First we change η_k to $X_{t_{k-1}}$. Taking the difference

$$\sum_{k=1}^{m}[f''(\eta_k) - f''(X_{t_{k-1}})](M_{t_k} - M_{t_{k-1}})^2$$
$$\leq \sup_{1\leq k\leq m} |f''(\eta_k) - f''(X_{t_{k-1}})| \cdot \sum_{k=1}^{m}(M_{t_k} - M_{t_{k-1}})^2.$$

By the Cauchy–Schwarz inequality

$$\left|\mathbf{E}\sum_{k=1}^{m}[f''(\eta_k) - f''(X_{t_{k-1}})](M_{t_k} - M_{t_{k-1}})^2\right|$$
$$\leq \sqrt{\mathbf{E}\sup_{1\leq k\leq m}(f''(\eta_k) - f''(X_{t_{k-1}}))^2}\sqrt{\mathbf{E}\left(\sum_{k=1}^{m}(M_{t_k} - M_{t_{k-1}})^2\right)^2}. \tag{16}$$

{eq:i3-3}

The first term tends to 0 because (X_t) is continuous and f'' is bounded. The second is bounded by the following lemma.

{lemma:Ito-aux}

Lemma 7. *Let (M_t) be a continuous bounded martingale on $[0,t]$, that is $\sup_{s,\omega}|M_s(\omega)| \leq K$, and let $\Pi = \{0 = t_0 < t_1 < \ldots < t_m = t\}$ be a partition. Then*

$\alpha < \infty$

$$\mathbf{E}\left[\left(\sum_{i=1}^{m}(M_{t_i} - M_{t_{i-1}})^2\right)^2\right] \leq 6K^4.$$

$M_t = \int_0^t W_s dW_s$

Proof. Expanding the square

quadratic var.

$$\mathbf{E}\left[\left(\sum_{i=1}^{m}(M_{t_i} - M_{t_{i-1}})^2\right)^2\right]$$
$$= \sum_{i=1}^{m}\mathbf{E}(M_{t_i} - M_{t_{i-1}})^4 + \sum_{i\neq j}\mathbf{E}(M_{t_i} - M_{t_{i-1}})^2(M_{t_j} - M_{t_{j-1}})^2.$$

mixed terms.

Using several times that

$$\mathbf{E}[(M_t - M_s)^2|\mathcal{F}_s] = \mathbf{E}[M_t^2 - M_s^2|\mathcal{F}_s], \quad s < t,$$

$0 = t_0 < \ldots < t_n = t$

$$\sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})^2 \xrightarrow{L^2} t$$

$$E\left[(M_t - M_s)^2 \mid \mathcal{F}_s\right] = E\left[M_t^2 - M_s^2 \mid \mathcal{F}_s\right]$$

$$\Vert$$

$$E\left[M_t^2 \underbrace{- 2M_t M_s} + M_s^2 \mid \mathcal{F}_s\right]$$

$$E\left[M_t \cdot M_s \mid \mathcal{F}_s\right] = M_s \underbrace{E\left[M_t \mid \mathcal{F}_s\right]}_{M_s} = M_s^2$$

we obtain

$$\sum \text{mixed terms} = \sum_{i\neq j} \mathbf{E}\left[(M_{t_i} - M_{t_{i-1}})^2 (M_{t_j} - M_{t_{j-1}})^2\right]$$

$$= 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \mathbf{E}\left[(M_{t_i} - M_{t_{i-1}})^2 (M_{t_j} - M_{t_{j-1}})^2\right]$$

tower rule

$$= 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \mathbf{E}\left[\mathbf{E}[\underbrace{(M_{t_i} - M_{t_{i-1}})^2}_{\text{meas}} (M_{t_j} - M_{t_{j-1}})^2 | \mathcal{F}_{t_{j-1}}]\right] = 2\sum\sum \mathbf{E}\left[(M_{t_i} - M_{t_{i-1}})^2 \cdot \underbrace{\mathbf{E}[(M_{t_j} - M_{t_{j-1}})^2 | \mathcal{F}_{t_{j-1}}]}\right]$$

