This is the heat equation.

For the forward equation we need again the adjoint of S. Let μ be absolutely continuous with respect to the Lebesgue measure, $\mu(\mathrm{d}y) = g(y)\mathrm{d}y$, and let $f \in C_c^2$. Integration by parts twice gives

$$\int f''(y)g(y)\mathrm{d}y = \int f(y)g''(y)\mathrm{d}y.$$

That is $(S^*\mu)(\mathrm{d}y) = \frac{1}{2}g''(y)\mathrm{d}y$. The forward equation is

$$\frac{\partial}{\partial t}p_t(y|x)\mathrm{d}y = \frac{1}{2}\frac{\partial^2}{\partial y^2}p_t(y|x)\mathrm{d}y,$$

which for the densities gives

$$\frac{\partial}{\partial t}\rho_t(y|x) = \frac{1}{2}\frac{\partial^2}{\partial y^2}\rho_t(y|x),$$

again the heat equation.

Recall that the *fundamental solution* to the heat equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x)$$

is

$$F(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}},$$

which is exactly the transition density of the SBM.

6.4 Diffusion processes

Diffusions can be handled as solution to SDEs. We showed that under general conditions unique strong solution to SDEs exists, implying the existence of diffusion processes. This is the probabilistic approach due to Lévy and Itô. Another more analytical approach to such processes was applied by Kolmogorov and Feller. They treated diffusions as general Markov processes and using tools from the theory of partial differential equations, they showed that under suitable conditions the Kolmogorov backward and forward equations have a unique solution. Then the existence of a desired Markov process follows from Kolmogorov's consistency theorem, and the continuity property

Theory SDE $\Rightarrow$ $\exists!$ strong sol.

of the process can be treated by Kolmogorov's continuity theorem (Theorem 12). Here we look a bit into the latter approach.

A diffusion process *locally* behaves as a Wiener process, in the sense that it satisfies the SDE

(W_t) SBM

$$\mathrm{d}Y_t = \mu(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t.$$

That is, for $h > 0$

$\Updownarrow$ def. of SDE

$$\Delta Y_t = Y_{t+h} - Y_t = \int_t^{t+h} \mu(Y_s)\mathrm{d}s + \int_t^{t+h} \sigma(Y_s)\mathrm{d}W_s$$
$$\approx h\mu(Y_t) + \sigma(Y_t)(W_{t+h} - W_t),$$

determin. drift; μ is cont.; σ cont.; Wiener

thus

$$\mathbf{E}\left[\Delta Y_t | Y_t = y\right] = \mu(y)h + o(h),$$
$$\mathbf{E}\left[(\Delta Y_t)^2 | Y_t = y\right] = \sigma^2(y)h + o(h).$$

$h \downarrow 0$

A *diffusion process* (Y_t) is a continuous Markov process satisfying as $h \downarrow 0$

(i) $\mathbf{P}(|\Delta Y_t| > \varepsilon | Y_t = y) = o(h)$; the truncation does not matter

(ii) $\mathbf{E}(\Delta_\varepsilon Y_t | Y_t = y) = \mu(y)h + o(h)$;

(iii) $\mathbf{E}((\Delta_\varepsilon Y_t)^2 | Y_t = y) = \sigma^2(y)h + o(h)$,

where $\Delta Y_t = Y_{t+h} - Y_t$, and

$$E_x(Y_t) = x + \mu(x) \cdot t + o(t)$$

$$\Delta_\varepsilon Y_t = \begin{cases} \Delta Y_t, & \text{if } |\Delta Y_t| \le \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

truncation at ε

The definition determines the infinitesimal generator of the process. For $f \in C^2$

$f(x + \underbrace{(Y_t - x)}_{h})$

$$\mathbf{E}_x f(Y_t) = \mathbf{E}_x \left[f(x) + (Y_t - x)f'(x) + (Y_t - x)^2 \frac{f''(x)}{2} + o((Y_t - x)^2) \right]$$
$$= f(x) + t\mu(x)f'(x) + t\sigma^2(x)\frac{f''(x)}{2} + o(t).$$

