where $B(x, r) = \{y : |x - y| \leq r\}$ is the ball of radius r, and $|B(x, r)|$ is the volume of the ball.

If $\mu \geq 0$ then S_n is a submartingale. Applying the first result to $\xi_i' = \xi_i - \mu$ we see that $S_n - n\mu$ is a martingale.

Example 4.2.2. Quadratic martingale. Suppose now that $\mu = E\xi_i = 0$ and $\sigma^2 = \text{var}(\xi_i) < \infty$. In this case $S_n^2 - n\sigma^2$ is a martingale.

Since $(S_n + \xi_{n+1})^2 = S_n^2 + 2S_n\xi_{n+1} + \xi_{n+1}^2$ and ξ_{n+1} is independent of $\mathcal{F}_n$, we have

$$E(S_{n+1}^2 - (n+1)\sigma^2|\mathcal{F}_n) = S_n^2 + 2S_nE(\xi_{n+1}|\mathcal{F}_n) + E(\xi_{n+1}^2|\mathcal{F}_n) - (n+1)\sigma^2$$
$$= S_n^2 + 0 + \sigma^2 - (n+1)\sigma^2 = S_n^2 - n\sigma^2$$

Example 4.2.3. Exponential martingale. Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables with $EY_m = 1$. If $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$ then $M_n = \prod_{m \leq n} Y_m$ defines a martingale. To prove this note that

$$E(M_{n+1}|\mathcal{F}_n) = M_nE(X_{n+1}|\mathcal{F}_n) = Y_n$$

Suppose now that $Y_i = e^{\theta\xi_i}$ and $\phi(\theta) = Ee^{\theta\xi_i} < \infty$. $Y_i = \exp(\theta\xi)/\phi(\theta)$ has mean 1 so $EY_i = 1$ and

$$M_n = \prod_{i=1}^{n} Y_i = \exp(\theta S_n)/\phi(\theta)^n \quad \text{is a martingale.}$$

We will see many other examples below, so we turn now to deriving properties of martingales. Our first result is an immediate consequence of the definition of a supermartingale. We could take the conclusion of the result as the definition of supermartingale, but then the definition would be harder to check.

Theorem 4.2.4. *If X_n is a supermartingale then for $n > m$, $E(X_n|\mathcal{F}_m) \leq X_m$.*

Proof. The definition gives the result for $n = m + 1$. Suppose $n = m + k$ with $k \geq 2$. By Theorem 4.1.2,

$$E(X_{m+k}|\mathcal{F}_m) = E(E(X_{m+k}|\mathcal{F}_{m+k-1})|\mathcal{F}_m) \leq E(X_{m+k-1}|\mathcal{F}_m)$$

by the definition and (4.1.2). The desired result now follows by induction. □

Theorem 4.2.5. *(i) If X_n is a submartingale then for $n > m$, $E(X_n|\mathcal{F}_m) \geq X_m$.*
(ii) If X_n is a martingale then for $n > m$, $E(X_n|\mathcal{F}_m) = X_m$.

Proof. To prove (i), note that $-X_n$ is a supermartingale and use (4.1.1). For (ii), observe that X_n is a supermartingale and a submartingale. □

$(\Omega, \mathcal{F}, P)$ $(\mathcal{F}_n)_n$ filtration sequence of σ-alg.

$E[X_n|\mathcal{F}_{n-1}] = X_{n-1}$ (a.s.) $\ldots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \ldots$

$\xi_1, \xi_m, \xi_2, \ldots$ iid
$E(\xi) = 0$

$X_n = \xi_1 + \ldots + \xi_n$

$\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$

X_n^2 submtg.

Remark. The idea in the proof of Theorem 4.2.5 will be used many times below. To keep from repeating ourselves, we will just state the result for either supermartingales or submartingales and leave it to the reader to translate the result for the other two.

Theorem 4.2.6. *If X_n is a martingale w.r.t. $\mathcal{F}_n$ and φ is a convex function with $E|\varphi(X_n)| < \infty$ for all n then $\varphi(X_n)$ is a submartingale w.r.t. $\mathcal{F}_n$. Consequently, if $p \geq 1$ and $E|X_n|^p < \infty$ for all n, then $|X_n|^p$ is a submartingale w.r.t. $\mathcal{F}_n$.*