$$= \mathbf{E}\left[M_{t_j}^2 - M_{t_{j-1}}^2 | \mathcal{F}_{t_{j-1}}\right]$$

$$= 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \mathbf{E}\left[(M_{t_i} - M_{t_{i-1}})^2 (M_{t_j}^2 - M_{t_{j-1}}^2)\right]$$

$$\sum_{j=i+1}^{m} (M_{t_j}^2 - M_{t_{j-1}}^2) = \; = M_t^2 - M_{t_i}^2$$

$$= 2\sum_{i=1}^{m-1} \mathbf{E}\left[(M_{t_i} - M_{t_{i-1}})^2 (M_t^2 - M_{t_i}^2)\right]$$

$$\le 2K^2 \sum_{i=1}^{m-1} \underbrace{\mathbf{E}(M_{t_i} - M_{t_{i-1}})^2}_{=}$$

boundedness

$$= 2K^2 \sum_{i=1}^{m-1} \mathbf{E}(M_{t_i}^2 - M_{t_{i-1}}^2) \le 2K^4. = 2K^2 \mathbf{E}(M_t^2 - M_0^2)$$

While, for the sum of 4th powers

$$\sum_{i=1}^{m} \mathbf{E}(M_{t_i} - M_{t_{i-1}})^4 \le 4K^2 \cancel{\mathbf{E}} \sum_{i=1}^{m} \mathbf{E}(M_{t_i} - M_{t_{i-1}})^2 = 4K^2 \sum \mathbf{E}(M_{t_i}^2 - M_{t_{i-1}}^2)$$

$$= 4K^2 \mathbf{E}(M_t^2 - M_0^2) \le 4K^4.$$

telescopic sum

□

Summarizing from I_3 we have the sum

$$\sum_{k=1}^{m} f''(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})^2.$$

We claim that

$$\sum_{k=1}^{m} f''(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})^2 \xrightarrow{L^1} \int_0^t \underbrace{f''(X_s)}_{1} H_s^2 \mathrm{d}s. \qquad (17) \quad \texttt{\{eq:i3-negyzetesva}$$

$$\text{|||} \quad 1$$

$$M_t = \int_0^t H_s dW_s$$

Since X and f'' are continuous

$$\sum_{k=1}^m f''(X_{t_{k-1}}) \int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \to \int_0^t f''(X_s) H_s^2 \mathrm{d}s \quad \text{a.s.}$$

Thus it is enough to show that

$$\sum_{k=1}^m f''(X_{t_{k-1}}) \left((M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \right) \xrightarrow{L^2} 0.$$

Theorem 25 (ii) implies

$$\begin{aligned}
\mathbf{E}\left[(M_{t_k} - M_{t_{k-1}})^2 | \mathcal{F}_{t_{k-1}}\right] &= \mathbf{E}\left[\left(\int_{t_{k-1}}^{t_k} H_s \, \mathrm{d}W_s \right)^2 | \mathcal{F}_{t_{k-1}} \right] \\
&= \mathbf{E}\left[\int_{t_{k-1}}^{t_k} H_s^2 \, \mathrm{d}s | \mathcal{F}_{t_{k-1}} \right],
\end{aligned}$$

so in

$$\mathbf{E}\left(\sum_{k=1}^m f''(X_{t_{k-1}}) \left((M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \right) \right)^2$$

the expectation of the mixed term is 0. Thus this equals

$$\begin{aligned}
&= \mathbf{E} \sum_{k=1}^m f''(X_{t_{k-1}})^2 \left((M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \right)^2 \\
&\le \|f\|_\infty^2 \left[\mathbf{E} \sum_{k=1}^m (M_{t_k} - M_{t_{k-1}})^4 + 2\mathbf{E} \sum_{k=1}^m (M_{t_k} - M_{t_{k-1}})^2 \int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \right. \\
&\qquad \left. + \mathbf{E} \sum_{k=1}^m \left(\int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \right)^2 \right] \\
&\le \|f\|_\infty^2 \left[\mathbf{E} \sum_{k=1}^m (M_{t_k} - M_{t_{k-1}})^4 + 2K^2 t \mathbf{E} \sup_{1 \le k \le m} (M_{t_k} - M_{t_{k-1}})^2 + K^4 t \|\Pi\| \right].
\end{aligned}$$

