Financial mathematics

Péter Kevei

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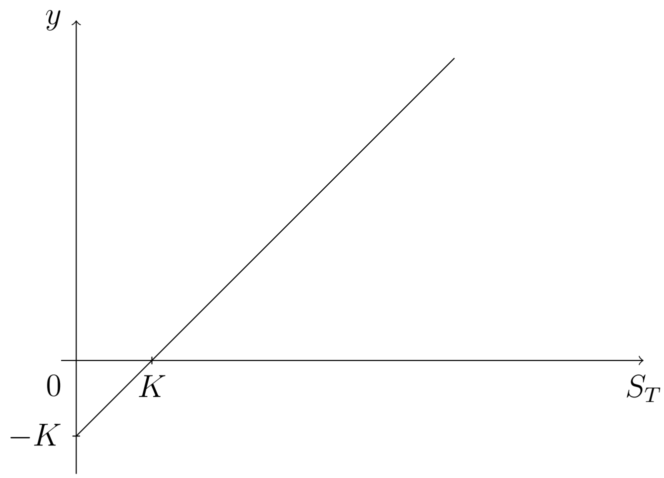

Figure 1: Payoff of a forward

1 Introduction

These notes are based on the Hungarian lecture notes by Gáll and Pap [3], on Shiryaev's monograph [4], and on Elliott and Kopp [2].

There are two type of financial instruments: the basic financial units and their derivatives.

Underlying:

- bond: risk-free asset, basically money. Its price is deterministic B_t;
- stock: risky asset. Its price is a random, modeled by a stochastic process $S_t = (S_t^1, \ldots, S_t^d)$.

Derivatives are bets on the underlying. They are used to share or reduce risk. Here we consider forward contracts and options.

1.1 Forward

A forward contract is an agreement to buy or sell an asset (stock) for a price previously agreed K in the future time T.

From the buyers point of view, at time T his wealth is $S_T - K$, that is the payoff function is $f(s) = s - K$.

We want to determine the fair price of this contract, and to understand the meaning of 'fair'. Assume $B_0 = 1$.

Seller's point of view: At time 0, we can buy a stock for S_0. Then at time T selling a stock for K and paying back the loan $S_0 \cdot B_T$, we have $K - S_0 B_T$. Therefore,

$$K \geq S_0 B_T.$$

Buyer's point of view: At time 0, we sell a stock for S_0. At time T we pay K for a stock, and the our wealth is $S_0 B_T - K$. Thus,

$$K \leq S_0 B_T.$$

We see that the fair price has to be $K = S_0 B_T$. Otherwise, either the seller or the buyer would have a strategy providing riskless profit (arbitrage).

Example 1. Let $S_0 = 40$, $B_t = e^{rt}$, $r = 0.1$ being the annual interest, $T = 1$ year. What is the fair price of this forward, and what is the value of the contract after half a year if $S_{0.5} = 45$?

The forward price at time 0 is

$$K = S_0 B_1 = 40 \cdot e^{0.1} = 44.2.$$

At time $t = 0.5$ the forward price

$$K_2 = S_{0.5} B_{0.5} = 45 \cdot e^{\frac{1}{2}0.1} = 47.3.$$

Thus the current value of the contract

$$e^{-\frac{1}{2}r}(47.3 - 44.2) = 2.9.$$

1.2 Options

An option is *right* to do something but not an obligation. European option can be executed only at the expiration date, while American options can be executed at any time.

