One can define stochastic integral with respect to more general processes. The process (X_t) is a continuous *semimartingale* if

$$X_t = M_t + A_t,$$

where M_t is a continuous martingale and A_t is of bounded variation, and both are adapted. As in Lemma 6 it can be shown that this decomposition is essentially unique.

We can define stochastic integral with respect to semimartingales. Indeed, integral with respect to A_t can be defined pathwise, since A is of bounded variation, and integration with respect to continuous M_t can be defined similarly as for SBM.

The following version of Itô's formula holds.

Theorem 31 (Itô formula for semimartingales)**.** *Let $X_t = M_t + A_t$ be a continuous semimartingale, and let $f \in C^2$. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\mathrm{d}\langle M\rangle_s.$$

5 Stochastic differential equations

5.1 Existence and uniqueness

We define the strong solution of SDEs and obtain existence and uniqueness results.

The followings are given:

- probability space $(\Omega, \mathcal{A}, \mathbf{P})$;
- with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$;
- r a ~~d~~-dimensional SBM $W_t = (W_t^1, \ldots, W_t^r)$ with respect to the filtration $(\mathcal{F}_t)$;
- measurable functions $f : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{d\times r}$;
- $\mathcal{F}_0$-measurable rv $\xi : \Omega \to \mathbb{R}^d$.

The (d-dimensional) process (X_t) is *strong solution to the SDE*

$$\begin{aligned} \mathrm{d}X_t &= \underbrace{f(X_t,t)}_{\text{det. drift}}\,\mathrm{d}t + \sigma(X_t,t)\,\mathrm{d}W_t, \\ X_0 &= \xi, \end{aligned} \qquad (22) \quad \texttt{\{eq:sde\}}$$

$$\Delta X_t = X_{t+h} - X_t = \underbrace{h \cdot f(X_t)} + (W_{t+h} - W_h)\cdot\sigma(X_t,t)$$

$$\to \quad \underset{\substack{\shortparallel \\ \xi}}{X_t - X_0} = \int_0^t f(X_s, s)\, ds + \int_0^t \sigma(X_s, s)\, dW_s.$$

if $\int_0^t f(X_s, s)\mathrm{d}s$ are $\int_0^t \sigma(X_s, s)\mathrm{d}W_s$ well-defined for all $t \in [0, T]$ and the integral version of (22) holds, i.e.

$$X_t = \xi + \int_0^t f(X_s, s)\,\mathrm{d}s + \int_0^t \sigma(X_s, s)\,\mathrm{d}W_s, \quad \text{for all } t \in [0, T] \text{ a.s.}$$

Written coordinatewise

$$X_t^i = \xi^i + \int_0^t f^i(X_s, s)\,\mathrm{d}s + \int_0^t \sum_{j=1}^r \sigma_{i,j}(X_s, s)\,\mathrm{d}W_s^j, \quad i = 1, 2, \ldots, d.$$

It is important to emphasize that with strong solutions not only the SDE (22) is given, but the driving SBM, the initial condition (not just distribution!) ξ and the filtration.

For d-dimensional vectors $|x| = \sqrt{x_1^2 + \ldots + x_d^2}$ stands for the usual Euclidean norm, and for a matrix $\sigma \in \mathbb{R}^{d \times r}$, define $|\sigma| = \sqrt{\sum_{i,j} \sigma_{ij}^2}$,

Picard-Lindelöf thm. for ODE

{thm:sde-exuni}

Theorem 32. *Assume that for the functions in* (22) *the following hold:*

ODE
$$\left\{\begin{array}{l} |f(x,t) - f(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq K|x - y|, \\ |f(x,t)|^2 + |\sigma(x,t)|^2 \leq K_0(1 + |x|^2), \end{array}\right.$$

Lipschitz-cont.

initial value $\to$ $\mathbf{E}|\xi|^2 < \infty.$

Then (22) *has a unique strong solution* X*, and*

$$\mathbf{E} \sup_{0 \leq t \leq T} |X_t|^2 \leq C(1 + \mathbf{E}|\xi|^2).$$

Proof. We only prove for $d = r = 1$. The general case is similar, but notationally messy.

Recall the following statement from the theory of ordinary differential equations.