The second and third term tend to 0, and for the first

$$\begin{aligned}
\mathbf{E}\sum_{k=1}^{m}(M_{t_k}-M_{t_{k-1}})^4 &\le \mathbf{E}\left[\sum_{k=1}^{m}(M_{t_k}-M_{t_{k-1}})^2\cdot \sup_{1\le k\le m}|M_{t_k}-M_{t_{k-1}}|^2\right]\\
&\le \sqrt{\mathbf{E}\left[\sum_{k=1}^{m}(M_{t_k}-M_{t_{k-1}})^2\right]^2}\sqrt{\mathbf{E}\sup_{1\le k\le m}|M_{t_k}-M_{t_{k-1}}|^4}\\
&\le \sqrt{6}K^2\sqrt{\mathbf{E}\sup_{1\le k\le m}|M_{t_k}-M_{t_{k-1}}|^4}\to 0.
\end{aligned}$$

Summarizing we obtained L^1, L^2 and almost sure convergence in (12)–(17). Since everything is bounded, L^1 convergence follows in each case, that is

$$\begin{aligned}
f(X_t)-f(X_0) &= \sum_{k=1}^{m}[f(X_{t_k})-f(X_{t_{k-1}})]\\
&\xrightarrow{L^1}\int_0^t f'(X_s)\mathrm{d}X_s+\frac{1}{2}\int_0^t f''(X_s)H_s^2\mathrm{d}s.
\end{aligned}$$

Convergence in L^1 implies a.s. convergence on a subsequence. As both sides are continuous we obtained that the two process are indistinguishable. □

{example:exp-2}

Example 12 (Continuation of Example 10). Let (X_t) adapted

$s<t$ $$\zeta_t^s=\int_s^t X_u\mathrm{d}W_u-\frac{1}{2}\int_s^t X_u^2\mathrm{d}u,\quad \zeta_t=\zeta_t^0,$$

where X_t is an adapted process. Then $\boxed{Z_t=e^{\zeta_t}}$ satisfies the stochastic differential equation

$$\boxed{Z_t=1+\int_0^t Z_sX_s\mathrm{d}W_s,}$$

or with a common notation

$$\Leftrightarrow\quad \mathrm{d}Z_t=Z_tX_t\mathrm{d}W_t,\quad \underbrace{Z_0=1}_{\text{initial cond.}}.$$

Writing ζ as an Itô process

$$\zeta_t=\underbrace{\int_0^t-\frac{1}{2}X_u^2\mathrm{d}u}_{\text{bounded var.}}+\underbrace{\int_0^t X_u\mathrm{d}W_u}_{\text{mtg.}}.$$

$X_u\equiv 1$

$Z_t=e^{\zeta_t}$

(X_u) simple

Z_t ✓

$Z_t=f(\zeta_t)\qquad f(x)=e^x$

$f'(x) = f''(x) = e^x$

$f(\zeta_t) = f(\zeta_0) + \int_0^t f'(\zeta_s)\, d\zeta_s + \frac{1}{2}\int_0^t f''(\zeta_s) X_s^2 ds$

Using Itô's formula with $f(x) = e^x$

$$\begin{aligned}
Z_t = e^{\zeta_t} &= 1 + \int_0^t e^{\zeta_s} \mathrm{d}\zeta_s + \frac{1}{2}\int_0^t e^{\zeta_s} X_s^2 \mathrm{d}s \\
&= 1 + \int_0^t e^{\zeta_s} \left(-\frac{1}{2} X_s^2 \mathrm{d}s + X_s \mathrm{d}W_s \right) + \frac{1}{2}\int_0^t e^{\zeta_s} X_s^2 \mathrm{d}s \\
&= 1 + \int_0^t e^{\zeta_s} X_s \mathrm{d}W_s \\
&= 1 + \int_0^t Z_s X_s \mathrm{d}W_s,
\end{aligned}$$

$\int_0^t$ anything dW_s mtg.

as claimed. We see that Z_t is martingale.

Exercise 28. Let ζ_t be as above. Show that $Y_t = e^{-\zeta_t}$ satisfies the SDE

$$\mathrm{d}Y_t = Y_t X_t^2 \mathrm{d}t - X_t Y_t \mathrm{d}W_t, \quad Y_0 = 1.$$

Similarly, one can show a more general version, where f depends on the time variable t.

