Next we prove (9). Since

$$\varepsilon W_{t_{i+1}} + (1-\varepsilon) W_{t_i} = \frac{W_{t_{i+1}} + W_{t_i}}{2} + \left(\varepsilon - \frac{1}{2}\right)\left(W_{t_{i+1}} - W_{t_i}\right),$$

we have to determine the limits

$$\sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2, \quad \sum_{i=0}^{n-1} (W_{t_{i+1}}^2 - W_{t_i}^2).$$

The first is exactly the quadratic variation of SBM, therefore converges to t in L^2, while the second is a telescopic sum, giving W_t^2.

{example:exp}

Example 10. Let X be simple process and W SBM. Let

$$\zeta_t^s(X) = \int_s^t X_u \mathrm{d}W_u - \frac{1}{2}\int_s^t X_u^2 \mathrm{d}u, \quad \zeta_t = \zeta_t^0.$$

special case: $X_u \equiv \lambda$

We show that $(Y_t = e^{\zeta_t})$ is martingale.

Since X is simple, we have

$\zeta_t = \lambda W_t - \frac{1}{2}\lambda^2 t$

$$X_t = \xi_0 \mathbf{I}_{\{0\}}(t) + \sum_{i=0}^{n-1} \xi_i \mathbf{I}_{(t_i, t_{i+1}]}(t),$$

$s < t$

$(e^{\lambda W_t - \frac{1}{2}\lambda^2 t})$

where ξ_i is $\mathcal{F}_{t_i}$-measurable. Thus if $s \in (t_k, t_{k+1}]$, $t \in (t_m, t_{m+1}]$, then

1st term

mtg.

s t_{s+1} t_m t t_{m+1}

$$\begin{aligned}\zeta_t^s &= \underbrace{\xi_k (W_{t_{k+1}} - W_s) - \frac{\xi_k^2}{2}(t_{k+1} - s) + \sum_{i=k+1}^{m-1}\left[\xi_i (W_{t_{i+1}} - W_{t_i}) - \frac{\xi_i^2}{2}(t_{i+1} - t_i)\right]}_{\mathcal{F}_{t_m}\text{-meas.}} \\ &\quad + \xi_m (W_t - W_{t_m}) - \frac{\xi_m^2}{2}(t - t_m).\end{aligned} \tag{10}$$

{eq:zeta-felbontas}

Since ζ_s is $\mathcal{F}_s$-measurable we obtain

$$\mathbf{E}[e^{\zeta_t} | \mathcal{F}_s] = e^{\zeta_s} \mathbf{E}[e^{\zeta_t^s} | \mathcal{F}_s].$$

$\zeta_t = \zeta_t^s + \underbrace{\zeta_s^0}$

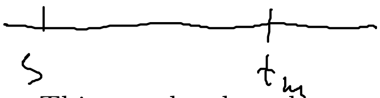
We only have to show that

$$\boxed{\mathbf{E}[e^{\zeta_t^s} | \mathcal{F}_s] = 1.}$$

measurable with respect to $\mathcal{F}_s$

$E[e^{\zeta_t} | \mathcal{F}_s] = e^{\zeta_s}$

$$E\left[e^{\xi_t^s} \,|\, \mathcal{F}_s\right] \overset{\text{tower rule}}{=} E\left[E\left[e^{\xi_t^s} \,|\, \mathcal{F}_{t_m}\right] \,|\, \mathcal{F}_s\right]$$



$$= E\left[e^{\xi_{t_m}^s} \cdot E\left[e^{\xi_m(W_t - W_{t_m}) - \frac{1}{2}\xi_m^2(t-t_m)} \,|\, \mathcal{F}_{t_m}\right] \,|\, \mathcal{F}_s\right]$$

This can be done by a repeated application of the tower rule. In (10) all terms but the last are $\mathcal{F}_{t_m}$-measurable and

$\mathcal{F}_{t_m}$-meas.