Lemma 8 (Gronwall–Bellman)**.** *Let* α, β *be integrable functions for which*

$$\underbrace{\alpha(t) \leq \beta(t) + H \int_a^t \alpha(s)\,\mathrm{d}s,}_{} \quad t \in [a, b],$$

for some $H \geq 0$*. Then*

$$\Downarrow$$

$$\alpha(t) \leq \beta(t) + H \int_a^t e^{H(t-s)} \beta(s)\,\mathrm{d}s.$$

everything is deterministic

$$\begin{cases} dX_t = f(X_t,t)\,dt + \sigma(X_t,t)\,dW_t \\ X_0 = \xi \end{cases}$$

$$X_t = \xi + \int_0^t f(X_s,s)\,ds + \int_0^t \sigma(X_s,s)\,dW_s.$$

Uniqueness. Let X_t, Y_t be solutions. Then

$$X_t - Y_t = \int_0^t (f(X_s,s) - f(Y_s,s))\,\mathrm{d}s + \int_0^t (\sigma(X_s,s) - \sigma(Y_s,s))\,\mathrm{d}W_s.$$

Since $(a+b)^2 \le 2a^2 + 2b^2$, by Theorem 25 (ii) and the Cauchy–Schwarz inequality

$$\mathbf{E}\Big(\int_0^t \sigma - \sigma \, dW_s\Big)^2$$

$$0 \le t \le T$$

$$\begin{aligned}
\mathbf{E}(X_t - Y_t)^2 &\le 2\,\mathbf{E}\left[\left(\int_0^t (f(X_s,s) - f(Y_s,s))\mathrm{d}s\right)^2\right] \\
&\quad + 2\,\mathbf{E}\int_0^t (\sigma(X_s,s) - \sigma(Y_s,s))^2 \mathrm{d}s \\
&\le 2(T+1)K^2 \int_0^t \mathbf{E}(X_s - Y_s)^2\,\mathrm{d}s.
\end{aligned}$$

$$K^2(X_s - Y_s)^2$$

With the notation $\varphi(t) = \mathbf{E}(X_t - Y_t)^2$ we obtained

$$\varphi(t) \le 2(T+1)K^2 \int_0^t \varphi(s)\,\mathrm{d}s.$$

By the Gronwall–Bellman lemma $\boxed{\varphi(t) \equiv 0,}$ i.e. $X_t = Y_t$ a.s. Since $X_t - Y_t$ is continuous, the two processes are indistinguishable, meaning

$$\mathbf{P}(X_t = Y_t,\ \forall t \in [0,T]) = 1.$$

Thus the uniqueness is proved.

Existence. Sketch. The proof goes similarly as the proof of the Picard–Lindelöf theorem for ODEs. We do Picard iteration. Let $X_t^{(0)} \equiv \xi$, and if $X_t^{(n)}$ is given, let

$$X_t^{(n)} \to X_t \text{ uniformly}$$

$$X_t^{(n+1)} = \xi + \int_0^t f(X_s^{(n)}, s)\mathrm{d}s + \int_0^t \sigma(X_s^{(n)}, s)\mathrm{d}W_s.$$

$$X_t = \xi + \int_0^t f(X_s,s)\,ds + \int_0^t \sigma(X_s,s)\,dW_s$$

Write

$$\begin{aligned}
X_t^{(n+1)} - X_t^{(n)} &= \int_0^t \big(f(X_s^{(n)}, s) - f(X_s^{(n-1)}, s)\big)\,\mathrm{d}s \\
&\quad + \int_0^t \big(\sigma(X_s^{(n)}, s) - \sigma(X_s^{(n-1)}, s)\big)\,\mathrm{d}W_s \\
&=: B_t^{(n)} + M_t^{(n)}.
\end{aligned}$$

$$E\left[\left(\int_0^t f(X_s,s) - f(Y_s,s)\,ds\right)^2\right] \leq$$

$$\leq E\left[\int_0^t \left(f(X_s,s)-f(Y_s,s)\right)^2 ds \cdot \int_0^t 1\,ds\right]$$

$$\int fg \leq \sqrt{\int f^2}\cdot\sqrt{\int g^2}$$

$$\leq t \cdot \int_0^t E\left(K^2 \cdot (X_s - Y_s)^2\right) ds$$

$$= t \cdot K^2 \int_0^t E\left[(X_s - Y_s)^2\right] ds$$

mtg $\rightarrow$ $M_t^{(n)} = \int_0^t \left(\sigma(X_s^{(n)}, s) - \sigma(X_s^{(n-1)}, s)\right) dW_s$

$$\mathbf{E}\left[(M_t)^2\right] = \int_0^t (\quad)^2 \, ds$$

By Doob's maximal inequality, as in the proof of uniqueness

$\left(\frac{p}{p-1}\right)^p = 4$, $p=2$

$$\begin{aligned}
\mathbf{E}\left(\sup_{s\in[0,t]} (M_s^{(n)})^2\right) &\le 4\mathbf{E}\int_0^t \left(\sigma(X_s^{(n)}, s) - \sigma(X_s^{(n-1)}, s)\right)^2 \mathrm{d}s \\
&\le 4K^2 \int_0^t \mathbf{E}\left[X_s^{(n)} - X_s^{(n-1)})^2\right] \mathrm{d}s.
\end{aligned}$$