$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$

Theorem 27 (More general Itô formula). *Let X_t be an Itô process and $f \in C^{1,2}$. Then*

$$\begin{aligned}
f(t, X_t) = f(0, X_0) &+ \int_0^t \frac{\partial}{\partial s} f(s, X_s) \mathrm{d}s + \int_0^t \frac{\partial}{\partial x} f(s, X_s) \mathrm{d}X_s \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} f(s, X_s) H_s^2 \mathrm{d}s.
\end{aligned}$$

4.4 Multidimensional Itô processes

Let $W = (W^1, W^2, \ldots, W^r)$ be an r-dimensional SBM, that is its component are iid SBM's. Then (X_t) is a *d-dimensional Itô process*, if

d and r can be different!!

$$X_t^i = X_0^i + \int_0^t K_s^i \mathrm{d}s + \sum_{j=1}^r \int_0^t H_s^{i,j} \mathrm{d}W_s^j, \qquad (18) \quad \{eq:multid-ito\}$$

where $\int_0^T |K_s^i| \mathrm{d}s < \infty$, $\int_0^T (H_s^{i,j})^2 \mathrm{d}s < \infty$ a.s., and $K^i, H^{i,j}$ are $\mathcal{F}_t$-adapted, $i = 1, 2, \ldots, d$, $j = 1, 2, \ldots, r$.

$$X_t^i = X_0^i + \int_0^t K_s^i \,ds + \sum_{j=1}^{r} \int_0^t H_s^{ij} \,dW_s^{(j)}$$

Theorem 28 (Multidimensional Itô formula). *Let (X_t) be a multidimensional Itô process and $f : \mathbb{R}^{1+d} \to \mathbb{R}$, $f \in C^{1,2}$. Then*

$$\begin{aligned} f(\underbrace{t}_{C^1}, \underbrace{X_t^1, \ldots, X_t^d}_{C^2}) &= f(0, X_0^1, \ldots, X_0^d) + \int_0^t \frac{\partial}{\partial s} f(s, X_s^1, \ldots, X_s^d) \,\mathrm{d}s \quad \text{time} \\ &+ \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} f(s, X_s^1, \ldots, X_s^d) \,\mathrm{d}X_s^i \Leftarrow \text{space} \\ &+ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s, X_s^1, \ldots, X_s^d) \sum_{k=1}^{r} H_s^{i,k} H_s^{j,k} \,\mathrm{d}s. \end{aligned}$$

4.5 Applications

Example 13 (Integration by parts I). Let (X, Y) be a two-dimensional Itô process with representation

$$\begin{aligned} X_t &= X_0 + \int_0^t K_s \,\mathrm{d}s + \int_0^t H_s \,\mathrm{d}W_s \\ Y_t &= Y_0 + \int_0^t L_s \,\mathrm{d}s + \int_0^t G_s \,\mathrm{d}W_s, \end{aligned}$$

$\underbrace{r=1}_{\text{number of diff. BM}}$ $\underbrace{d=2}_{\text{dimension of the Itô proc.}}$

where K, L, H, G are as usual. Then

$$\int_0^t X_s \mathrm{d}Y_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s \mathrm{d}X_s - \underbrace{\int_0^t H_s G_s \mathrm{d}s}_{\text{extra term}}.$$

$[X_s Y_s]_{s=0}^{t}$

Note that in the deterministic integration by parts formula the last term is missing.

For the proof apply Itô's formula for (X, Y) and $f(x, y) = xy$. Then

$$r = 1,\ d = 2,\ K_s^1 = K_s,\ K_s^2 = L_s,\ H_s^{1,1} = H_s,\ H_s^{2,1} = G_s.$$

Since $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$, $\frac{\partial^2 f}{\partial^2 x} = \frac{\partial^2 f}{\partial^2 y} = 0$, and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 1$, we obtain

$$X_t Y_t = X_0 Y_0 + \int_0^t Y_s \mathrm{d}X_s + \int_0^t X_s \mathrm{d}Y_s + \frac{1}{2} 2 \int_0^t H_s G_s \mathrm{d}s,$$

$$f(X_t, Y_t) = f(X_0, Y_0) + \int_0^t \frac{\partial}{\partial x} f(X_s, Y_s)\, dX_s + \int_0^t \frac{\partial}{\partial y} f(X_s, Y_s)\, dY_s + \frac{1}{2} 2 \int_0^t H_s G_s \,ds.$$

as claimed.