Therefore,

$$(Sf)(x) = \lim_{t \to 0} \frac{1}{t} \mathbf{E}_x [f(Y_t) - f(x)] = \mu(x)f'(x) + \sigma^2(x)\frac{f''(x)}{2}.$$

starting position

Taylor $f(x+h) = f(x) + h \cdot f'(x) + \frac{h^2}{2} \cdot f''(x) + o(h^2)$

$$(Sf)(x) = \mu(x) f'(x) + \frac{\sigma^2(x)}{2} f''(x)$$

$$P_t(B|x) = P(Y_t \in B \mid Y_0 = x) \qquad p_t(dy|x) = \rho_t(y|x)\,dy$$

ρ dens.

Kolmogorov backward equation is

$$\frac{\partial}{\partial t} p_t(y|x) = \mu(x) \frac{\partial}{\partial x} p_t(y|x) + \frac{\sigma^2(x)}{2} \frac{\partial^2}{\partial x^2} p_t(y|x).$$

$$\frac{\partial}{\partial t} \rho_t(y|x) = \mu(x) \frac{\partial}{\partial x} \rho_t(y|x) + \frac{\sigma^2(x)}{2} \frac{\partial^2}{\partial x^2} \rho_t(y|x)$$

For the forward equation we need the adjoint of S. This can be determined as for the SBM. Let $\rho_t(y|x)$ denote the density of the process, i.e. $p_t(\mathrm{d}y|x) = \rho_t(y|x)\mathrm{d}y$. Let $\mu(\mathrm{d}y) = g(y)\mathrm{d}y$. If f has compact support then in the integration by parts formula the increment disappears and we get

$$\int (Sf)(y)\,\mu(dy) = \int f(y)\,(S^*\mu)(dy)$$

$f \in C_c^2$

$g \in C_c^2$

$$\int (Sf)(y) \underbrace{g(y)\mathrm{d}y}_{\mu(dy)} = \int \left[\mu(y) f'(y) + \frac{\sigma^2(y)}{2} f''(y) \right] g(y)\mathrm{d}y$$

$$= \int f(y) \left[-\frac{\mathrm{d}}{\mathrm{d}y} \left(\mu(y) g(y) \right) + \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}y^2} \left(\sigma^2(y) g(y) \right) \right] \mathrm{d}y.$$

Thus

$$\underbrace{\left(S^* p_t(\cdot|x) \right)}_{\text{measure}} (\mathrm{d}y) = \left[-\frac{\mathrm{d}}{\mathrm{d}y} \left(\mu(y) \rho_t(y|x) \right) + \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}y^2} \left(\sigma^2(y) \rho_t(y|x) \right) \right] \mathrm{d}y,$$

and the forward equation is $\quad p_t(dy|x) = \rho_t(y|x)\,dy$

$$\frac{\partial}{\partial t} \rho_t(y|x) = -\frac{\partial}{\partial y} \left(\mu(y) \rho_t(y|x) \right) + \frac{1}{2} \frac{\partial^2}{\partial y^2} \left(\sigma^2(y) \rho_t(y|x) \right).$$

Example 23 (Ornstein–Uhlenbeck process)**.** Consider the *Langevin equation*

$$\mathrm{d}Y_t = -\mu Y_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t,$$

$Y_t > 0$

where $\mu > 0$, $\sigma > 0$, and Y_0 is independent of $\sigma(W_s : s \geq 0)$.