The writer of a European call option agrees to sell a stock for a previously agreed price K. Clearly, the buyer of this option will not use his right if $S_T < K$. The payoff function for the buyer is $f(s) = (s - K)_+$

In case of a put option the writer agrees to buy a stock for K. The payoff function of the buyer

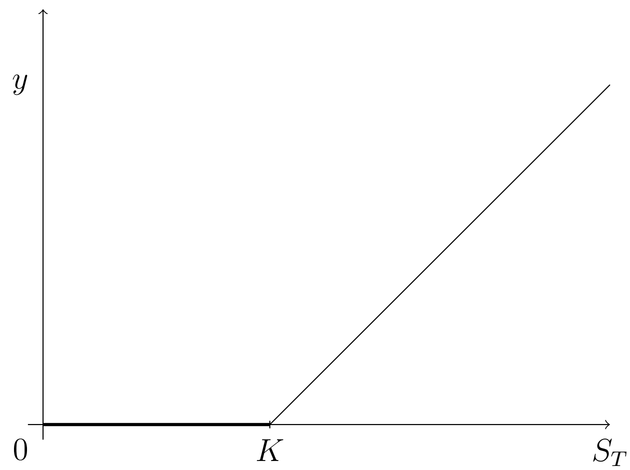

Figure 2: Payoff a call option

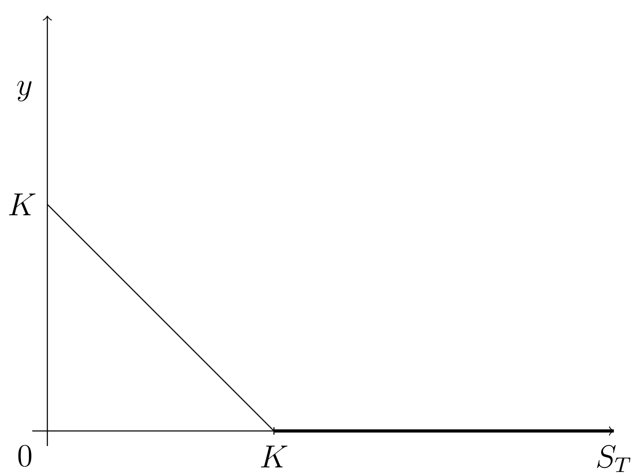

Figure 3: Payoff of a put option

1.3 Put–call parity

The aim of the course is to determine the fair price of an option, and understand the fairness. However, there is a simple relation between call and put prices regardless of the underlying market model.

Let C_K be the fair price of the call, and P_K be the fair price of the put, both with strike price K. Then, from the payoff functions it is easy to see that having put, a stock, and -1 call results at the expiration date (regardless of the stock price) a wealth K. That is, after discounting

$$\frac{K}{B_T} = P + S_0 - C.$$

This is the put-call parity.

2 Portfolio, claim, and hedging in discrete time

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. In the discrete time case we always assume (if not stated otherwise) that Ω is finite, and $\mathbf{P}(\{\omega\}) > 0$ for each $\omega \in \Omega$. We assume that transactions are made only at the time instants $0, 1, \ldots, N$. Let $(\mathcal{F}_n)_{n=0,1,\ldots,N}$ be a filtration, an increasing sequence of σ-algebras, such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_N = \mathcal{F}$. Assume that there are d risky assets and a bond. The price of the risky asset i at time is S_n^i, an $\mathcal{F}_n$-measurable random variable, and the bond price at time n is B_n.

2.1 Portfolio

An investment portfolio (strategy) is $\pi_n = (\beta_n, \gamma_n)$, where $\beta_n \in \mathbb{R}$ represents the amount of bonds in the portfolio at time n, while $\gamma_n = (\gamma_n^1, \ldots, \gamma_n^d) \in \mathbb{R}^d$, where γ_n^i represents the amount of type-i stock at time n. The random variables (β_n, γ_n) are $\mathcal{F}_{n-1}$-measurable, which means the investor has to decide at time $n-1$ how to invest on time n. That is the sequence (β_n, γ_n) is *predictable*. For simplicity

$$\gamma_n S_n = \sum_{i=1}^{d} \gamma_n^i S_n^i.$$

The wealth of the investor at time n under the strategy π is

$$X_n^\pi = \beta_n B_n + \gamma_n S_n.$$

This is the *value process* of the investment portfolio.