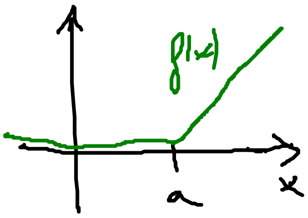
Proof By Jensen's inequality and the definition

cond. Jensen   mtg

$$\underline{E(\varphi(X_{n+1})|\mathcal{F}_n)} \geq \varphi(E(X_{n+1}|\mathcal{F}_n)) = \underline{\varphi(X_n)} \qquad \square$$

Theorem 4.2.7. *If X_n is a submartingale w.r.t. $\mathcal{F}_n$ and φ is an increasing convex function with $E|\varphi(X_n)| < \infty$ for all n, then $\varphi(X_n)$ is a submartingale w.r.t. $\mathcal{F}_n$. Consequently (i) If X_n is a submartingale then $(X_n - a)^+$ is a submartingale. (ii) If X_n is a supermartingale then $X_n \wedge a$ is a supermartingale.*

$f(x) = (x-a)_+$



conditional Jensen

$E[X_{n+1}|\mathcal{F}_n] \geq X_n$

Proof By Jensen's inequality and the assumptions

φ is increasing + submtg

$$\underline{E(\varphi(X_{n+1})|\mathcal{F}_n)} \geq \varphi(E(X_{n+1}|\mathcal{F}_n)) \geq \underline{\varphi(X_n)} \qquad \square$$

Let $\mathcal{F}_n$, $n \geq 0$ be a filtration. H_n, $n \geq 1$ is said to be a **predictable sequence** if $H_n \in \mathcal{F}_{n-1}$ for all $n \geq 1$. In words, the value of H_n may be predicted (with certainty) from the information available at time $n-1$. In this section, we will be thinking of H_n as the amount of money a gambler will bet at time n. This can be based on the outcomes at times $1, \ldots, n-1$ but not on the outcome at time n!

$(\mathcal{F}_n)$

(H_n) predictable

Once we start thinking of H_n as a gambling system, it is natural to ask how much money we would make if we used it. Let X_n be the net amount of money you would have won at time n if you had bet one dollar each time. If you bet according to a gambling system H then your winnings at time n would be

if H_n is $\mathcal{F}_{n-1}$ meas.

discrete stoch. integral $\rightarrow$

$$(H \cdot X)_n = \sum_{m=1}^{n} H_m (X_m - X_{m-1})$$

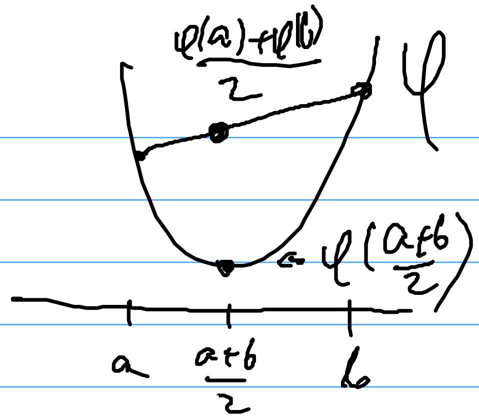
since if at time m you have wagered \$3 the change in your fortune would be 3 time that of a person who wagered \$1. Alternatively you can think of X_m is the value of a stock and H_m the number of shares you hold from time $m-1$ to time m.

Suppose now that $\xi_m = X_m - X_{m-1}$ have $P(\xi_m = 1) = p$ and $P(\xi_m = -1) = 1 - p$. A famous gambling system called the "martingale" is defined by $H_1 = 1$ and for $n \geq 2$,

$$H_n = \begin{cases} 2H_{n-1} & \text{if } \xi_{n-1} = -1 \\ 1 & \text{if } \xi_{n-1} = 1 \end{cases}$$

Jensen inequality:

$$\varphi(E(X)) \leq E\,\varphi(X)$$



$$(H \cdot X)_n = \sum_{m=1}^{n} H_m \cdot (X_m - X_{m-1})$$

In words, we double our bet when we lose, so that if we lose k times and then win, our net winnings will be 1. To see this consider the following concrete situation

H_n	1	2	4	8	16
ξ_n	-1	-1	-1	-1	1
$(H \cdot X)_n$	-1	-3	-7	-15	1

This system seems to provide us with a "sure thing" as long as $P(\xi_m = 1) > 0$. However, the next result says there is no system for beating an unfavorable game.