$$\mathbf{E}\left[\exp\left\{\xi_m(W_t - W_{t_m}) - \frac{\xi_m^2}{2}(t - t_m)\right\} |\mathcal{F}_{t_m}\right]$$
$$= e^{-\frac{\xi_m^2}{2}(t-t_m)}\mathbf{E}\left[\exp\{\xi_m(W_t - W_{t_m})\}|\mathcal{F}_{t_m}\right].$$

independent $\sim N(0, t-t_m)$

meas. $\mathcal{F}_{t_m}$

In the exponent of the RHS ξ_m is $\mathcal{F}_{t_m}$-measurable and $W_t - W_{t_m}$ is independent of $\mathcal{F}_{t_m}$, therefore (by the next exercise) ξ_m can be handled as a constant. We have

$$\mathbf{E}e^{\lambda Z} = e^{\frac{\lambda^2}{2}}, \qquad Z \sim N(0,1)$$

therefore

$$\mathbf{E}\left[\exp\{\xi_m(W_t - W_{t_m})\}|\mathcal{F}_{t_m}\right] = e^{\frac{\xi_m^2}{2}(t-t_m)}.$$

$$E\left[e^{\xi_m(W_t - W_{t_m})} \,|\, \mathcal{F}_{t_m}\right] = E\left[e^{\xi_m\sqrt{t-t_m}\,Z} \,|\, \mathcal{F}_{t_m}\right]$$

Summarizing

$$\mathbf{E}\left[\exp\{\xi_m(W_t - W_{t_m}) - \frac{\xi_m^2}{2}(t - t_m)\}|\mathcal{F}_{t_m}\right] = 1.$$

Applying repeatedly the tower rule first to the σ-algebra $\mathcal{F}_{t_{m-1}}$, then to $\mathcal{F}_{t_{m-2}}$, ..., we obtain that each factor equals 1.

Z indep. of $\mathcal{F}_{t_m}$

Using the Itô formula we show that Y is martingale for more general processes and it satisfies a certain stochastic differential equation.

Exercise 27. Let X, Y be random variables, X is $\mathcal{G}$-measurable, and Y is independent of $\mathcal{G}$. Then

$$\mathbf{E}[h(X,Y)|\mathcal{G}] = \int h(X,y)\mathrm{d}F(y),$$

where $F(y) = \mathbf{P}(Y \le y)$ is the distribution function of Y.

4.3 Itô's formula

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $(\mathcal{F}_t)$ a filtration, and (W_t) SBM for this filtration. Then (X_t) *is Itô process* if

$$X_t = X_0 + \underbrace{\int_0^t K_s \mathrm{d}s}_{\text{bounded var.}} + \underbrace{\int_0^t H_s \mathrm{d}W_s}_{\text{mtg.}}, \qquad (11) \quad \texttt{\{eq:ito-proc\}}$$

where

K_s, H_s adapted

- X_0 $\mathcal{F}_0$-measurable;
- K, H are $\mathcal{F}_t$-adapted processes;
- $\int_0^T |K_u| \mathrm{d}u < \infty$, $\int_0^T H_s^2 \mathrm{d}s < \infty$ a.s.

The part $\int_0^t K_s \mathrm{d}s$ is the bounded variation part of the process, while $\int_0^t H_s \mathrm{d}W_s$ is the martingale part.

{lemma:korlatosval

Lemma 6. *If $M_t = \int_0^t K_s \mathrm{d}s$ is a continuous martingale and $\int_0^T |K_s| \mathrm{d}s < \infty$ almost surely then $M_t \equiv 0$.*

Proof. Assume that $\int_0^T |K_s| \mathrm{d}s \le C$ for some $C < \infty$. Then for a sequence of partitions ($\Pi_n = \{0 = t_0 < t_1 < \ldots < t_n = T\}$) of $[0, T]$

$$\mathbf{E}\sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2 \le \mathbf{E}\left(\sup_{0 \le i \le n-1} |M_{t_{i+1}} - M_{t_i}| \int_0^T |K_s| \mathrm{d}s\right) \quad \sum_{i=0}^{n-1} \left| \int_{t_i}^{t_{i+1}} K_s ds \right|$$
$$\le C\mathbf{E} \sup_{0 \le i \le n-1} |M_{t_{i+1}} - M_{t_i}| \to 0,$$

as $\|\Pi_n\| \to 0$. We used that continuous function is uniformly continuous on compacts and Lebesgue's dominated convergence can be used because of the boundedness.