A strategy is *self-financing (SF)* if the investor does not take out money from, and does not invest money to the portfolio after time 0. That is π is self-financing if

$$X_{n-1}^{\pi} = \beta_n B_{n-1} + \gamma_n S_{n-1} \quad \text{for all } n.$$

For a sequence a_n put $\Delta a_n = a_n - a_{n-1}$.

{lemma:SF}

Lemma 1. *The following are equivalent:*

(i) π *is SF;*

(ii) $\Delta X_n^{\pi} = \beta_n \Delta B_n + \gamma_n \Delta S_n$*;*

(iii) $B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n = 0$.

Proof. We have

$$\begin{aligned}
&\Delta X_n = X_n - X_{n-1} \\
&= \beta_n B_n - \beta_{n-1} B_{n-1} + \gamma_n S_n - \gamma_{n-1} S_{n-1} \\
&= \beta_n (B_n - B_{n-1}) + (\beta_n - \beta_{n-1}) B_{n-1} + \gamma_n (S_n - S_{n-1}) + (\gamma_n - \gamma_{n-1}) S_{n-1} \\
&= \beta_n \Delta B_n + \Delta\beta_n B_{n-1} + \gamma_n \Delta S_n + \Delta\gamma_n S_{n-1},
\end{aligned}$$

and the equivalence follows. □

In what follows, unless otherwise stated all the strategies are meant to be SF.

We can decompose the value process as

$$\begin{aligned}
X_n^{\pi} &= X_{n-1}^{\pi} + \Delta X_n^{\pi} = \dots \\
&= X_0^{\pi} + \sum_{i=1}^{n} (\beta_i \Delta B_i + \gamma_i \Delta S_i) \\
&=: X_0^{\pi} + G_n^{\pi},
\end{aligned}$$

where G_n^{π} is the *gain process*. So the value of the strategy is the initial investment plus the gain.

2.2 Strategies in a more general market

Previously, we assumed that there are no transaction cost (market is frictionless), shares pay no dividend, and apart from time 0, there is neither investment, nor consumption. Here we see how to handle this.

2.2.1 Dividend

Assume that stock-i pays a dividend $\delta_n^i = D_n^i - D_{n-1}^i \geq 0$ at n, where δ_n^i, and D_n^i are adapted processes. Then the change in the value process is

$$\Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n(\Delta S_n + \delta_n),$$

and the value of the portfolio

$$X_n^\pi = \beta_n B_n + \gamma_n(S_n + \delta_n).$$

Then, π is self-financing portfolio if

$$B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n = \delta_{n-1}\gamma_{n-1}.$$

Indeed, the dividend obtained in time $n-1$ is reinvested in the portfolio.

2.2.2 Consumption and investment

The consumption and investment can be included as well. Let (C_n), (I_n) be adapted nondecreasing random sequences with $C_0 = I_0 = 0$. Then, if an investor takes out ΔC_n and invests ΔI_n then

$$\begin{aligned} \Delta X_n^\pi &= \beta_n \Delta B_n + \gamma_n \Delta S_n + \Delta I_n - \Delta C_n \\ X_n^\pi &= \beta_n B_n + \gamma_n S_n. \end{aligned}$$

2.2.3 Transaction costs

If $\Delta\gamma_n > 0$ then we buy share, and pay an extra cost λ, that is we pay $(1+\lambda)S_{n-1}\Delta\gamma_n$. While if $\Delta\gamma_n < 0$ we sell, and paying transaction cost means receiving less money, say $-(1-\mu)S_{n-1}\Delta\gamma_n$. Then an SF strategy satisfies (see (iii) in Lemma 1)

$$B_{n-1}\Delta\beta_n + (1+\lambda)S_{n-1}\Delta\gamma_n I(\Delta\gamma_n > 0) + (1-\mu)S_{n-1}\Delta\gamma_n I(\Delta\gamma_n < 0) = 0.$$

2.3 Claim and hedging

Let f_N be a nonnegative random variable, which is the *payoff function*, or *obligation*, or *contingent claim*. A strategy π is an *upper* (x, f_N)*-hedge*, if $\mathbf{P}$-almost surely

$$X_0^\pi = x, \quad X_N^\pi \geq f_N.$$

It is a *lower* (x, f_N)*-hedge*, if a.s.

$$X_0^\pi = x, \quad X_N^\pi \leq f_N.$$

The hedge is perfect if $=$ holds a.s.