Theorem 4.2.8. *Let X_n, $n \geq 0$, be a supermartingale. If $H_n \geq 0$ is predictable and each H_n is bounded then $(H \cdot X)_n$ is a supermartingale.*

Proof. Using the fact that conditional expectation is linear, $(H \cdot X)_n \in \mathcal{F}_n$, $H_n \in \mathcal{F}_{n-1}$, and (4.1.14), we have

$$\begin{aligned} E((H \cdot X)_{n+1}|\mathcal{F}_n) &= (H \cdot X)_n + E(H_{n+1}(X_{n+1} - X_n)|\mathcal{F}_n) \\ &= (H \cdot X)_n + H_{n+1}E((X_{n+1} - X_n)|\mathcal{F}_n) \leq (H \cdot X)_n \end{aligned}$$

since $E((X_{n+1} - X_n)|\mathcal{F}_n) \leq 0$ and $H_{n+1} \geq 0$. □

Remark. The same result is obviously true for submartingales and for martingales (in the last case, without the restriction $H_n \geq 0$). mtg./super not needed for mtg.

$(\Omega, \mathcal{F}, P)$

$N : \Omega \to \{0, 1, \dots\}$

We will now consider a very special gambling system: bet \$1 ar each time $n \leq N$ then stop playing. A random variable N is said to be a **stopping time** if $\{N = n\} \in \mathcal{F}_n$ for all $n < \infty$, i.e., the decision to stop at time n must be measurable with respect to the information known at that time. If we let $H_n = 1_{\{N \geq n\}}$, then $\{N \geq n\} = \{N \leq n-1\}^c \in \mathcal{F}_{n-1}$, so H_n is predictable, and it follows from Theorem 4.2.8 that $(H \cdot X)_n = X_{N \wedge n} - X_0$ is a supermartingale. Since the constant sequence $Y_n = X_0$ is a supermartingale and the sum of two supermartingales is also, we have:

$a \wedge b = \min\{a, b\}$

Theorem 4.2.9. *If N is a stopping time and X_n is a supermartingale, then $X_{N \wedge n}$ is a supermartingale.*

Although Theorem 4.2.8 implies that you cannot make money with gambling systems, you can prove theorems with them. Suppose X_n, $n \geq 0$, is a submartingale. Let $a < b$, let $N_0 = -1$, and for $k \geq 1$ let

HW

$$\begin{aligned} N_{2k-1} &= \inf\{m > N_{2k-2} : X_m \leq a\} \\ N_{2k} &= \inf\{m > N_{2k-1} : X_m \geq b\} \end{aligned}$$

$N_1 = \min\{m > -1 : X_m \leq a\}$

first time X goes below a

The N_j are stopping times and $\{N_{2k-1} < m \leq N_{2k}\} = \underbrace{\{N_{2k-1} \leq m - 1\}}_{\in \mathcal{F}_{m-1}} \cap \underbrace{\{N_{2k} \leq m - 1\}^c}_{\in \mathcal{F}_{m-1}} \in \mathcal{F}_{m-1}$, so

$$H_m = \begin{cases} 1 & \text{if } N_{2k-1} < m \leq N_{2k} \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

(X_n) sup.mtg. $\Rightarrow \left(X_{N \wedge n}\right)_{n=0}^{\infty}$ is sup-mtg.

$N_2 = \min\{m > N_1 : X_m \geq b\}$

first time after N_1 X goes above b.

$$(H\cdot X)_n = \sum_{m=1}^{n} H_m(X_m - X_{m-1})$$

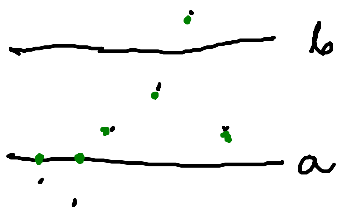
$$E\left[(H\cdot X)_{n+1} \,\middle|\, \mathcal{F}_n\right] = E\Big[\underbrace{(H\cdot X)_n}_{\mathcal{F}_n \text{ meas.}} + H_{n+1}(X_{n+1}-X_n) \,\Big|\, \mathcal{F}_n\Big]$$

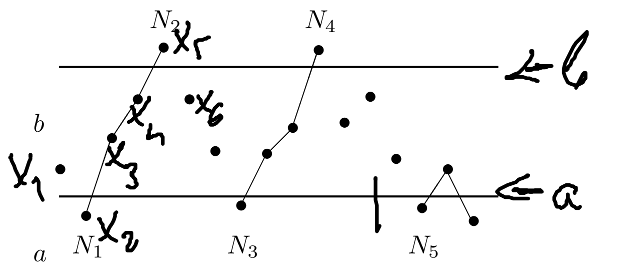
$$= (H\cdot X)_n + E\Big[\underbrace{H_{n+1}}_{\mathcal{F}_n\text{-meas.}}(X_{n+1}-X_n) \,\Big|\, \mathcal{F}_n\Big]$$