The solution of the homogeneous equation is $e^{-\mu t}$. Taking the derivative of $e^{\mu t} Y_t$ we obtain

$$\mathrm{d}\left(e^{\mu t} Y_t \right) = e^{\mu t}\,\mathrm{d}Y_t + \mu\, e^{\mu t} Y_t\,\mathrm{d}t = e^{\mu t}\,\sigma\,\mathrm{d}W_t,$$

$$e^{\mu t} Y_t - e^{\mu \cdot 0} Y_0 = \int_0^t \sigma e^{\mu s}\, dW_s$$

which gives

$$Y_t = e^{-\mu t} \left(Y_0 + \int_0^t e^{\mu s}\,\sigma\,\mathrm{d}W_s \right).$$

This is the *Ornstein–Uhlenbeck process.* The integral of a deterministic function with respect to SBM is Gaussian, thus

$$Y_t - e^{-\mu t} Y_0 = e^{-\mu t} \int_0^t e^{\mu s} \sigma\, dW_s$$

$$d(e^{\mu t} Y_t) = e^{\mu t} dY_t + Y_t\, de^{\mu t} = e^{\mu t}(-\mu Y_t\, dt + \sigma\, dW_t) + Y_t\, \mu e^{\mu t}\, dt = \sigma \cdot e^{\mu t}\, dW_t.$$

$$\int S f(y)\, g(y)\, dy = \int \left(\mu(y) f'(y) + \frac{\sigma^2(y)}{2} f''(y) \right) g(y)\, dy$$

$$= \int \mu(y)\, g(y)\, f'(y)\, dy + \int \frac{\sigma^2(y)}{2} g(y)\, f''(y)\, dy$$

Want: get f instead of f'', f'.

$$\int \mu(y)\, g(y)\, f'(y)\, dy = \overbrace{\Big[f(y) \cdot \mu(y) g(y) \Big]_{-\infty}^{\infty}}^{=0} - \int f(y) \cdot \frac{d}{dy}\big(\mu(y) g(y)\big)\, dy$$

$$= - \int f(y)\, \frac{d}{dy}\big(\mu(y) g(y)\big)\, dy.$$

$$\int g(y) \sigma^2(y)\, f''(y)\, dy = \underbrace{\Big[f'(y)\, g(y) \sigma^2(y) \Big]_{-\infty}^{\infty}}_{=0} - \int f'(y) \cdot \frac{d}{dy}\big(g(y)\sigma^2(y)\big)\, dy$$

integration by parts

$$= -\Big[f(y)\, \frac{d}{dy}\big(g(y)\sigma^2(y)\big) \Big]_{-\infty}^{\infty} + \int f(y)\, \frac{d^2}{dy^2}\big(g(y)\sigma^2(y)\big)\, dy$$

$$= \int f(y)\, \frac{d^2}{dy^2}\big(g(y)\sigma^2(y)\big)\, dy.$$

$$\int S f(y)\, g(y)\, dy = -\int f(y) \frac{d}{dy}\big(\mu(y)\, g(y)\big)\, dy$$

$$+ \frac{1}{2} \int f(y) \frac{d^2}{dy^2}\big(g(y)\,\sigma^2(y)\big)\, dy$$

$$= \int f(y) \cdot \underbrace{\left[-\frac{d}{dy}\big(\mu(y)\, g(y)\big) + \frac{1}{2} \frac{d^2}{dy^2}\big(g(y)\,\sigma^2(y)\big)\right]} dy$$

$$= \int f(y) \quad S^{*} g(y)\, dy \qquad \text{adjoint of } S.$$

$$I_t = \int_0^t \overset{\text{deterministic}}{f(s)}\, dW_s = \lim_{n\to\infty} \underbrace{\sum_i f(t_i^{(n)}) \left(W_{t_{i+1}^{(n)}} - W_{t_i^{(n)}}\right)}$$

(I_t) process is Gaussian

$$\left(I_{t_1}, I_{t_2}, \dots, I_{t_k}\right) \sim \underset{\substack{\text{k dimensional}\\ \text{normal vector}}}{N_k}(0, \Sigma)$$