Put

$$C^*(f_N) = \inf\{x : \exists \text{ upper } (x, f_N)\text{-hedge }\},$$

and similarly

$$C_*(f_N) = \sup\{x : \exists \text{ lower } (x, f_N)\text{-hedge }\}.$$

For the class of upper (x, f_N)-hedge strategies put $H^*(x, f_N, \mathbf{P})$, and for the lower $H_*(x, f_N, \mathbf{P})$.

{lemma:hedge}

Lemma 2. *For any payoff function f_N there exists an x such that there is an upper (x, f_N)-hedge.*

Proof. Put

$$x = \frac{B_0}{B_N} \max_{\omega \in \Omega} |f_N(\omega)|.$$

Then the (trivial) strategy $\pi_n \equiv (\frac{x}{B_0}, 0)$ (start with enough money and don't do anything) is an upper hedge. □

2.4 Binomial market

{ss:bin}

2.4.1 One-step market

Consider a one-step binomial market with $d = 1$ stock. That is $\Omega = \{0, 1\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F} = 2^\Omega$. Assume that $\mathbf{P}(\{0\}) \in (0, 1)$. The bond price $B_1 = (1 + r)B_0$, that is $r > -1$ is the interest rate, and for some $a < b$, $S_1 = (1 + \rho)S_0$, $\rho \in \{a, b\}$. Say, $\rho(1) = b$, $\rho(0) = a$. Let f be a payoff, that is $f(0) = f_0$, $f(1) = f_1$. We construct a perfect hedge.

Using the strategy $\pi_1 = (\beta_1, \gamma_1)$ we want that

$$X_1^\pi = \beta_1 B_1 + \gamma_1 S_1 = f \quad \text{a.s.}$$

Since there are only two possibilities, a.s. means

$$\beta_1 B_0(1 + r) + \gamma_1 S_0(1 + a) = f_0$$
$$\beta_1 B_0(1 + r) + \gamma_1 S_0(1 + b) = f_1.$$

Solving the linear system

$$\gamma_1 = \frac{1}{S_0}\frac{f_1 - f_0}{b - a}, \quad \beta_1 = \frac{f_1 - (1+b)\frac{f_1 - f_0}{b-a}}{B_0(1+r)}.$$

This is deterministic, so $\mathcal{F}_0$-measurable, as it should be. The initial cost of this strategy is

$$X_0^\pi = B_0\beta_1 + S_0\gamma_1 = \frac{1}{1+r}\left(\frac{r-a}{b-a}f_1 + \frac{b-r}{b-a}f_0.\right)$$

If $a < r < b$ this can be written as

$$X_0^\pi = \frac{1}{1+r}\mathbf{E}_{\mathbf{Q}} f,$$

with the probability measure $\mathbf{Q}(\{0\}) = (b-r)/(b-a)$, $\mathbf{Q}(\{1\}) = (r-a)/(b-a)$.

This shows that the 'fair' price of the payoff is $\mathbf{E}_{\mathbf{Q}} f/(1+r)$. Note that this does not depend on the probability measure $\mathbf{P}$.

2.4.2 N-step market

Assume we have only one stock, $d = 1$. For the bond $B_n = (1 + r_n)B_{n-1}$, and for the share $S_n = (1 + \rho_n)S_{n-1}$, where $\rho_n \in \{a_n, b_n\}$.

Exercise 1. Give a concrete construction of the probability space and the filtration!