Furthermore, $E(M_t^2 + M_s^2 - 2 M_t M_s)$ $\quad s < t$

$$\mathbf{E}\left[(M_t - M_s)^2\right] = \mathbf{E}M_t^2 + \mathbf{E}M_s^2 - 2\mathbf{E}\left(\mathbf{E}[M_t M_s | \mathcal{F}_s]\right)$$
$$= \mathbf{E}M_t^2 - \mathbf{E}M_s^2,$$

$E[M_t | \mathcal{F}_s] = M_s$

mtg. prop.

for $s < t$, thus

$$0 \xleftarrow[\|\Pi\| \to 0]{n \to \infty} \mathbf{E}\sum_{i=0}^{n-1}(M_{t_{i+1}} - M_{t_i})^2 = \mathbf{E}(M_t^2 - M_0^2) = \mathbf{E}M_t^2.$$

$$= \sum_{i=0}^{n-1} \left(E(M^2_{t_{i+1}}) - E(M^2_{t_i}) \right)$$

Therefore $\mathbf{E}M_t^2 = 0$ for all t, and the statement follows. ∎

$\Rightarrow \boxed{M_t = 0}$

Corollary 7. *Representation* (11) *is unique.*

Proof. Indeed, if

$$\int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s = \int_0^t L_s \mathrm{d}s + \int_0^t G_s \mathrm{d}W_s,$$

$\forall t > 0$
$t \in [0, T]$

Cont. mtg & quadratic var. $\to 0 \Rightarrow$ trivial then

$$\int_0^t (K_s - L_s)\mathrm{d}s = \overbrace{\int_0^t (G_s - H_s)\mathrm{d}W_s}^{\text{mtg}}.$$

Lemma $\Rightarrow$ $K = L$

$G = H$.

The RHS is a continuous martingale, therefore by the previous lemma it has to be contant 0. $\square$

In what follows we use the *notation* $X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s$

$$\mathrm{d}X_t = K_t\mathrm{d}t + H_t\mathrm{d}W_t.$$

Theorem 26 (Itô formula (1944))**.** *Let* $X_t = X_0 + \int_0^t K_s\mathrm{d}s + \int_0^t H_s\mathrm{d}W_s$ *be an Itô process and* $f \in C^2$*. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\overbrace{\mathrm{d}X_s}^{K_s ds + H_s dW_s} + \frac{1}{2}\int_0^t f''(X_s)H_s^2\mathrm{d}s.$$

That is $(f(X_t))$ is an Itô process too, with representation (11)

$$f(X_t) = f(X_0) + \underbrace{\int_0^t \left(f'(X_s)K_s + \frac{1}{2}f''(X_s)H_s^2\right)\mathrm{d}s}_{\text{bounded variation}} + \overbrace{\int_0^t f'(X_s)H_s\mathrm{d}W_s}^{\text{mtg part.}}.$$

Example 11. We already calculated the stochastic integral $\int W_s\mathrm{d}W_s$ in Example 9. Now we determine it again.

The SBM as an Itô process can be represented with $K_s \equiv 0$, $H_s \equiv 1$. Let $f(x) = x^2$. Then

$$W_t^2 = W_0^2 + \int_0^t 2W_s\mathrm{d}W_s + \frac{1}{2}\int_0^t 2\mathrm{d}s.$$

From this we obtain

$$\int_0^t W_s\mathrm{d}W_s = \frac{W_t^2 - t}{2}.$$

We see immediately that $W_t^2 - t$ is martingale.

Proof. We only prove under the following extra assumptions: f is compactly supported; $\sup_{s,\omega}|K_s(\omega)| < K$, $\sup_{s,\omega}|H_s(\omega)| < K$ for some $K < \infty$. (This is not an essential restriction.)

$$X_t = X_0 + \int_0^t K_s\,ds + \int_0^t H_s\,dW_s$$

$$dX_t = K_t\,dt + H_t\,dW_t$$

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,H_s^2\,ds$$

Itô

$$d f(X_t) = \underbrace{f'(X_t)\,dX_t}_{\text{usual chain rule}} + \frac{1}{2} f''(X_t)\cdot H_t^2\,dt.$$

– o –

$$\int_0^t W_s\,dW_s \qquad \Bigg| \qquad X_t = 0 + \int_0^t \overset{K}{0}\,ds + \int_0^t \overset{H}{1}\,dW_s = W_t$$

$$X_t = W_t \qquad f(x) = x^2 \qquad f'(x) = 2x$$

$$f(X_t) = W_t^2 = f(X_0) + \int_0^t f'(X_s)\,dX_s + \frac{1}{2}\int_0^t f''(X_s)\,\overset{1^2}{\,}\,ds$$

$$= \int_0^t 2W_s\,dW_s + \frac{1}{2}\int_0^t 2\,ds$$

$$= 2\int_0^t W_s\,dW_s + t$$

$$\int_0^t W_s\,dW_s = \underbrace{\frac{W_t^2 - t}{2}}_{\text{mtg}}.$$

mtg $\rightarrow \int_0^t \text{anything}\,dW_s$

η_k between x_{k-1}, x_k.