On the other hand, by Cauchy–Schwarz & Lipschitz prop. $B_t = \int_0^t f(X_s^{(n)}, s) - f(\ldots) ds$

$$\mathbf{E}\left(\sup_{s\in[0,t]} (B_s^{(n)})^2\right) \le tK^2\,\mathbf{E}\int_0^t \left(X_s^{(n)} - X_s^{(n-1)}\right)^2 \mathrm{d}s.$$

This implies

$$\mathbf{E}\left(\sup_{s\in[0,t]} (X_s^{(n+1)} - X_s^{(n)})^2\right) \le L\int_0^t \mathbf{E}(X_s^{(n)} - X_s^{(n-1)})^2 \mathrm{d}s,$$

with $L = 2(T+4)K^2$. Iterating and changing the order of integration

$u \le s$

$\int_0^t du \int_u^t ds$ $\quad t-u$

$$\begin{aligned}
\mathbf{E}\left(\sup_{s\in[0,t]} (X_s^{(n+1)} - X_s^{(n)})^2\right) &\le L\int_0^t \mathbf{E}(X_s^{(n)} - X_s^{(n-1)})^2 \,\mathrm{d}s \\
&\le L^2 \int_0^t \left(\int_0^s \mathbf{E}(X_u^{(n-1)} - X_u^{(n-2)})^2 \,\mathrm{d}u\right) \mathrm{d}s \\
&\le L^2 \int_0^t (t-s)\mathbf{E}(X_s^{(n-1)} - X_s^{(n-2)})^2 \,\mathrm{d}s.
\end{aligned}$$

Continuing, and using the assumption on ξ we obtain induction

$$\begin{aligned}
&\mathbf{E}\left(\sup_{s\in[0,t]} (X_s^{(n+1)} - X_s^{(n)})^2\right) \\
&\le L^n \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\mathbf{E}(X_s^1 - \xi)^2 \,\mathrm{d}s \le C\frac{(LT)^n}{n!}.
\end{aligned}$$

summable $\sum_{n=1}^{\infty} \frac{(LT)^n}{n!} < \infty$

By Chebyshev

$$\sum_{n=1}^{\infty} \mathbf{P}\left(\sup_{0\le t\le T} |X_t^{(n+1)} - X_t^n| > n^{-2}\right) \le \sum_{n=1}^{\infty} C' n^4 \frac{(LT)^n}{n!} < \infty.$$

$\Rightarrow$ I. Borel-Cantelli

$X_t^{(n)} \to X_t$ uniformly in $t \in [0,T]$

Therefore, applying the first Borel–Cantelli lemma the infinite sum

$$\sum_{n=0}^{\infty}(X_t^{(n+1)} - X_t^n)$$

converges a.s. Clearly the sum is a solution to the SDE (22). □

5.2 Examples

Most of the examples and exercises are from Evans [4].

Example 16. Let g be a continuous function, and consider the SDE

$$\begin{cases} \mathrm{d}X_t = g(t)X_t\mathrm{d}W_t \\ X_0 = 1. \end{cases}$$

Show that the unique solution is

$$X_t = \exp\left\{-\frac{1}{2}\int_0^t g(s)^2\mathrm{d}s + \int_0^t g(s)\mathrm{d}W_s\right\}.$$

The uniqueness follows from Theorem 32, assuming g is nice enough. To check that X_t is indeed a solution, we use Itô's formula. Let

$$Y_t = -\frac{1}{2}\int_0^t g(s)^2\mathrm{d}s + \int_0^t g(s)\mathrm{d}W_s.$$

$f(Y_t) = f(Y_0) +$

With $f(x) = e^x$, we have

$$\begin{aligned} X_t = e^{Y_t} &= 1 + \int_0^t e^{Y_s}\mathrm{d}Y_s + \frac{1}{2}\int_0^t e^{Y_s}g^2(s)\mathrm{d}s \\ &= 1 + \int_0^t X_s g(s)\mathrm{d}W_s, \end{aligned}$$

$+ \int_0^t f'(Y_s)dY_s +$

as claimed.