Solution 1. Let

$$\Omega = \{0,1\}^N = \{\omega = (\omega_1, \dots, \omega_N) \, : \omega_i \in \{0,1\}\}.$$

Define the random variables $\rho_n : \Omega \to \{a_n, b_n\}$ as

$$\rho_n(\omega) = \begin{cases} a_n, & \text{if } \omega_n = 0, \\ b_n, & \text{if } \omega_n = 1. \end{cases}$$

For the filtration let $\mathcal{F}_n = \sigma(\rho_1, \dots, \rho_n)$, i.e. the natural filtration generated by the variables $\rho_1, \dots, \rho_n$.

Consider any payoff function f_N. A perfect hedge can be constructed recursively, using the simple one-step market. Indeed, a two-step model can be seen as 3 one-step markets.

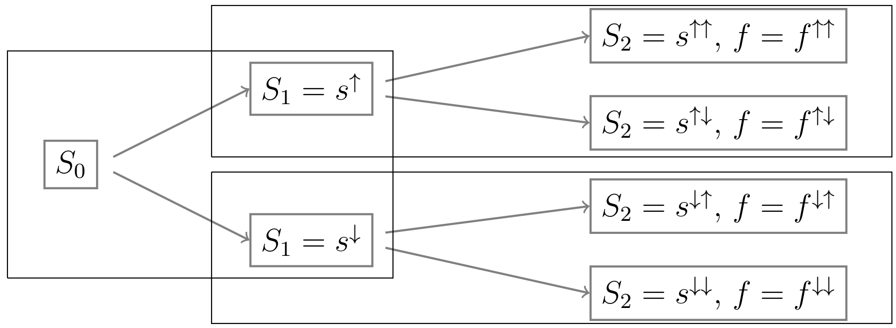

Figure 4: 2-step binary market as 3 1-step binary market

3 Arbitrage and pricing in discrete time

3.1 Arbitrage

A SF strategy π is an *arbitrage strategy* if

- $X_0^\pi = 0$;
- $X_n^\pi \geq 0$ for all $n = 0, 1, \ldots, N$;
- $\mathbf{P}(X_N^\pi > 0) > 0$.

That is, using the strategy π with 0 money we have riskless profit.

If the second assumption only holds for $n = N$ then π is a *weak arbitrage strategy*. According to the following if weak arbitrage strategy exists, then also arbitrage strategy exists.

{lemma:arbitrage}

Lemma 3. *Assume that π is a weak arbitrage strategy. Then there exists an arbitrage strategy π'.*

Proof. If $X_n^\pi \geq 0$ a.s. for all n, then we are ready. Otherwise, there exists $m < N$ such that $\mathbf{P}(X_m^\pi < 0) > 0$, and $X_n^\pi \geq 0$ for any $n \geq m+1$. Let

$$A_m = \{X_m < 0\} \in \mathcal{F}_m.$$

Consider the strategy

$$\beta_n' = I_{A_m} I_{n>m} \left(\beta_n - \frac{X_m}{B_m} \right), \quad \gamma_n' = I_{A_m} I_{n>m} \gamma_n.$$

It is easy to check that this strategy is predictable, SF, and arbitrage strategy. Indeed,

(i) predictable: for $n \le m$ this is clear, since $\beta'_n = 0$ and $\gamma'_n = 0$, while for $n > m$ A_m is $\mathcal{F}_m$-measurable and thus $\mathcal{F}_{n-1}$-measurable as well, and β_n, γ_n are $\mathcal{F}_{n-1}$-measurable by the assumption.

(ii) SF: for $n \le m$ this is again clear. For $n = m+1$

$$B_m \Delta\beta'_{m+1} + S_m \Delta\gamma'_{m+1} = I_{A_m} \left(B_m \beta_{m+1}(\omega) - X^\pi_m(\omega) + S_m \gamma_{m+1}(\omega) \right) = 0,$$

since π is SF. For $n > m+1$ we have $\Delta\beta'_n = I_{A_m} \Delta\beta_n$, and $\Delta\gamma'_n = I_{A_m} \Delta\gamma_n$, and the result follows, using again that π is SF.