$$f(x_k) - f(x_{k-1}) = f'(x_{k-1}) \cdot (x_k - x_{k-1}) + f''(\eta_k) \cdot \frac{(x_k - x_{k-1})^2}{2}$$

Take $\Pi = \{0 = t_0 < t_1 < \ldots < t_m = T\}$. Using the Taylor formula

$$\begin{aligned}
f(X_t) - f(X_0) &= \sum_{k=1}^{m} \left[f(X_{t_k}) - f(X_{t_{k-1}}) \right] \\
&= \sum_{k=1}^{m} f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) + \frac{1}{2} \sum_{k=1}^{m} f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2 \\
&= \sum_{k=1}^{m} f'(X_{t_{k-1}}) \int_{t_{k-1}}^{t_k} K_s \mathrm{d}s + \sum_{k=1}^{m} f'(X_{t_{k-1}}) \int_{t_{k-1}}^{t_k} H_s \mathrm{d}W_s \\
&\quad + \frac{1}{2} \sum_{k=1}^{m} f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2 \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where $\eta_k(\omega)$ is between $X_{t_{k-1}}(\omega)$ and $X_{t_k}(\omega)$.

It is easy to handle I_1. As f' and X_t are continuous

$$I_1 = \sum_{k=1}^{m} f'(X_{t_{k-1}}) \int_{t_{k-1}}^{t_k} K_s \mathrm{d}s \longrightarrow \int_0^t f'(X_s) K_s \mathrm{d}s \quad \text{a.s.}, \tag{12}$$ {eq:i1}

as $\|\Pi\| \to 0$.

Rewrite I_2 as

$$I_2 = \sum_{k=1}^{m} f'(X_{t_{k-1}}) \int_{t_{k-1}}^{t_k} H_s \mathrm{d}W_s = \int_0^t \overbrace{\sum_{k=1}^{m} f'(X_{t_{k-1}}) \mathbf{I}_{(t_{k-1}, t_k]}(s)}^{\uparrow f'(X_s)} H_s \mathrm{d}W_s.$$

As $\|\Pi\| \to 0$ Want to show: $I_2 \longrightarrow \int_0^t f(X_s) H_s dW_s$

$$\mathbf{E} \int_0^t \underbrace{\left(f'(X_s)H_s - \sum_{k=1}^{m} f'(X_{t_{k-1}}) \mathbf{I}_{(t_{k-1}, t_k]}(s) H_s \right)^2}_{} \mathrm{d}s \to 0.$$

Indeed, for any $\omega \in \Omega$ fix the integrand is bounded and by continuity goes to 0, therefore the dominated Lebesgue convergence theorem applies. Theorem 25 (ii) implies

$$I_2 = \int_0^t \sum_{k=1}^{m} f'(X_{t_{k-1}}) I_{(t_{k-1}, t_k]}(s) H_s \mathrm{d}W_s \xrightarrow{L^2} \int_0^t f'(X_s) H_s \mathrm{d}W_s. \tag{13}$$ {eq:i2-konv}

$$I_2 - \int_0^t f'(X_s) H_s dW_s = \int_0^t \left(\sum f'(X_{t_{k-1}}) \cdot I_{(t_{k-1}, t_k]}(s) - f'(X_s) \right) H_s dW_s$$

$$E\left[\left(I_2-\int_0^t f'(X_s)H_s\,dW_s\right)^2\right] = E\int\left(\sum_i f'(X_{t_{k-1}})I_{\dots} - f'(X_s)\right)^2 H_s^2\,ds$$