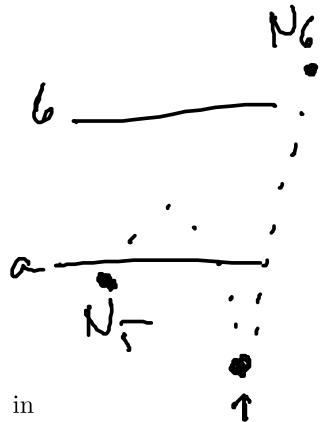
$$= (H\cdot X)_n + \underbrace{\underbrace{H_{n+1}}_{\geq 0} \cdot \underbrace{E\left[X_{n+1}-X_n \,\middle|\, \mathcal{F}_n\right]}_{\text{supermart. } \leq 0}}_{\leq 0} \leq (H\cdot X)_n$$

H_m is $\mathcal{F}_{m-1}$ meas. i.e. predictable

defines a predictable sequence. $X(N_{2k-1}) \le a$ and $X(N_{2k}) \ge b$, so between times N_{2k-1} and N_{2k}, X_m crosses from below a to above b. H_m is a gambling system that tries to take advantage of these "upcrossings." In stock market terms, we buy when $X_m \le a$ and sell when $X_m \ge b$, so every time an upcrossing is completed, we make a profit of $\ge (b-a)$. Finally, $U_n = \sup\{k : N_{2k} \le n\}$ is the number of upcrossings completed by time n.







n $U_n = 2$

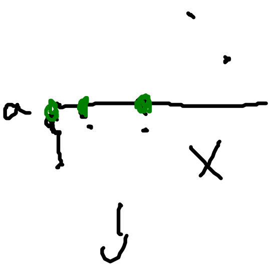
Figure 4.2: Upcrossings of (a,b). Lines indicate increments that are included in $(H \cdot X)_n$. In Y_n the points $< a$ are moved up to a.

(X_n) $(\mathcal{F}_n)$ submtg, fix $a<b$

Theorem 4.2.10. Upcrossing inequality. *If X_m, $m \ge 0$, is a submartingale then*

upcrossings up to time n

$$(b-a)EU_n \le E(X_n - a)^+ - E(X_0 - a)^+$$



Proof. Let $Y_m = a + (X_m - a)^+$. By Theorem 4.2.7, Y_m is a submartingale. Clearly, it upcrosses $[a,b]$ the same number of times that X_m does, and we have $(b-a)U_n \le (H \cdot Y)_n$, since each upcrossing results in a profit $\ge (b-a)$ and a final incomplete upcrossing (if there is one) makes a nonnegative contribution to the right-hand side. It is for this reason we had to replace X_m by Y_m.

Let $K_m = 1 - H_m$. Clearly, $Y_n - Y_0 = (H \cdot Y)_n + (K \cdot Y)_n$, and it follows from Theorem 4.2.8 that $E(K \cdot Y)_n \ge E(K \cdot Y)_0 = 0$ so $E(H \cdot Y)_n \le E(Y_n - Y_0)$, proving the desired inequality. $\square$

$Y_m = X_m$ if $X_m \ge a$

$Y_m = a$ if $X_m \le a$

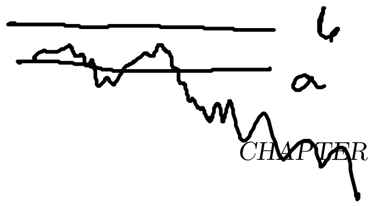
We have proved the result in its classical form, even though this is a little misleading. The key fact is that $E(K \cdot Y)_n \ge 0$, i.e., no matter how hard you try you can't lose money betting on a submartingale. From the upcrossing inequality, we easily get

Theorem 4.2.11. Martingale convergence theorem. *If X_n is a submartingale with $\sup_n EX_n^+ < \infty$ then as $n \to \infty$, X_n converges a.s. to a limit X with $E|X| < \infty$.*

$Y_n - Y_0 = (X_n - a)^+ - (X_0 - a)^+$

$E(H \cdot Y)_n \ge E(b-a)U_n = (b-a)E(U_n)$

$a < b$



Proof. Since $(X-a)^+ \le X^+ + |a|$, Theorem 4.2.10 implies that

$$EU_n \le (|a| + EX_n^+)/(b-a)$$

As $n \uparrow \infty$, $U_n \uparrow U$ the number of upcrossings of $[a, b]$ by the whole sequence, so if $\sup EX_n^+ < \infty$ then $EU < \infty$ and hence $U < \infty$ a.s. Since the last conclusion holds for all rational a and b, → for any fix a, b this has prob 0.