$t_1 < t_2 < \dots < t_k$

same proof

for any fix n this is Gaussian

$$\sim N\left(0, \sum_i f(t_i^{(n)})^2 \left(t_{i+1}^{(n)} - t_i^{(n)}\right)\right)$$

$$\downarrow$$

$$N\left(0, \int_0^t f^2(s)\, ds\right)$$

$$Y_t = e^{-\mu t}\left(Y_0 + \int_0^t e^{\mu s}\sigma \, dW_s\right)$$

$$E(Y_t) = E(Y_0)\cdot e^{-\mu t}$$

$$E\left[\left(\int_0^t e^{\mu s}\sigma \, dW_s\right)^2\right] = E\left(\int_0^t e^{2\mu s}\sigma^2 ds\right) = \int_0^t \sigma^2 e^{2\mu s} ds$$

is normal with mean and variance

$$\mathbf{E}Y_t = e^{-\mu t}\,\mathbf{E}Y_0,$$

$$\mathbf{E}Y_t^2 = e^{-2\mu t}\,\mathbf{E}Y_0^2 + e^{-2\mu t}\int_0^t \sigma^2\, e^{2\mu s}\,\mathrm{d}s = e^{-2\mu t}\,\mathbf{E}Y_0^2 + \frac{\sigma^2}{2\mu}\,(1 - e^{-2\mu t}).$$

We see that as $t \to \infty$

$$Y_t \xrightarrow{\mathcal{D}} N(0, \sigma^2/(2\mu)).$$

Taking the limit for the initial distribution Y_0 we see that (Y_t) is Gaussian and

$$Y_t \sim \mathrm{N}\left(0, \frac{\sigma^2}{2\mu}\right).$$

Let $Y_0 \sim N\left(0, \frac{\sigma^2}{2\mu}\right)$

Next we determine the covariance function of Y. Since

$$Y_t = e^{-\mu t}\left(Y_0 + \int_0^t \sigma\, e^{\mu u}\,\mathrm{d}W_u\right)$$

$EY_t = 0$ ✓

we get

$$Y_t - e^{-\mu(t-s)}Y_s = e^{-\mu t}\int_s^t \sigma\, e^{\mu u}\,\mathrm{d}W_u,\ t > s, \qquad (30)$$

which is independent of $\sigma(W_u : u \le s)$ σ. Therefore,

$$\begin{aligned}\mathbf{Cov}(Y_t, Y_s) &= \mathbf{E}Y_tY_s = \mathbf{E}\left(Y_t - e^{-\mu(t-s)}Y_s + e^{-\mu(t-s)}Y_s\right)Y_s \\ &= e^{-\mu(t-s)}\,\mathbf{E}Y_s^2 = \frac{\sigma^2}{2\mu}e^{-\mu(t-s)},\end{aligned}$$

which depends only on $t - s$. That is (Y_t) is stationary.

Using formula (30) for $A \in \mathcal{B}(\mathbb{R})$

$$\begin{aligned}&\mathbf{P}(Y_t \in A | Y_u : u \le s, Y_s = x) \\ &= \mathbf{P}(Y_t - e^{-\mu(t-s)}Y_s \in A - e^{-\mu(t-s)}x | Y_u : u \le s, Y_s = x) \\ &= \mathbf{P}(Y_t - e^{-\mu(t-s)}Y_s \in A - e^{-\mu(t-s)}x).\end{aligned}$$

The variable $Y_t - e^{-\mu(t-s)}Y_s$ is mean zero Gaussian with variance

$$\mathbf{E}\left(Y_t - e^{-\mu(t-s)}Y_s\right)^2 = e^{-2\mu t}\underbrace{\int_s^t \sigma^2 e^{2\mu u}\mathrm{d}u}_{\frac{\sigma^2}{2\mu}\left(e^{2\mu t} - e^{2\mu s}\right)} = \frac{\sigma^2}{2\mu}\left(1 - e^{-2\mu(t-s)}\right).$$

$$Y_t = e^{-\mu t}\left(Y_0 + \int_0^t e^{\mu s}\sigma\, dW_s\right)$$

$$Y_0 \sim N\left(0, \frac{\sigma^2}{2\mu}\right) \text{ independent of } (W_s)_{s\geq 0}$$