Integration by parts:

$$F(s)' = f(s) \qquad G'(s) = g(s)$$

$$\int_0^t G(s)\, f(s)\, ds = \left[F(s)\, G(s)\right]_{s=0}^{t} - \int_0^t g(s) F(s)\, ds$$

$$f(s)\,ds = dF(s) \qquad g(s)\,ds = dG(s)$$

$$\int_0^t G(s)\, dF(s) = \left[F(s)\, G(s)\right]_{s=0}^{t} - \int_0^t F(s)\, dG(s)$$

(integration by parts for Lebesgue-Stieltjes ints

Example 14 (Integration by parts II)**.** To change a bit let $\widetilde{W}$ be another SBM independent of W and (X, Y)

$$X_t = X_0 + \int_0^t K_s \, ds + \int_0^t H_s \, dW_s \qquad + \int_0^t \tilde{H}_s \, d\tilde{W}_s$$

$$Y_t = Y_0 + \int_0^t L_s \, ds + \int_0^t G_s \, d\widetilde{W}_s.$$

Then

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s.$$

The proof is the same but here $d = r = 2$, and no extra term appears.

{pelda:exp-BM}

Example 15 (Geometric Brownian motion)**.** Let $\mu \in \mathbb{R}$, $\sigma > 0$. Solve the SDE

$$dX_t = \mu X_t dt + \sigma X_t dW_t. \tag{19}$$ {eq:exp-BM-sde}

We have

$$X_t = X_0 + \cancel{\int_0^t \mu X_s ds} + \int_0^t \sigma X_s dW_s.$$

Applying Itô's formula with $f(x) = \log x$

$$\begin{aligned}\log X_t &= \log X_0 + \int_0^t \frac{1}{X_s} \left(\mu X_s ds + \sigma X_s dW_s\right) + \frac{1}{2} \int_0^t -\frac{1}{X_s^2} \sigma^2 X_s^2 ds \\ &= \log X_0 + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t.\end{aligned}$$

Thus

$$X_t = X_0 \cdot e^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t}. \tag{20}$$ {eq:exp-BM}

This is martingale iff $\mu = 0$.

Note that $\log x$ is not defined at 0, so the proof is not complete. It only gives us a potential solution.

Exercise 29. Show that X_t in (20) is indeed a solution to the SDE (19). time dependent Itô

A more constructive solution is to apply Itô's formula with a general f, and then choose f to obtain a simple equation. With $f(x) = \log x$ the integrand in the martingale part is constant.

Exercise 30. Show that $Y(t) = e^{t/2} \cos W_t$ is martingale.

time dependent

SDE: $dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$

$\rightarrow \quad X_t = X_0 + \int_0^t \mu X_s \, ds + \int_0^t \sigma X_s \, dW_s.$

$f \in C^2$

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \, dX_s + \frac{1}{2} \int_0^t f''(X_s)(\sigma X_s)^2 ds$$

$$= f(X_0) + \int_0^t f'(X_s)\left(\mu X_s \, ds + \sigma X_s \, dW_s\right)$$

$$+ \frac{1}{2} \int_0^t f''(X_s) \, \sigma^2 X_s^2 \, ds =$$

$$f(X_t) = f(X_0) + \int_0^t \left(f'(X_s) \mu X_s + \frac{1}{2} f''(X_s) \sigma^2 X_s^2\right) ds$$

$$+ \int_0^t \sigma f'(X_s) X_s \, dW_s$$

$f(x) = \log x \qquad f'(x) = \frac{1}{x} \qquad f'(x) x = 1.$

$$\int_0^t \sigma \, dW_s = \sigma W_t$$

$$f(x) = \log x$$

$$f'(x) = \frac{1}{x} \qquad f''(x) = -\frac{1}{x^2}$$

$$\log X_t = \log X_0 + \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) ds + \sigma W_t$$

