap \{\tau \le n\} \in \mathcal{F}_n, \; n = 1, 2, \ldots\}. \qquad (1) \quad \texttt{\{eq:pretau\}}$$

A is a σ-alg. on Ω if:
- $\emptyset \in \mathcal{A}$, $\Omega \in \mathcal{A}$
- $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$.
- $A_i \in \mathcal{A} \Rightarrow \bigcup A_i \in \mathcal{A}$.

$- \; \Omega \in \mathcal{F}_\tau : \quad \underset{\substack{\| \\ \{\tau \leq n\}}}{\Omega \cap \{\tau \leq n\}} \overset{?}{\in} \mathcal{F}_n \checkmark$

$- \; \checkmark$

It is easy to see that $\mathcal{F}_\tau$ is indeed a σ-algebra. Clearly, $\Omega \in \mathcal{F}_\tau$, and if $A \in \mathcal{F}_\tau$, then

def. stopping time

$$A^c \cap \{\tau \leq n\} = (\Omega - A) \cap \{\tau \leq n\} = \underbrace{\{\tau \leq n\}}_{\in \mathcal{F}_n} - \underbrace{(A \cap \{\tau \leq n\})}_{\in \mathcal{F}_n \text{ def. } \mathcal{F}_\tau} \in \mathcal{F}_n, \; n \in \mathbb{N}.$$

Finally, if $A_1, A_2, \ldots \in \mathcal{F}_\tau$, then

$$(\cup_{k=1}^{\infty} A_k) \cap \{\tau \leq n\} = \cup_{k=1}^{\infty} \underbrace{(A_k \cap \{\tau \leq n\})}_{\in \mathcal{F}_n} \in \mathcal{F}_n$$

for any $n = 1, 2, \ldots$.

Exercise 3. Show that if $\tau \equiv k$ for some $k \in \mathbb{N}$ then $\mathcal{F}_\tau = \mathcal{F}_k$, so the notaion (t) is consistent.

Some simple properties are summarized in the next statement.

Lemma 2. *Let σ, τ be stopping times.*

(i) τ is $\mathcal{F}_\tau$-measureable.

(ii) $\sigma \wedge \tau = \min(\sigma, \tau)$ and $\sigma \vee \tau = \max(\sigma, \tau)$ are stopping times. (HW)

(iii) If $\sigma \leq \tau$, then $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$.

(iv) If $(X_n)_n$ is an adapted sequence then X_τ is $\mathcal{F}_\tau$-measurable.

{thm:opt-stop}

Theorem 4 (Optional stopping theorem, Doob)**.** *Let $(X_n)_n$ be a supermartingale, and $\sigma \leq \tau$ stopping times such that*

$$\mathbf{E}(|X_\sigma|) < \infty\,, \qquad \mathbf{E}(|X_\tau|) < \infty \tag{2}$$

{eq:opt-stop-1}

and

$$\liminf_{n \to \infty} \int_{\{\tau > n\}} |X_n| \, \mathrm{d}\mathbf{P} = 0. \tag{3}$$

{eq:opt-stop-2}

Then $\overbrace{\mathbf{E}(X_\tau | \mathcal{F}_\sigma) \leq X_\sigma}$ almost surely.

Furthermore, if $(X_n)_n$ is martingale then $\mathbf{E}(X_\tau | \mathcal{F}_\sigma) = X_\sigma$.

Clearly, conditions (2) and (3) hold if the stopping times are bounded.

Proof. Since X_σ is $\mathcal{F}_\sigma$-measurable, $X_\sigma = \mathbf{E}(X_\sigma | \mathcal{F}_\sigma)$, therefore it is enough to show that

$$\mathbf{E}(X_\tau - X_\sigma | \mathcal{F}_\sigma) \leq 0. \quad \Longleftrightarrow \; E[X_\tau | \mathcal{F}_\sigma] \leq X_\sigma$$

This is the same as

$$\int_A (X_\tau - X_\sigma) \, \mathrm{d}\mathbf{P} \leq 0 \text{ for all } A \in \mathcal{F}_\sigma. \tag{4}$$

{eq:opt-aux1}

$[\; (\Omega, \mathcal{F}, P) \quad Z$ r.v.

$Z \geq 0$ a.s. $\Longleftrightarrow \int_A Z \, dP \geq 0 \quad \forall A \in \mathcal{F}.$

i) τ is $\mathcal{F}_\tau$ meas.