$s<t$

$$Cov(Y_s, Y_t) = E(Y_s \cdot Y_t)$$

$$= E\left(e^{-\mu s}\cdot\left(Y_0 + \int_0^s e^{\mu u}\sigma\, dW_u\right) e^{-\mu t}\left(Y_0 + \int_0^t e^{\mu u}\sigma\, dW_u\right)\right)$$

$$= e^{-\mu(t+s)}\Big[E(Y_0^2) + \underbrace{E\left(Y_0 \int_0^t \dots dW_u\right)}_{=0} + \underbrace{E\left(Y_0 \int_0^s \dots dW_u\right)}_{=0}$$

$$+ E\left(\int_0^s \dots dW_u \cdot \int_0^t \dots dW_u\right)\Big]$$

$$= e^{-\mu(t+s)}\left[\frac{\sigma^2}{2\mu} + 0 + 0 + \frac{\sigma^2}{2\mu}\left(e^{2\mu s} - 1\right)\right] \neq$$

$$E\left(Y_0 \cdot \int_0^t \dots dW_u\right) = 0$$

$$E\left(\int_0^s \dots dW_u \underbrace{\int_0^t \dots dW_u}_{\int_0^s + \int_s^t}\right) = E\left(\int_0^s \dots dW_u\right)^2 + 0$$

$$= E \int_0^s (\;)^2 du = \sigma^2 \int_0^s e^{2\mu u} du$$

$$= \frac{\sigma^2}{2\mu}\left(e^{2\mu s} - 1\right)$$

$$= e^{-\mu(t+s)} \frac{\sigma^2}{2\mu}\left[\left(e^{2\mu s} - 1\right) + 1\right]$$

$$= \frac{\sigma^2}{2\mu} \cdot e^{-\mu(t-s)}$$ ← depends only on $t-s$

$\Downarrow$

stationary

$\frac{1}{\sqrt{2\pi}\tilde{\sigma}} \cdot e^{-\frac{(x-\tilde{\mu})^2}{2\tilde{\sigma}^2}}$ ← these are not the same as in the prev

Substituting $s = 0$

$$p_t(\cdot|x) \sim \mathrm{N}\left(e^{-\mu t}x, \frac{\sigma^2}{2\mu}\left(1 - e^{-2\mu t}\right)\right),$$

that is, the transition density

$$\rho_t(y|x) = \sqrt{\frac{\mu}{\pi\sigma^2(1 - e^{-2\mu t})}} \exp\left\{-\frac{\mu(y - e^{-\mu t}x)^2}{\sigma^2(1 - e^{-2\mu t})}\right\}.$$

We proved that (Y_t) is a continuous stationary Markov process. It can be shown that this characterizes the OU process.

Finally, we spell out the Kolmogorov equations. The backward is

$$\frac{\partial}{\partial t}\rho_t(y|x) = -\mu x \frac{\partial}{\partial x}\rho_t(y|x) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\rho_t(y|x),$$

which is called *Fokker–Planck equation.* The forward is

$$\frac{\partial}{\partial t}\rho_t(y|x) = -\frac{\partial}{\partial y}\left(-\mu y \rho_t(y|x)\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\rho_t(y|x).$$

It is important to emphasize that in general explicit formulas for the transition densities cannot be obtained. For simulation results the Kolmogorov equations are important, because solutions can be approximated numerically.

7 Brownian motion and PDEs

This part is from Karatzas and Shreve [5].

We showed that the infinitesimal generator of the SBM is the Laplacian operator Δ. Furthermore the transition density of SBM is the fundamental solution to the heat equation. These facts already show the intrinsic connection between Brownian motion and partial differential equations. Here we spell out this connection a bit more.