$$= \log X_0 + t \cdot \left(\mu - \frac{\sigma^2}{2}\right) + \sigma W_t .$$

$$\left[\, X_t = X_0 \cdot \exp\left\{ t \cdot \left(\mu - \frac{\sigma^2}{2}\right) + \sigma W_t \right\} \right.$$

$$\log x = f(x) \qquad f \in C^2$$

Exercise 31. Show that

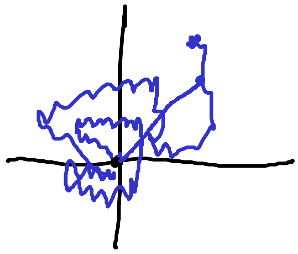
$$\int_0^t W_s^2 \mathrm{d}W_s = \frac{1}{3}W_t^3 - \int_0^t W_s \mathrm{d}s,$$

and

$$\int_0^t W_s^3 \mathrm{d}W_s = \frac{1}{4}W_t^4 - \frac{3}{2}\int_0^t W_s^2 \mathrm{d}s.$$

Exercise 32. Let $\mathbf{W} = (W^1, \ldots, W^r)$ be an r-dimensional SBM, $r \geq 2$, and let

$$R_t = \sqrt{\sum_{i=1}^r (W_t^i)^2}.$$



Show that R satisfies the SDE

$$\mathrm{d}R_t = \frac{r-1}{2R_t}\mathrm{d}t + \sum_{i=1}^r \frac{W_t^i}{R_t}\mathrm{d}W_t^i.$$

This is the Bessel equation and R is the *Bessel process*.

4.6 Quadratic variation and the Doob–Meyer decomposition

We proved that

$$\mathbf{E}\left[\left(\int_s^t X_u \mathrm{d}W_u\right)^2 |\mathcal{F}_s\right] = \mathbf{E}\left[\int_s^t X_u^2 \,\mathrm{d}u|\mathcal{F}_s\right],$$

which means that the process

$$\left(\int_0^t X_u \,\mathrm{d}W_u\right)^2 - \int_0^t X_u^2 \,\mathrm{d}u \qquad (21) \quad \texttt{\{eq:doob-meyer\}}$$

mtg

is a continuous martingale. In the decomposition

$$\text{submtg:}\quad \left(\int_0^t X_u \mathrm{d}W_u\right)^2 = \int_0^t \underbrace{X_u^2}_{\geq 0} \,\mathrm{d}u + \overbrace{\left(\int_0^t X_u \mathrm{d}W_u\right)^2 - \int_0^t X_u^2 \,\mathrm{d}u}^{\text{mtg}}$$

the first term is an increasing process and the second term is a martingale, that is we obtained the Doob–Meyer decomposition of $I_t(X)^2$.

$$E\left[\left(\int_0^t X_u dW_u\right)^2 - \int_0^t X_u^2 du \,\Big|\, \mathcal{F}_s\right] =$$

$$= E\Big[\Big(\int_0^s + \int_s^t\Big)^2 - \int_0^s - \int_s^t \,\Big|\, \mathcal{F}_s\Big]$$

$\mathcal{F}_s$- meas.

$$= \Big(\int_0^s X_u\,dW_u\Big)^2 - \int_0^s X_u^2\,du +$$

$$+ E\Big[\Big(\int_s^t X_u\,dW_u\Big)^2 - \int_s^t X_u^2\,du \,\Big|\, \mathcal{F}_s\Big] = 0$$

$$+ E\Big[\int_0^s \dots dW_u \cdot \int_s^t \dots dW_u \,\Big|\, \mathcal{F}_s\Big] = 0$$

$$E\Big[\int_0^t - \int_0^s \,\Big|\, \mathcal{F}_s\Big] = 0.$$

$$= \Big(\int_0^s X_u\,dW_u\Big)^2 - \int_0^s X_u^2\,du,$$ so it is mtg.

<=> =

Doob-Meyer dec.

(Y_t) submtg. $\quad E[Y_t \mid \mathcal{F}_s] \geq Y_s$

$$Y_t = \underbrace{M_t}_{\text{mtg.}} + \underbrace{A_t}_{\text{increasing}}$$

$\sum' (M_{t_i} - M_{t_{i-1}})^2$

On the other hand, at the proof of Itô's formula we showed (see (17)) that

$$\sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i} X_u \mathrm{d}W_u \right)^2 \xrightarrow{L^1} \int_0^t X_u^2 \mathrm{d}u, \quad \text{as } \|\Pi_n\| \to 0.$$

The left-hand side is exactly the *quadratic variation process* of the martingale $I_t(X)$.