$+ \frac{1}{2}\int_0^t f''(Y_s)g^2(s)ds$

Exercise 34. Let f and g be continuous functions, and consider the SDE

$$\begin{cases} \mathrm{d}X_t = f(t)X_t\mathrm{d}t + g(t)X_t\mathrm{d}W_t \\ X_0 = 1. \end{cases}$$

Show that the unique solution is

$$X_t = \exp\left\{\int_0^t \left[f(s) - \frac{1}{2}g(s)^2\right]\mathrm{d}s + \int_0^t g(s)\mathrm{d}W_s\right\}.$$

Exercise 35 (Brownian bridge)**.** Show that

$$B_t = (1-t)\int_0^t \frac{1}{1-s}\mathrm{d}W_s$$

is the unique solution of the SDE

$$\begin{cases} \mathrm{d}B_t = -\frac{B_t}{1-t}\mathrm{d}t + \mathrm{d}W_t \\ B_0 = 0. \end{cases}$$

Calculate the mean and covariance function of B.

A mean zero Gaussian process B_t on $[0,1]$ is called *Brownian bridge* if its covariance function is

$$\mathbf{Cov}(B_s, B_t) = \min(s,t) - st.$$

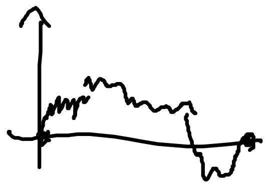
$(W_t, \mathcal{F}_t)$

Exercise 36. Show that if W is SBM then $B_t = W_t - tW_1$ is Brownian bridge.

$t \in [0,1]$

Exercise 37. Solve the SDE

$$\begin{cases} \mathrm{d}X_t = -\frac{1}{2}e^{-2X_t}\mathrm{d}t + e^{-X_t}\mathrm{d}W_t \\ X(0) = 0 \end{cases}$$



and show that it explodes in a finite random time. *Hint: Look for a solution* $X_t = u(W_t)$.

Exercise 38. Solve the SDE

$$\mathrm{d}X_t = -X_t\mathrm{d}t + e^{-t}\mathrm{d}W_t.$$

Exercise 39. Show that $(X_t, Y_t) = (\cos W_t, \sin W_t)$ is a solution to the SDE

$$\begin{cases} \mathrm{d}X_t = -\frac{1}{2}X_t\mathrm{d}t - Y_t\mathrm{d}W_t \\ \mathrm{d}Y_t = -\frac{1}{2}Y_t\mathrm{d}t + X_t\mathrm{d}W_t. \end{cases}$$

Show that $\sqrt{X_t^2 + Y_t^2}$ is a constant for any solution (X, Y)!

35. $\begin{cases} dB_t = -\frac{B_t}{1-t}\,dt + dW_t \\ B_0 = 0 \end{cases}$

$B_t = (1-t)\cdot\int_0^t \frac{1}{1-s}\,dW_s$ is a solution.

$t < 1$

$-$ $\int_0^t \left(\frac{1}{1-s}\right)^2 ds < \infty$

$t < 1$ ✓

$t = 1$: $\int_0^1 \frac{1}{(1-s)^2}ds = \int_0^1 \frac{1}{s^2}ds = \left[-\frac{1}{s}\right]_0^1 = \infty$

$-$. problem

$-$

$dB_t = -\frac{B_t}{1-t}\,dt + dW_t$, $B_0 = 0$

$B_t = \underbrace{(1-t)}_{X_t}\cdot\underbrace{\int_0^t \frac{1}{1-s}\,dW_s}_{Y_t}$

$f(X_t, Y_t) = X_t\cdot Y_t = X_0\cdot Y_0 + \int_0^t Y_s\,dX_s + \int_0^t X_s\,dY_s + \frac{1}{2}\int_0^t 2\cdot 1\cdot 0\,ds$

$f(x,y) = x\cdot y \quad \frac{\partial}{\partial x}f = y \quad \frac{\partial}{\partial y}f = x$

$\frac{\partial^2}{\partial x^2}f = \frac{\partial^2}{\partial y^2}f = 0 \quad \frac{\partial^2}{\partial x\partial y}f = 1$

$$= -\int_0^t Y_s \, ds + \int_0^t (1-s) \frac{1}{1-s} \, dW_s$$

$$= -\int_0^t \frac{B_s}{1-s} \, ds + W_t .$$

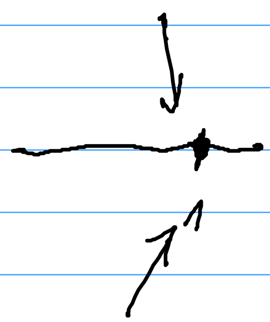
$$B_t = -\int_0^t \frac{B_s}{1-s} \, ds + W_t$$