Theorem on Itô integral

Next comes I_3, the difficult part. We have to show that

$$I_3 \to \frac{1}{2}\int_0^t f''(X_s)H_s^2 \mathrm{d}s.$$

$$I_3 = \frac{1}{2}\sum_k f''(\eta_k)(X_{t_k}-X_{t_{k-1}})^2$$

Write

$$X_t = X_0 + \int_0^t K_s\,ds + \int_0^t H_s\,dW_s$$

$$\begin{aligned}
(X_{t_k}-X_{t_{k-1}})^2 &= \left(\int_{t_{k-1}}^{t_k} K_s \mathrm{d}s + \int_{t_{k-1}}^{t_k} H_s \mathrm{d}W_s\right)^2 \\
&= \left(\int_{t_{k-1}}^{t_k} K_s \mathrm{d}s\right)^2 + 2\int_{t_{k-1}}^{t_k} K_s \mathrm{d}s \cdot \int_{t_{k-1}}^{t_k} H_s \mathrm{d}W_s \\
&\quad + \left(\int_{t_{k-1}}^{t_k} H_s \mathrm{d}W_s\right)^2.
\end{aligned}$$

$$K(t_k - t_{k-1})^2 \geq$$

We show that the contribution of the first two terms is negligible to the whole sum. For the first

$$\leq \|\Pi\|_n \cdot \sum(t_k-t_{k-1}) = \|\Pi\|_n \cdot T.$$

$$\left|\sum_{k=1}^m f''(\eta_k)\left(\int_{t_{k-1}}^{t_k} K_s \mathrm{d}s\right)^2\right| \leq \|f''\|_\infty \cdot K^2 \sum_{k=1}^m (t_k-t_{k-1})^2 \to 0 \quad \text{a.s.} \tag{14}$$ {eq:i3-1}

To handle the second introducte $M_t = \underbrace{\int_0^t H_s \mathrm{d}W_s}_{\text{cont.}}$. Then

$$\begin{aligned}
&\left|\sum_{k=1}^m f''(\eta_k)\int_{t_{k-1}}^{t_k} K_s \mathrm{d}s \cdot \int_{t_{k-1}}^{t_k} H_s \mathrm{d}W_s\right| \\
&\leq \|f''\|_\infty \cdot K \sup_{1\leq k\leq m}|M_{t_k}-M_{t_{k-1}}| \cdot \sum_{k=1}^m (t_k-t_{k-1}) \\
&= \|f''\|_\infty \cdot K\cancel{T} \underbrace{\sup_{1\leq k\leq m}|M_{t_k}-M_{t_{k-1}}|}_{\text{uniformly cont.}} \to 0, \quad \text{a.s.},
\end{aligned} \tag{15}$$ {eq:i3-2}

since $M_t = \int_0^t H_s \mathrm{d}W_s$ is a continuous martingale.

We have to deal with the sum

$$\underbrace{\sum_{k=1}^m f''(\eta_k)\left(\int_{t_{k-1}}^{t_k} H_s \mathrm{d}W_s\right)^2}.$$

$$X_{t_{k-1}}$$

First we change η_k to $X_{t_{k-1}}$. Taking the difference

$$\begin{aligned}
&\sum_{k=1}^{m}[f''(\eta_k) - f''(X_{t_{k-1}})](M_{t_k} - M_{t_{k-1}})^2 \\
(*)\qquad &\le \sup_{1\le k\le m} |f''(\eta_k) - f''(X_{t_{k-1}})| \cdot \sum_{k=1}^{m}(M_{t_k} - M_{t_{k-1}})^2. \qquad \text{quadratic var.}
\end{aligned}$$

By the Cauchy–Schwarz inequality and $(*)$

$$\begin{aligned}
&\left| \mathbf{E}\sum_{k=1}^{m}[f''(\eta_k) - f''(X_{t_{k-1}})](M_{t_k} - M_{t_{k-1}})^2 \right| \\
&\le \sqrt{\mathbf{E}\sup_{1\le k\le m}(f''(\eta_k) - \underbrace{f''(X_{t_{k-1}})}_{\text{close}})^2}\sqrt{\mathbf{E}\left[\left(\sum_{k=1}^{m}(M_{t_k} - M_{t_{k-1}})^2\right)^2\right]}.
\end{aligned} \tag{16}$$

{eq:i3-3}

The first term tends to 0 because (X_t) is continuous and f'' is bounded. The second is bounded by the following lemma.