$\sigma(\tau) \subseteq \mathcal{F}_\tau$

$\tau : \Omega \to \mathbb{N} = \{0, 1, 2, \dots\}$

Have to check: $\{\tau = k\} \in \mathcal{F}_\tau \quad \forall k \in \mathbb{N}$.

$$\underbrace{\{\tau = k\} \cap \{\tau \leq n\}}_{} \in \mathcal{F}_n \qquad \forall n \in \mathbb{N}.$$

$k > n : \; = \emptyset \in \mathcal{F}_n$ ✓

$k \leq n : \; = \{\tau = k\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$ ✓.

$E|X_\sigma| < \infty$

If σ is bounded: $\sigma \leq 3$ a.s.

$E X_\sigma \leq E X_3$

$$\int_{\{\tau > n\}} |X_n| \, dP = 0 \quad \text{if } n \text{ is large.}$$

$\tau \leq 3$ if $n \geq 3$ $\{\tau > n\} = \emptyset$

$\sigma \leq \tau \leq m$

First assume that τ is bounded, that is $\tau \leq m$ for some m. For any $A \in \mathcal{F}_\sigma$

$\mathcal{F}_{k-1}$

$$A \cap \{\sigma < k \leq \tau\} = A \cap \underbrace{\{\sigma \leq k-1\}}_{\in \mathcal{F}_{k-1}} \cap \{\tau > k-1\} \in \mathcal{F}_{k-1}\,, \quad k \geq 2,$$

thus

$$\begin{aligned}
&\int_A (X_\tau - X_\sigma)\,\mathrm{d}\mathbf{P} \\
&= \int_A \Big(\sum_{k=\sigma+1}^{\tau} (X_k - X_{k-1}) \Big)\,\mathrm{d}\mathbf{P} \\
&= \int_A \Big(\sum_{k=2}^{m} \mathbf{I}(\sigma < k \leq \tau)(X_k - X_{k-1}) \Big)\,\mathrm{d}\mathbf{P} \\
&= \sum_{k=2}^{m} \int_{A\cap\{\sigma<k\leq\tau\}} (X_k - X_{k-1})\,\mathrm{d}\mathbf{P} \\
&= \sum_{k=2}^{m} \int_{A\cap\{\sigma<k\leq\tau\}} \underbrace{\mathbf{E}(X_k - X_{k-1}|\mathcal{F}_{k-1})}_{\leq 0}\,\mathrm{d}\mathbf{P} \leq 0,
\end{aligned}$$

$\int_B X_k \, dP = \int_B E[X_k | \mathcal{F}_{k-1}] \, dP$

$B \in \mathcal{F}_{k-1}$

supermtg.

proving (4).

Consider the general case. For any n we can write

$$\begin{aligned}
&\int_A (X_\tau - X_\sigma)\mathrm{d}\mathbf{P} \\
&= \int_A (X_{\tau\wedge n} - X_{\sigma\wedge n})\mathrm{d}\mathbf{P} + \int_A (X_\tau - X_{\tau\wedge n})\mathrm{d}\mathbf{P} - \int_A (X_\sigma - X_{\sigma\wedge n})\mathrm{d}\mathbf{P}.
\end{aligned}$$

On the event $\{\sigma \geq n\}$ we have $X_{\tau\wedge n} = X_n = X_{\sigma\wedge n}$, therefore

$$\int_A (X_{\tau\wedge n} - X_{\sigma\wedge n})\mathrm{d}\mathbf{P} = \int_{A\cap\{\sigma<n\}} (X_{\tau\wedge n} - X_{\sigma\wedge n})\mathrm{d}\mathbf{P} \leq 0\,, \quad n \in \mathbb{N}, \qquad (5)$$ {eq:opt-aux2}

where the inequality follows from the previous case. ← have to check.

By condition (3) there exists a sequence $n_k \to \infty$ such that

$A \cap \{\sigma < n\} \in \mathcal{F}_{\sigma\wedge n}$

$$\lim_{k\to\infty} \int_{\{\tau>n_k\}} |X_{n_k}|\,\mathrm{d}\mathbf{P} = 0.$$

HW

$$\liminf_{n\to\infty} \int_{\{\tau>n\}} |X_n| \, dP = 0$$

It is enough to show that on this subsequence the second and third terms in decomposition (5) tends to 0. For the second term

$E|X_\tau| < \infty$

$$\begin{aligned}\left|\int_A (X_\tau - X_{\tau\wedge n_k})\mathrm{d}\mathbf{P}\right| &= \left|\int_{A\cap\{\tau>n_k\}} (X_\tau - X_{\tau\wedge n_k})\mathrm{d}\mathbf{P}\right| \\ &\leq \int_{A\cap\{\tau>n_k\}} (|X_\tau| + |X_{n_k}|)\mathrm{d}\mathbf{P} \\ &\leq \int_{\{\tau>n_k\}} |X_\tau|\,\mathrm{d}\mathbf{P} + \int_{\{\tau>n_k\}} |X_{n_k}|\,\mathrm{d}\mathbf{P}.\end{aligned}$$