$$a < b \quad \cup_{a,b\in\mathbf{Q}} \{\liminf X_n < a < b < \limsup X_n\} \quad \text{has probability } 0$$

and hence $\limsup X_n = \liminf X_n$ a.s., i.e., $\lim X_n$ exists a.s. Fatou's lemma guarantees $EX^+ \le \liminf EX_n^+ < \infty$, so $X < \infty$ a.s. To see $X > -\infty$, we observe that

$f_n \ge 0$

$\int \liminf f_n \le \liminf \int f_n$

$$EX_n^- = EX_n^+ - EX_n \le EX_n^+ - EX_0$$

(since X_n is a submartingale), so another application of Fatou's lemma shows

$$EX^- \le \liminf_{n\to\infty} EX_n^- \le \sup_n EX_n^+ - EX_0 < \infty$$

and completes the proof. □

Remark. To prepare for the proof of Theorem 4.7.1, the reader should note that we have shown that if the number of upcrossings of (a, b) by X_n is finite for all $a, b \in \mathbf{Q}$, then the limit of X_n exists.

An important special case of Theorem 4.2.11 is

Theorem 4.2.12. *If $X_n \ge 0$ is a supermartingale then as $n \to \infty$, $X_n \to X$ a.s. and $EX \le EX_0$.*

Proof. $Y_n = -X_n \le 0$ is a submartingale with $EY_n^+ = 0$. Since $EX_0 \ge EX_n$, the inequality follows from Fatou's lemma. □

In the next section, we will give several applications of the last two results. We close this one by giving two "counterexamples."

Example 4.2.13. The first shows that the assumptions of Theorem 4.2.12 (or 4.2.11) do not guarantee convergence in L^1. Let S_n be a symmetric simple random walk with $S_0 = 1$, i.e., $S_n = S_{n-1} + \xi_n$ where $\xi_1, \xi_2, \ldots$ are i.i.d. with $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. Let $N = \inf\{n : S_n = 0\}$ and let $X_n = S_{N\wedge n}$. Theorem 4.2.9 implies that X_n is a nonnegative martingale. Theorem 4.2.12 implies X_n converges to a limit $X_\infty < \infty$ that must be $\equiv 0$, since convergence to $k > 0$ is impossible. (If $X_n = k > 0$ then $X_{n+1} = k \pm 1$.) Since $EX_n = EX_0 = 1$ for all n and $X_\infty = 0$, convergence cannot occur in L^1.

$\forall n$

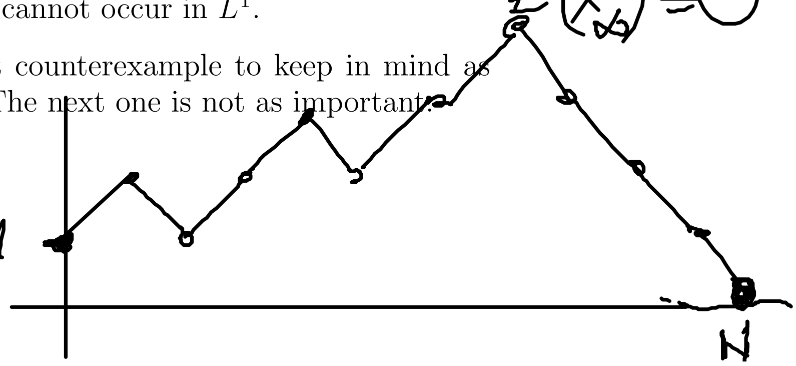
$E(X_n) = 1$

$E(X_\infty) = 0$

Example 4.2.13 is an important counterexample to keep in mind as you read the rest of this chapter. The next one is not as important.

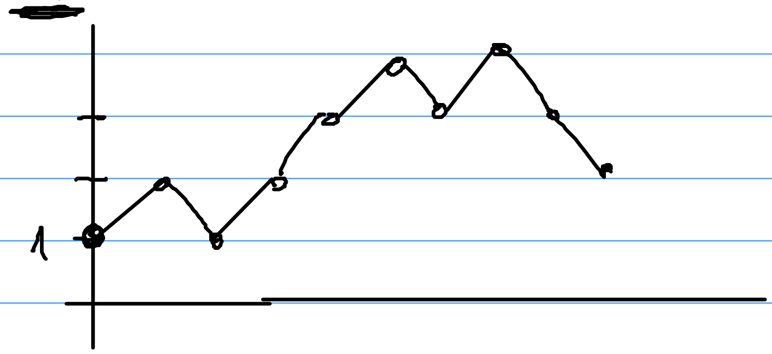
$X_n = S_{N\wedge n}$

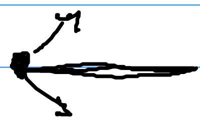
N stopping time



$\lim_{n\to\infty} X_n$ exists a.s. $\Big\} \Rightarrow X_n \to 0$ a.s.