$$dB_t = -\frac{B_t}{1-t} \, dt + dW_t .$$

$$B_t = (1-t) \cdot \int_0^t \frac{1}{1-s} \, dW_s$$

$$dB_t = -1 \cdot dt \int_0^t \frac{1}{1-s} \, dW_s + (1-t) \frac{1}{1-t} \, dW_t$$

$$= -\frac{B_t}{1-t} \, dt + dW_t .$$



$$B_t = (1-t)\cdot \int_0^t \frac{1}{1-s}\, dW_s \to \text{Gaussian process}$$

$$E(B_t) = E(B_0) = 0.$$

$\int_0^t \text{anything}\, dW_s$ mtg

$$Cov(B_t, B_s) = E(B_s B_t) =$$

$s < t$

$$= (1-t)(1-s) E\left(\int_0^s \frac{1}{1-u}\, dW_u \cdot \underbrace{\int_0^t \frac{1}{1-u}\, dW_u}_{\int_0^s + \int_s^t}\right)$$

independence

$$= (1-t)(1-s)\left(E\left[\left(\int_0^s \frac{1}{1-u}\, dW_u\right)^2\right] + \underbrace{E\left(\int_0^s \frac{1}{1-u}\, dW_u\right) E\left(\int_s^t \frac{1}{1-u}\, dW_u\right)}_{=0}\right)$$

$$= (1-t)(1-s)\, E\left[\left(\int_0^s \frac{1}{1-u}\, dW_u\right)^2\right]$$

$$= (1-t)(1-s) E \underbrace{\int_0^s \frac{1}{(1-u)^2}\, du}_{\left[\frac{1}{1-u}\right]_0^s = \frac{1}{1-s} - 1} = (1-t)(1-s)\left(\frac{1}{1-s} - 1\right) =$$

$$= 1 - t - (1-s)(1-t)$$

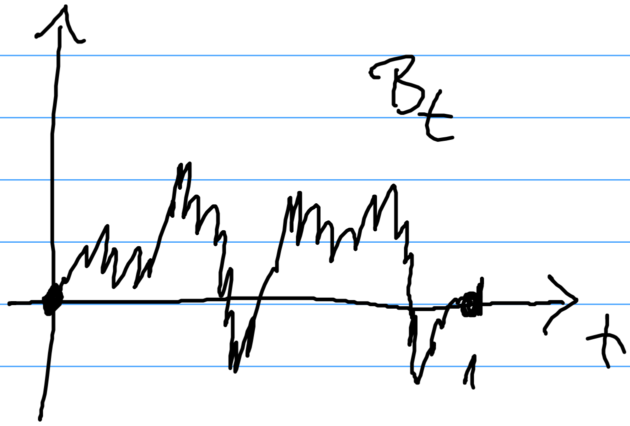
$$= \cancel{1} - \cancel{t} - \cancel{1} - st + s + \cancel{t}$$

$= s - st.$

$$\mathrm{Cov}(B_s, B_t) = (s \wedge t) - st.$$

$$E(B_1^2) = \mathrm{Cov}(B_1, B_1) = 0.$$

$B_1 = 0$ a.s.



Brownian bridge

Satz.: $U_1, U_2, \ldots$ iid Uniform$(0,1)$

$$\sqrt{n}\,\big(F_n(t) - t\big) \xrightarrow[n\to\infty]{} \text{Brownian bridge.}$$

36. $\begin{cases} dX_t = \underbrace{-\frac{1}{2} e^{-2X_t}} dt + \underbrace{e^{-X_t}} dW_t \\ X_0 = 0 \end{cases}$

$X_t = u(W_t)$. $\quad u \in C^2$

$$X_t = u(W_t) = u(W_0) + \int_0^t \boxed{u'(W_s)} dW_s +$$

$$+ \frac{1}{2} \int_0^t \underbrace{u''(W_s)} ds$$

$$u'(W_s) = e^{-X_s} = e^{-u(W_s)}$$

$\Rightarrow$ $\boxed{u'(x) = e^{-u(x)}}$ & $u(0) = 0$.

and $\frac{1}{2} u''(x) = -\frac{1}{2} e^{-2u(x)}$

then $X_t = u(W_t)$ is indeed a solution.

$$\begin{cases} u' = e^{-u} \\ u(0) = 0 \end{cases} \Bigg|$$

$$(e^u)' = e^u u' = 1$$

$$e^{u(t)} = t + C$$

$$u(t) = \lg(t+C)$$

$$u(0) = 0 \Rightarrow c = 1$$

$$u(t) = \lg(1+t)$$

the 2nd eq. ✓

—

$$X_t = \lg(1 + W_t)$$

the solution explodes when W_t hits -1

$$\tau = \inf\{t : W_t = -1\}$$

→ .