$\downarrow 0$

Similarly, for the third term

$$\begin{aligned}\left|\int_A (X_\sigma - X_{\sigma\wedge n_k})\mathrm{d}\mathbf{P}\right| &= \left|\int_{A\cap\{\sigma>n_k\}} (X_\sigma - X_{n_k})\mathrm{d}\mathbf{P}\right| \\ &\leq \int_{\{\sigma>n_k\}} |X_\sigma|\,\mathrm{d}\mathbf{P} + \int_{\{\tau>n_k\}} |X_{n_k}|\,\mathrm{d}\mathbf{P}.\end{aligned}$$

Using (2) both upper bounds tend to 0. □

Corollary 2. *Assume that (X_n) is (super-, sub-) martingale, τ is a stopping time, $\mathbf{E}(|X_\tau|) < \infty$ and (3) holds. Then*

$\sigma = 1$

(i) $\mathbf{E}(X_\tau|\mathcal{F}_1) \leq X_1$ *and* $\mathbf{E}(X_\tau) \leq \mathbf{E}(X_1)$ *for supermartingales;*

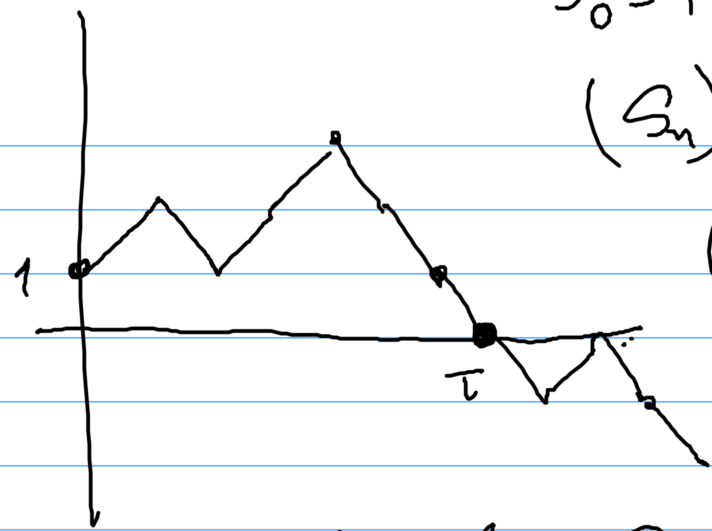
(ii) $\mathbf{E}(X_\tau|\mathcal{F}_1) \geq X_1$ *and* $\mathbf{E}(X_\tau) \geq \mathbf{E}(X_1)$ *for submartingales;*

(iii) $\mathbf{E}(X_\tau|\mathcal{F}_1) = X_1$ *and* $\mathbf{E}(X_\tau) = \mathbf{E}(X_1)$ *for martingales.*

Some conditions are needed for the optional stopping to hold.

Example 2 (Simple symmetric random walk)**.** Let $\xi, \xi_1, \xi_2, \ldots$ are iid random variables with $\mathbf{P}(\xi = \pm 1) = 1/2$. Let $S_0 = 1$ and $S_n = S_{n-1} + \xi_n$. Then (S_n) is martingale. Let $\tau = \min\{n : S_n = 0\}$. Then τ is a stopping time and the martingale $(S_{\tau\wedge n})_n$ tends to 0 a.s. The optional stopping does not hold as $S_\tau \equiv 0$ a.s., while $S_0 = 1$. Clearly, condition (3) does not hold.

Theorem 5 (Wald identity)**.** *Let $X, X_1, X_2, \ldots$ be iid random variables with $\mathbf{E}X = \mu \in \mathbb{R}$, and let τ be a stopping time with $\mathbf{E}(\tau) < \infty$. Put $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$. Then $\mathbf{E}(S_\tau) = \mu\mathbf{E}(\tau)$.*



$S_0 = 1$

(S_n) mtg

$(S_{n\wedge\tau})$ mtg

nonneg.

mtg. conv. thm. $\lim_{n\to\infty} S_{n\wedge\tau}$ $\exists$ a.s.

$\Downarrow$

$S_{n\wedge\tau} \to 0$ a.s. $n\to\infty$.

$\Downarrow$

$\tau < \infty$ a.s.

$S_\tau = 0$ a.s.