0 can be a limit





Example 4.2.14. We will now give an example of a martingale with $X_k \to 0$ in probability but not a.s. Let $X_0 = 0$. When $X_{k-1} = 0$, let $X_k = 1$ or -1 with probability $1/2k$ and $= 0$ with probability $1 - 1/k$. When $X_{k-1} \neq 0$, let $X_k = kX_{k-1}$ with probability $1/k$ and $= 0$ with probability $1 - 1/k$. From the construction, $P(X_k = 0) = 1 - 1/k$ so $X_k \to 0$ in probability. On the other hand, the second Borel-Cantelli lemma implies $P(X_k = 0 \text{ for } k \geq K) = 0$, and values in $(-1, 1) - \{0\}$ are impossible, so X_k does not converge to 0 a.s.

Exercises

4.2.1. Suppose X_n is a martingale w.r.t. $\mathcal{G}_n$ and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Then $\mathcal{G}_n \supset \mathcal{F}_n$ and X_n is a martingale w.r.t. $\mathcal{F}_n$.

4.2.2. Give an example of a submartingale X_n so that X_n^2 is a supermartingale. Hint: X_n does not have to be random.

4.2.3. Generalize (i) of Theorem 4.2.7 by showing that if X_n and Y_n are submartingales w.r.t. $\mathcal{F}_n$ then $X_n \vee Y_n$ is also.

4.2.4. Let X_n, $n \geq 0$, be a submartingale with $\sup X_n < \infty$. Let $\xi_n = X_n - X_{n-1}$ and suppose $E(\sup \xi_n^+) < \infty$. Show that X_n converges a.s.

4.2.5. Give an example of a martingale X_n with $X_n \to -\infty$ a.s. Hint: Let $X_n = \xi_1 + \cdots + \xi_n$, where the ξ_i are independent (but not identically distributed) with $E\xi_i = 0$.

4.2.6. Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables with $EY_m = 1$ and $P(Y_m = 1) < 1$. By example 4.2.3 that $X_n = \prod_{m \leq n} Y_m$ defines a martingale. (i) Use Theorem 4.2.12 and an argument by contradiction to show $X_n \to 0$ a.s. (ii) Use the strong law of large numbers to conclude $(1/n) \log X_n \to c < 0$.

4.2.7. Suppose $y_n > -1$ for all n and $\sum |y_n| < \infty$. Show that $\prod_{m=1}^{\infty}(1 + y_m)$ exists.

4.2.8. Let X_n and Y_n be positive integrable and adapted to $\mathcal{F}_n$. Suppose

$$E(X_{n+1}|\mathcal{F}_n) \leq (1 + Y_n)X_n$$

with $\sum Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit by finding a closely related supermartingale to which Theorem 4.2.12 can be applied.

4.2.9. The switching principle. Suppose X_n^1 and X_n^2 are supermartingales with respect to $\mathcal{F}_n$, and N is a stopping time so that $X_N^1 \geq X_N^2$. Then

$$Y_n = X_n^1 1_{(N>n)} + X_n^2 1_{(N \leq n)} \text{ is a supermartingale.}$$
$$Z_n = X_n^1 1_{(N \geq n)} + X_n^2 1_{(N<n)} \text{ is a supermartingale.}$$

4.2.10. Dubins' inequality. For every positive supermartingale X_n, $n \geq 0$, the number of upcrossings U of $[a, b]$ satisfies

$$P(U \geq k) \leq \left(\frac{a}{b}\right)^k E \min(X_0/a, 1)$$

To prove this, we let $N_0 = -1$ and for $j \geq 1$ let

$$\begin{aligned} N_{2j-1} &= \inf\{m > N_{2j-2} : X_m \leq a\} \\ N_{2j} &= \inf\{m > N_{2j-1} : X_m \geq b\} \end{aligned}$$

Let $Y_n = 1$ for $0 \leq n < N_1$ and for $j \geq 1$

$$Y_n = \begin{cases} (b/a)^{j-1}(X_n/a) & \text{for } N_{2j-1} \leq n < N_{2j} \\ (b/a)^j & \text{for } N_{2j} \leq n < N_{2j+1} \end{cases}$$