7.1 Harmonic functions and the Dirichlet problem

Let D be an open subset of $\mathbb{R}^d$. Let W be a d-dimensional standard Brownian motion, and let

$$\tau_D = \inf\{t \geq 0 : W_t \in D^c\}$$ ← exit time

the first exit time from D. Let B_r be the open ball centered at the origin, V_r its volume and S_r its surface. The normalized surface measure on B_r is μ_r

$$\mu_r(\mathrm{d}x) = \mathbf{P}_0(W_{\tau_{B_r}} \in \mathrm{d}x).$$

Then

Fubini

$$\int_{B_r} f(x)\mathrm{d}x = \int_0^r S_\rho \int_{\partial B_\rho} f(x)\mu_\rho(\mathrm{d}x)\mathrm{d}\rho. \tag{31}$$

A function u is *harmonic* in D if

$$\Delta u = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} u = 0$$

in D. A function $u : D \to \mathbb{R}$ satisfies the *mean-value property*, if for every $a \in D$ and $r > 0$ such that $a + \overline{B}_r \subset D$,

$$u(a) = \int_{\partial B_r} u(a+x)\mu_r(\mathrm{d}x).$$

We know that u is harmonic if and only if it satisfies the mean-value property. We give a simple proof to one direction using Itô formula.

Proposition 11. *If u is harmonic in D, then it satisfies the mean-value property there.*

Proof. By Itô's formula

$$u(W_{t\wedge\tau_{a+B_r}}) = u(W_0) + \sum_{i=1}^d \int_0^{t\wedge\tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s)\mathrm{d}W_s^{(i)} + \frac{1}{2}\int_0^{t\wedge\tau_{a+B_r}} \underbrace{\Delta u(W_s)}_{=0}\mathrm{d}s.$$

Taking expectation $\mathbf{E}_a$ and letting $t \to \infty$

$W_0 = a$

$$\mathbf{E}_a u(W_{\tau_{a+B_r}}) = u(a),$$

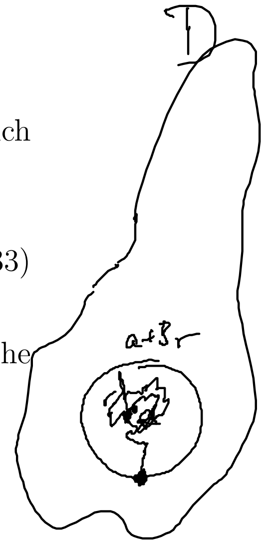
as stated. □

Let D be an open set of $\mathbb{R}^d$ and $f : \partial D \to \mathbb{R}$ be a continuous function. Consider the Dirichlet problem

$$\begin{aligned} \Delta u &= 0, \quad \text{in } D, \\ u &= f, \quad \text{on } \partial D. \end{aligned} \tag{32}$$

$\tau_D = 0$ on ∂D

$x \in \partial D$ $\quad u(x) = f(x)$ ✓



A solution to the Dirichlet problem is a continuous function $u : \overline{D} \to \mathbb{R}$ which satisfies the equation above.

Then one can guess that

exit

$$u(x) = \mathbf{E}_x f(W_{\tau_D}) \tag{33}$$

should be a solution, provided that the expectation exists.

Indeed, the boundary condition holds by the definition of τ_D. Using the strong Markov property

$$\begin{aligned} \boxed{u(a)} = \mathbf{E}_a f(W_{\tau_D}) &= \mathbf{E}_a \left[\mathbf{E}_a[f(W_{\tau_D}) | \mathcal{F}_{\tau_{a+B_r}}] \right] \\ = \mathbf{E}_a u(W_{\tau_{a+B_r}}) &= \int_{\partial B_r} u(a+x) \mu_r(\mathrm{d}x), \end{aligned}$$

that is the mean-value property holds, which means that u is indeed harmonic.

We proved the following.

Proposition 12. *If u in* (33) *is well-defined then it is harmonic.*

The proof of Proposition 11 shows in fact uniqueness.