$S_0 = 1$

$0 = E S_\tau \neq E(S_0) = 1$

Proof. First assume $X \geq 0$. We have

$$\begin{aligned}\mathbf{E}(S_\tau) &= \mathbf{E}\biggl(\sum_{k=1}^{\infty} I_{\{\tau\geq k\}} X_k\biggr)\\ &= \sum_{k=1}^{\infty} \mathbf{E}(I_{\{\tau\geq k\}} X_k)\\ &= \sum_{k=1}^{\infty} \mathbf{E}(I_{\{\tau\geq k\}})\mathbf{E}(X_k),\end{aligned}$$

that is

$$\mu \sum_{k=1}^{\infty} \mathbf{P}\{\tau \geq k\} = \mu\mathbf{E}(\tau).$$

To see the general case consider the decomposition

$$S_\tau^{(+)} = \sum_{k=1}^{\infty} X_k^+ \mathbf{I}(\tau \geq k)$$

and

$$S_\tau^{(-)} = \sum_{k=1}^{\infty} X_k^- \mathbf{I}(\tau \geq k).$$

□

Example 3 (Gambler's ruin)**.** Let $X, X_1, X_2, \ldots$ be iid random variables such that $\mathbf{P}(X = 1) = p = 1 - \mathbf{P}(X = -1)$, $0 < p < 1$, and put $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$. Fix $a, b \in \mathbb{N}$ and let

$$\tau = \tau_{a,b}(p) = \inf\{n : S_n \geq b \text{ or } S_n \leq -a\},$$

with the convention $\inf \emptyset = \infty$. Let $(\mathcal{F}_n)$ be the natural filtration, i.e. $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n \in \mathbb{N}$.

It is easy to show that $\mathbf{P}(\tau < \infty) = 1$, and τ is a stopping time. Furthermore, $|S_\tau| \leq \max(a, b)$, in particular $\mathbf{E}|S_\tau| < \infty$ and

$$\liminf_{n\to\infty} \int_{\{\tau > n\}} |S_n| \,\mathrm{d}\mathbf{P} \leq \liminf_{n\to\infty} \max(a, b)\mathbf{P}\{\tau > n\} = 0.$$

First assume that $p = 1/2$. Then $\mathbf{E}X = 0$ and (S_n) is a martingale. Therefore, by the optional stopping theorem

$$\begin{aligned}0 = \mathbf{E}S_0 = \mathbf{E}S_\tau &= -a\mathbf{P}(S_\tau = -a) + b\mathbf{P}(S_\tau = b)\\ &= -a(1 - \mathbf{P}(S_\tau =)) + b\mathbf{P}(S_\tau = b).\end{aligned}$$

Thus

$$\mathbf{P}(S_\tau = b) = \frac{a}{a+b} \quad \text{and} \quad \mathbf{P}(S_\tau = -a) = \frac{b}{a+b}.$$

Furthermore, we proved that $(S_n^2 - n)$ is a martingale, thus

$$0 = \mathbf{E}(S_0^2 - 0) = \mathbf{E}(S_\tau^2 - \tau)$$

which implies

$$\mathbf{E}\tau = \mathbf{E}S_\tau^2 = a^2\mathbf{P}(S_\tau = -a) + b^2\mathbf{P}(S_\tau = b) = a^2\frac{b}{a+b} + b^2\frac{a}{a+b} = ab.$$

The case $p \neq 1/2$ is different. Introduce

$$Z_n = s^{S_n} = \prod_{k=1}^{n} s^{X_k}$$

with $s = (1-p)/p = 1/r$. Then (Z_n) is a martingale and

$$Z_\tau = s^b\mathbf{I}(S_n = b) + s^{-a}\mathbf{I}(S_n = -a) \le s^b + s^{-a},$$

thus $\mathbf{E}Z_\tau < \infty$ and

$$\liminf_{n\to\infty} \int_{\{\tau > n\}} |Z_n| \,\mathrm{d}\mathbf{P} \le (s^b + s^{-a}) \liminf_{n\to\infty} \mathbf{P}\{\tau > n\} = 0.$$

Again, by the optional sampling theorem

$$\begin{aligned}&s^{-a}\mathbf{P}(S_\tau = -a) + s^b\left(1 - \mathbf{P}(S_\tau = -a)\right)\\ &= s^{-a}\mathbf{P}(S_\tau = -a) + s^b\mathbf{P}(S_\tau = b)\\ &= \mathbf{E}(s^{S_\tau}) = \mathbf{E}(Z_\tau)\\ &= \mathbf{E}(Z_1) = \mathbf{E}(s^X) = 1.\end{aligned}$$

Rearranging we obtain

$$\mathbf{P}(S_\tau = -a) = \frac{1 - s^b}{s^{-a} - s^b}\,\frac{r^b}{r^b} = \frac{r^b - 1}{r^{a+b} - 1} = \frac{1 - r^b}{1 - r^{a+b}}.$$

Exercise 4. Show that $\tau < \infty$ a.s.