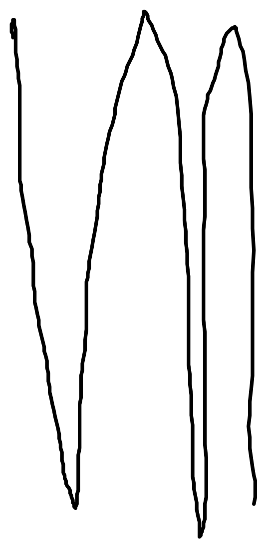
(i) Use the switching principle in the previous exercise and induction to show that $Z_n^j = Y_{n \wedge N_j}$ is a supermartingale. (ii) Use $EY_{n \wedge N_{2k}} \leq EY_0$ and let $n \to \infty$ to get Dubins' inequality.

4.3 Examples

In this section, we will apply the martingale convergence theorem to generalize the second Borel-Cantelli lemma and to study Polya's urn scheme, Radon-Nikodym derivatives, and branching processes. The four topics are independent of each other and are taken up in the order indicated.

4.3.1 Bounded Increments

Our first result shows that martingales with bounded increments either converge or oscillate between $+\infty$ and $-\infty$.



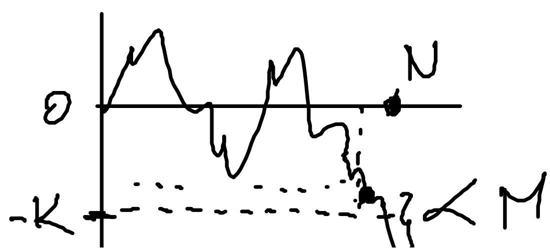
Theorem 4.3.1. *Let $X_1, X_2, \ldots$ be a martingale with $|X_{n+1} - X_n| \leq M < \infty$. Let*

$$\begin{aligned} C &= \{\lim X_n \textit{ exists and is finite}\} \\ D &= \{\limsup X_n = +\infty \textit{ and } \liminf X_n = -\infty\} \end{aligned}$$

Then $P(C \cup D) = 1$.

Proof. Since $X_n - X_0$ is a martingale, we can without loss of generality suppose that $X_0 = 0$. Let $0 < K < \infty$ and let $N = \inf\{n : X_n \leq -K\}$. $X_{n \wedge N}$ is a martingale with $X_{n \wedge N} \geq -K - M$ a.s. so applying Theorem 4.2.12 to $X_{n \wedge N} + K + M$ shows $\lim X_n$ exists on $\{N = \infty\}$. Letting $K \to \infty$, we see that the limit exists on $\{\liminf X_n > -\infty\}$. Applying the last conclusion to $-X_n$, we see that $\lim X_n$ exists on $\{\limsup X_n < \infty\}$ and the proof is complete. □

K large



Joseph Leo Doob

To prepare for an application of this result we need

Theorem 4.3.2. Doob's decomposition. *Any submartingale X_n, $n \geq 0$, can be written in a unique way as $X_n = M_n + A_n$, where M_n is a martingale and A_n is a predictable increasing sequence with $A_0 = 0$.*

Proof. We want $X_n = M_n + A_n$, $E(M_n|\mathcal{F}_{n-1}) = M_{n-1}$, and $A_n \in \mathcal{F}_{n-1}$. So we must have

F_n meas.

$$\begin{aligned} E(X_n|\mathcal{F}_{n-1}) &= E(M_n|\mathcal{F}_{n-1}) + E(A_n|\mathcal{F}_{n-1}) \\ &= M_{n-1} + A_n = X_{n-1} - A_{n-1} + A_n \end{aligned}$$

and it follows that

$$A_n - A_{n-1} = E(X_n|\mathcal{F}_{n-1}) - X_{n-1} \geq 0 \quad (X_n \text{ is submartg}) \tag{4.3.1}$$

Since $A_0 = 0$, we have ← has to be

$$A_n = \sum_{m=1}^{n} E(X_{\cancel{n}} - X_{\cancel{n}-1}|\mathcal{F}_{\cancel{n}-1}) \tag{4.3.2}$$

m m m

To check that our recipe works, we observe that $A_n - A_{n-1} \geq 0$ since X_n is a submartingale and $A_n \in \mathcal{F}_{n-1}$. To prove that $M_n = X_n - A_n$ is a martingale, we note that using $A_n \in \mathcal{F}_{n-1}$ and (4.3.1)

$$\begin{aligned} E(M_n|\mathcal{F}_{n-1}) &= E(X_n - A_n|\mathcal{F}_{n-1}) \\ &= E(X_n|\mathcal{F}_{n-1}) - A_n \overset{(4.3.1)}{=} X_{n-1} - A_{n-1} = M_{n-1} \end{aligned}$$

which completes the proof. □

The illustrate the use of this result we do the following important example.