{lemma:Ito-aux}

Lemma 7. *Let (M_t) be a continuos bounded martingale on $[0,t]$, that is $\sup_{s,\omega}|M_s(\omega)| \le K$, and let $\Pi = \{0 = t_0 < t_1 < \ldots < t_m = t\}$ be a partition. Then*

$$\mathbf{E}\left[\left(\sum_{i=1}^{m}(M_{t_i} - M_{t_{i-1}})^2\right)^2\right] \le \underline{6K^4}.$$

Proof. Expanding the square

$$\begin{aligned}
&\mathbf{E}\left(\sum_{i=1}^{m}(M_{t_i} - M_{t_{i-1}})^2\right)^2 \\
&= \sum_{i=1}^{m}\mathbf{E}(M_{t_i} - M_{t_{i-1}})^4 + \sum_{i\ne j}\mathbf{E}(M_{t_i} - M_{t_{i-1}})^2(M_{t_j} - M_{t_{j-1}})^2.
\end{aligned}$$

Using several times that

$$\mathbf{E}[(M_t - M_s)^2|\mathcal{F}_s] = \mathbf{E}[M_t^2 - M_s^2|\mathcal{F}_s], \quad s < t,$$

we obtain

$$
\begin{aligned}
&\sum_{i\neq j} \mathbf{E}(M_{t_i} - M_{t_{i-1}})^2 (M_{t_j} - M_{t_{j-1}})^2 \\
&= 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \mathbf{E}(M_{t_i} - M_{t_{i-1}})^2 (M_{t_j} - M_{t_{j-1}})^2 \\
&= 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \mathbf{E}\left[\mathbf{E}[(M_{t_i} - M_{t_{i-1}})^2 (M_{t_j} - M_{t_{j-1}})^2 | \mathcal{F}_{t_{j-1}}]\right] \\
&= 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \mathbf{E}(M_{t_i} - M_{t_{i-1}})^2 (M_{t_j}^2 - M_{t_{j-1}}^2) \\
&= 2\sum_{i=1}^{m-1} \mathbf{E}(M_{t_i} - M_{t_{i-1}})^2 (M_t^2 - M_{t_i}^2) \\
&\leq 2K^2 \sum_{i=1}^{m-1} \mathbf{E}(M_{t_i} - M_{t_{i-1}})^2 \\
&= 2K^2 \sum_{i=1}^{m-1} \mathbf{E}(M_{t_i}^2 - M_{t_{i-1}}^2) \leq 2K^4.
\end{aligned}
$$

While, for the sum of 4th powers

$$
\begin{aligned}
\sum_{i=1}^{m} \mathbf{E}(M_{t_i} - M_{t_{i-1}})^4 &\leq 4K^2 \mathbf{E} \sum_{i=1}^{m} \mathbf{E}(M_{t_i} - M_{t_{i-1}})^2 \\
&= 4K^2 \mathbf{E}(M_t^2 - M_0^2) \leq 4K^4.
\end{aligned}
$$

□

Summarizing from I_3 we have the sum

$$
\sum_{k=1}^{m} f''(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})^2.
$$

We claim that

$$
\sum_{k=1}^{m} f''(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})^2 \xrightarrow{L^1} \int_0^t f''(X_s) H_s^2 \mathrm{d}s. \tag{17}
$$

{eq:i3-negyzetesva

Since X and f'' are continuous

$$\sum_{k=1}^{m} f''(X_{t_{k-1}}) \int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \to \int_0^t f''(X_s) H_s^2 \mathrm{d}s \quad \text{a.s.}$$

$M_t = \int_0^t H_s \, dW_s$

Thus it is enough to show that

$$(**) \qquad \sum_{k=1}^{m} f''(X_{t_{k-1}}) \left((M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \right) \xrightarrow{L^2} 0.$$

$E(M_t^2) = = E\int_0^t H_s^2 \, ds$

Theorem 25 (ii) implies

$$\mathbf{E}\left[(M_{t_k} - M_{t_{k-1}})^2 | \mathcal{F}_{t_{k-1}}\right] = \mathbf{E}\left[\left(\int_{t_{k-1}}^{t_k} H_s \, \mathrm{d}W_s\right)^2 | \mathcal{F}_{t_{k-1}}\right]$$
$$= \mathbf{E}\left[\int_{t_{k-1}}^{t_k} H_s^2 \, \mathrm{d}s | \mathcal{F}_{t_{k-1}}\right],$$

so in

$$\mathbf{E}\left[\left(\sum_{k=1}^{m} f''(X_{t_{k-1}}) \left((M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \right)\right)^2\right] \longrightarrow 0$$