Summarizing, we proved the following.

$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$

{thm:quad-DM}

Theorem 29. *For any Itô process X_t, the quadratic variation of $I_t(X)$ and the increasing process in the Doob–Meyer decomposition of $I_t(X)^2$ are the same.*

This result holds in a more general setup.

Let (X_t) be a (continuous) square integrable martingale, $X \in \mathcal{M}_2$ (or $X \in \mathcal{M}_2^c$). Then X_t^2 is a submartingale, so by the Doob–Meyer decomposition there exists a unique (up to indistinguishibility) adapted increasing process A_t, such that $A_0 = 0$ a.s. and $X_t^2 - A_t$ is a martingale. The process $\langle X \rangle_t = A_t$ is the *quadratic variation of X*.

$\langle . \rangle$

With this notation, Theorem 29 states that

$$\left\langle \int_0^\cdot X_u \mathrm{d}W_u \right\rangle_t = \langle I(X) \rangle_t = \int_0^t X_u^2 \, \mathrm{d}u.$$

Without proof we mention that Theorem 29 holds not only for Itô processes but for *continuous square integrable martingales.*

Theorem 30. *Let $X \in \mathcal{M}_2^c$. For partition Π of $[0,t]$ we have*

$$V_t^{(2)}(\Pi) := \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2 \xrightarrow{\mathbf{P}} \langle X \rangle_t \quad \textit{as} \quad \|\Pi\| \to 0.$$

For square integrable martingales X, Y the *crossvariation process of X and Y* is

$$\langle X, Y \rangle_t = \frac{1}{4} \left(\langle X + Y \rangle_t - \langle X - Y \rangle_t \right).$$

← polarization parallelogram rule

The processes X and Y are *orthogonal* if $\langle X, Y \rangle_t = 0$ a.s. for any t.

Exercise 33. Show that if $X, Y \in \mathcal{M}_2$, then $XY - \langle X, Y \rangle$ is a martingale.

$\int \dots dA_t$ pathwise because we can define LS integral

One can define stochastic integral with respect to more general processes. The process (X_t) is a continuous *semimartingale* if

$$X_t = M_t + A_t,$$

where M_t is a continuous martingale and A_t is of bounded variation, and both are adapted. As in Lemma 6 it can be shown that this decomposition is essentially unique.

W_s

$\int_0^t X_s \, dM_s$

simple proc.

↓

extension

We can define stochastic integral with respect to semimartingales. Indeed, integral with respect to A_t can be defined pathwise, since A is of bounded variation, and integration with respect to continuous M_t can be defined similarly as for SBM.

The following version of Itô's formula holds.

Theorem 31 (Itô formula for semimartingales)**.** *Let $X_t = M_t + A_t$ be a continuous semimartingale, and let $f \in C^2$. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \mathrm{d}X_s + \frac{1}{2} \int_0^t f''(X_s) \underbrace{\mathrm{d}\langle M \rangle_s}_{H_s^2 ds}$$

$\int_0^t H_s \, dW_s$

Itô

$\langle M \rangle_t = \int_0^t H_s^2 \, ds$

5 Stochastic differential equations

We define the strong solution of SDEs and obtain existence and uniqueness results.

The followings are given:

- probability space $(\Omega, \mathcal{A}, \mathbf{P})$;
- with a filtration $(\mathcal{F}_t)_{t \in [0,T]}$;
- a d-dimensional SBM $W_t = (W_t^1, \dots, W_t^r)$ with respect to the filtration $(\mathcal{F}_t)$;
- measurable functions $f : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{d \times r}$;
- $\mathcal{F}_0$-measurable rv $\xi : \Omega \to \mathbb{R}^d$.

The (d-dimensional) process (X_t) is *strong solution to the SDE*

$$\begin{aligned} \mathrm{d}X_t &= f(X_t, t) \, \mathrm{d}t + \sigma(X_t, t) \, \mathrm{d}W_t, \\ X_0 &= \xi, \end{aligned} \tag{22}$$

{eq:sde}