Proposition 13. *If f is bounded and $\mathbf{P}_a(\tau_D < \infty) = 1$ for all $a \in D$, then any bounded solution to* (32) *has the form* (33).

Proof. Consider a bounded solution u. By Itô's formula

$$u(W_{t \wedge \tau_D}) = u(W_0) + \sum_{i=1}^{d} \int_0^{t \wedge \tau_{a+B_r}} \frac{\partial u}{\partial x_i}(W_s) \mathrm{d}W_s^{(i)}. \quad + \frac{1}{2} \int \underbrace{\Delta u}_{=0}$$

Taking expectation $\mathbf{E}_a$ and ~~letting $t \to \infty$~~

$$\boxed{\mathbf{E}_a u(W_{\tau_D}) = u(a),}$$

as stated. □

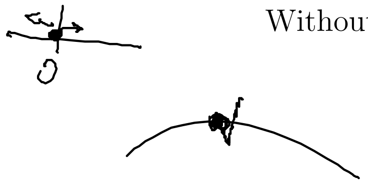
Note that a solution to the Dirichlet problem (32) is necessarily continuous. Therefore, we need conditions characterizing the points $a \in \partial D$ for which

$$\lim_{x \to a, x \in D} \mathbf{E}_x f(W_{\tau_D}) = f(a) \tag{34}$$

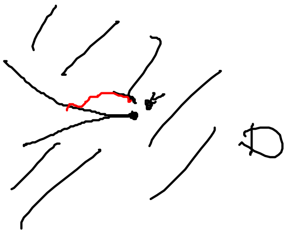
holds for any bounded measurable function, which is continuous at a.

Define the stopping time $\sigma_D = \inf\{t > 0 : W_t \in D^c\}$. Note the $>$ compared to $\geq$ in τ_D. A point $a \in \partial D$ is *regular* for D is $\mathbf{P}_a(\sigma_D = 0) = 1$.

Without proof we state the result on regularity.



$a \in \partial D$



Theorem 34. *Let $d \geq 2$ and fix $a \in \partial D$. The following are equivalent:*

(i) (34) *holds for every bounded, measurable function which continuous at a;*

(ii) *a is regular for D;*

(iii) *for all $\varepsilon > 0$ we have*

$$\lim_{x \to a, x \in D} \mathbf{P}_x(\tau_D > \varepsilon) = 0.$$

For $d = 1$ every point of ∂D is regular. The Dirichlet problem is always solvable, the solution is piecewise linear. For $d \geq 2$ consider the punctured unit ball $D = \{x \in \mathbb{R}^d : 0 < \|x\| < 1\}$. Clearly, the origin is irregular for D. For any $x \in D$ the SBM exits D on its outer boundary, therefore we do not see the value of f at 0. For this D the Dirichlet problem has a solution only if $f(0) = \widetilde{u}(0)$, where $\widetilde{u}$ is the solution for B_1.

7.2 Feynman–Kac formula

Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u, \tag{35}$$

with initial condition $u(0, x) = f(x)$.

The fundamental solution to the heat equation is in fact the transition probabilities of the d-dimensional SBM

$$\rho_t(y|x) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{\|x-y\|^2}{2t}}.$$

Under some growth condition on f, the unique solution to (35) has the form

$$u(t, x) = \mathbf{E}_x f(W_t) = \int f(y) \rho_t(y|x) \mathrm{d}y.$$

The probabilistic representation of the solution to certain PDEs holds in a more general setup.

Consider the equation

$$\begin{aligned} -\frac{\partial v}{\partial t} + kv &= \frac{1}{2} \Delta v + g \quad \text{on } [0, T) \times \mathbb{R}^d, \\ v(T, x) &= f(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{36}$$

where $f : \mathbb{R}^d \to \mathbb{R}$, $k : \mathbb{R}^d \to [0, \infty)$, and $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}$.