the expectation of the mixed term is 0. Thus this equals

$$= \mathbf{E}\left[\sum_{k=1}^{m} f''(X_{t_{k-1}})^2 \left((M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \right)^2\right]$$
$$\leq \|f\|_\infty^2 \left[\mathbf{E} \sum_{k=1}^{m} (M_{t_k} - M_{t_{k-1}})^4 + 2\mathbf{E} \sum_{k=1}^{m} (M_{t_k} - M_{t_{k-1}})^2 \int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \right.$$
$$\left. + \mathbf{E} \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} H_s^2 \mathrm{d}s \right)^2 \right]$$
$$\leq \|f\|_\infty^2 \left[\mathbf{E} \sum_{k=1}^{m} (M_{t_k} - M_{t_{k-1}})^4 + 2K^2 t \mathbf{E} \underbrace{\sup_{1 \leq k \leq m} (M_{t_k} - M_{t_{k-1}})^2}_{\to 0} + K^4 t \underbrace{\|\Pi\|}_{\to 0} \right].$$

$$E\left[\left(\sum_{k} f''(X_{t_{k-1}}) \cdot \left((M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2 \, ds\right)\right)^2\right]$$

$$E\left\{ \sum_{k=1}^{m} f''(X_{t_{k-1}})^2 \left((M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2 \, ds \right)^2 \right.$$

$$+ \sum_{k \neq l} f''(X_{t_{k-1}}) f''(X_{t_{l-1}}) \left((M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2 \, ds \right) \cdot$$

$$\left. \cdot \left((M_{t_l} - M_{t_{l-1}})^2 - \int_{t_{l-1}}^{t_l} H_s^2 \, ds \right) \right\}$$

$k < l$ mixed: $E\left[E\left[\cdot (M_{t_l} - M_{t_{l-1}})^2 \ldots \right) \middle| \mathcal{F}_{t_{l-1}} \right] \right]$

$$= 0 .$$

The second and third term tend to 0, and for the first

$$
\begin{aligned}
\mathbf{E}\sum_{k=1}^{m}(M_{t_k}-M_{t_{k-1}})^4 &\le \mathbf{E}\left[\sum_{k=1}^{m}(M_{t_k}-M_{t_{k-1}})^2\cdot \sup_{1\le k\le m}|M_{t_k}-M_{t_{k-1}}|^2\right]\\
&\le \sqrt{\mathbf{E}\left[\sum_{k=1}^{m}(M_{t_k}-M_{t_{k-1}})^2\right]^2}\sqrt{\mathbf{E}\sup_{1\le k\le m}|M_{t_k}-M_{t_{k-1}}|^4}\\
&\le \sqrt{6}K^2\sqrt{\mathbf{E}\sup_{1\le k\le m}|M_{t_k}-M_{t_{k-1}}|^4}\to 0.
\end{aligned}
$$

Summarizing we obtained L^1, L^2 and almost sure convegence in (12)–(17). Since everthing is bounded, L^1 convergence follows in each case, that is

$$
\begin{aligned}
\underline{f(X_t)-f(X_0)} &= \sum_{k=1}^{m}[f(X_{t_k})-f(X_{t_{k-1}})]\\
&\xrightarrow{L^1}\int_0^t f'(X_s)\underbrace{\mathrm{d}X_s}_{\zeta_s ds + H_s dW_s}+\frac{1}{2}\int_0^t f''(X_s)H_s^2\mathrm{d}s.
\end{aligned}
$$

$\forall t$ a.s.

Convergence in L^1 implies a.s. convergence on a subsequence. As both sides are continuous we obtained that the two process are indistinguishable. $\square$

{example:exp-2}

Example 12 (Continuation of Example 10)**.** Let

$$
\zeta_t^s=\int_s^t X_u\mathrm{d}W_u-\frac{1}{2}\int_s^t X_u^2\mathrm{d}u,\quad \zeta_t=\zeta_t^0,
$$

where X_t is an adapted process. Then $Z_t=e^{\zeta_t}$ satisfies the stochastic differential equation

$$
Z_t=1+\int_0^t Z_sX_s\mathrm{d}W_s,
$$

or with a common notation

$$
\mathrm{d}Z_t=Z_tX_t\mathrm{d}W_t,\quad Z_0=1.
$$

Writing ζ as an Itô process

$$
\zeta_t=\int_0^t -\frac{1}{2}X_u^2\mathrm{d}u+\int_0^t X_u\mathrm{d}W_u.
$$