Example 4.3.3. Let and suppose $B_n \in \mathcal{F}_n$. Using (4.3.2)

$$M_n = \sum_{m=1}^{n} 1_{B_m} - E(1_{B_m}|\mathcal{F}_{m-1})$$

Theorem 4.3.4. Second Borel-Cantelli lemma, II. *Let $\mathcal{F}_n$, $n \geq 0$ be a filtration with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and let B_n, $n \geq 1$ a sequence of events with $B_n \in \mathcal{F}_n$. Then*

$\sum 1_{B_n} = \infty$

$$\{B_n \ i.o.\} = \left\{ \sum_{n=1}^{\infty} P(B_n|\mathcal{F}_{n-1}) = \infty \right\}.$$

$\{B_n$ occurs infinitely often$\}$

Special case
(B_n) indep.
$F_n = \sigma(B_1, \ldots, B_n)$
B_n ind of F_{n-1}

Borel-Cantelli lemma:

I. $B_n \in \mathcal{F}$, $\sum_n P(B_n) < \infty$

$\Rightarrow P($ B_n occurs finitely many times $) = 1$

II. $B_n \in \mathcal{F}$, $\sum_n P(B_n) = \infty$ <u>and</u> B_n's <u>indep</u>

$\Rightarrow P($ B_n occur i.o. $) = 1$

Proof. If we let $X_0 = 0$ and $X_n = \sum_{m \le n} 1_{B_m}$, then X_n is a submartingale. (4.3.2) implies $A_n = \sum_{m=1}^{n} E(1_{B_m}|\mathcal{F}_{m-1})$ so if $M_0 = 0$ and

$$M_n = \sum_{m=1}^{n} \left(1_{B_m} - P(B_m|\mathcal{F}_{m-1}) \right)$$

for $n \ge 1$ then M_n is a martingale with $|M_n - M_{n-1}| \le 1$. Using the notation of Theorem 4.3.1 we have:

$$\text{on } C, \quad \sum_{n=1}^{\infty} 1_{B_n} = \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} P(B_n|\mathcal{F}_{n-1}) = \infty$$

$$\text{on } D, \quad \sum_{n=1}^{\infty} 1_{B_n} = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} P(B_n|\mathcal{F}_{n-1}) = \infty$$

Since $P(C \cup D) = 1$, the result follows. □

4.3.2 Polya's Urn Scheme

An urn contains r red and g green balls. At each time we draw a ball out, then replace it, and add c more balls of the color drawn. Let X_n be the fraction of green balls after the nth draw. To check that X_n is a martingale, note that if there are i red balls and j green balls at time n, then

$$X_{n+1} = \begin{cases} (j+c)/(i+j+c) & \text{with probability } j/(i+j) \\ j/(i+j+c) & \text{with probability } i/(i+j) \end{cases}$$

and we have

$$\frac{j+c}{i+j+c} \cdot \frac{j}{i+j} + \frac{j}{i+j+c} \cdot \frac{i}{i+j} = \frac{(j+c+i)j}{(i+j+c)(i+j)} = \frac{j}{i+j}$$

Since $X_n \ge 0$, Theorem 4.2.12 implies that $X_n \to X_\infty$ a.s. To compute the distribution of the limit, we observe (a) the probability of getting green on the first m draws then red on the next $\ell = n - m$ draws is

$$\frac{g}{g+r} \cdot \frac{g+c}{g+r+c} \cdots \frac{g+(m-1)c}{g+r+(m-1)c} \cdot \frac{r}{g+r+mc} \cdots \frac{r+(\ell-1)c}{g+r+(n-1)c}$$

and (b) any other outcome of the first n draws with m green balls drawn and ℓ red balls drawn has the same probability since the denominator remains the same and the numerator is permuted. Consider the special case $c = 1$, $g = 1$, $r = 1$. Let G_n be the number of green balls after the nth draw has been completed and the new ball has been added. It follows from (a) and (b) that

$$P(G_n = m+1) = \binom{n}{m} \frac{m!(n-m)!}{(n+1)!} = \frac{1}{n+1}$$