(iii) arbitrage: we have

$$X^{\pi'}_n = I_{A_m} I_{n>m} \left(\beta_n B_n + \gamma_n S_n - \frac{X^\pi_m B_n}{B_m} \right),$$

where the sum of the first two terms in the bracket is nonnegative by the definition of m and the last is strictly negative on A_m, which proves the statement.

□

Exercise 2. Assume that $a < b < r$ in the one-step binomial model. Give an arbitrage strategy.

Assume that $a_n < b_n < r_n$ for some n in the N-step binomial model. Give an arbitrage strategy.

3.2 Reminder: conditional expectation and martingales

3.2.1 Conditional expectation

Let $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Let X be a random variable with $\mathbf{E}|X| < \infty$. The *conditional expectation of X with respect to the σ-algebra $\mathcal{G}$*, $\mathbf{E}[X|\mathcal{G}]$ is the almost surely uniquely determined random variable which

- $\mathcal{G}$-measurable, i.e. for any Borel set $B \in \mathcal{B}$ $\mathbf{E}[X|\mathcal{G}])^{-1}(B) \in \mathcal{G}$;
- for any $G \in \mathcal{G}$

$$\int_G X \, \mathrm{d}\mathbf{P} = \int_G \mathbf{E}[X|\mathcal{G}] \, \mathrm{d}\mathbf{P}.$$

The conditional probability of $A \in \mathcal{A}$ with respect to $\mathcal{G}$ $\mathbf{P}(A|\mathcal{G}) = \mathbf{E}[I_A|\mathcal{G}]$. Then the second condition above can be written as for all $G \in \mathcal{G}$

$$\mathbf{P}(A \cap G) = \int_G \mathbf{P}(A|\mathcal{G}) \, \mathrm{d}\mathbf{P}.$$

3.2.2. Martingálok

Ebben a fejezetben összefoglaljuk a diszkrét idejű martingálokra vonatkozó legfontosabb állításokat. Csörgő Sándor jegyzetét [1] követjük.

Legyen $(\Omega, \mathcal{F}, \mathbf{P})$ egy valószínűségi mező, és ezen $(\mathcal{F}_n)_n$ egy *filtráció* (azaz σ-algebrák monoton bővülő rendszere, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \ldots \subset \mathcal{A}$), és $(X_n)_n$ véletlen változók sorozata. Az $(X_n)_n$ sorozat *adaptált* az $(\mathcal{F}_n)$ filtrációhoz, ha minden n esetén X_n mérhető $\mathcal{F}_n$ szerint. Az $(X_n, \mathcal{F}_n)$ sorozat *martingál*, ha

(i) (X_n) adaptált az $(\mathcal{F}_n)$ filtrációhoz;

(ii) $\mathbf{E}|X_n| < \infty$ minden n esetén;

(iii) $\mathbf{E}[X_{n+1}|\mathcal{F}_n] = X_n$ m.b.

Az $\{X_n, \mathcal{F}_n\}$ sorozat *szubmartingál (szupermartingál)*, ha (i), (ii) teljesül, és $\mathbf{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$ ($\mathbf{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$) m.b. minden n esetén.

Vegyük észre, hogy $(X_n, \mathcal{F}_n)$ pontosan akkor martingál, ha szubmartingál és szupermartingál. Továbbá, szubmartingál mínusz egyszerese szupermartingál, ezért minden szubmartingálokra bizonyított állítás megfelelője igaz szupermartingálokra, és martingálokra is.

1. Proposition. *Legyen I véges vagy végtelen intervallum a számegyenesen, $\mathbf{P}(X_n \in I) = 1$ minden n esetén, $\varphi : I \to \mathbb{R}$ konvex függvény, $\varphi(X_n)$ integrálható.*

(i) Ha $(X_n, \mathcal{F}_n)$ martingál, akkor $(\varphi(X_n), \mathcal{F}_n)$ szubmartingál.

(ii) Ha $(X_n, \mathcal{F}_n)$ szubmartingál és φ monoton nemcsökkenő függvény, akkor $(\varphi(X_n), \mathcal{F}_n)$ szubmartingál.

Bizonyítás. (i): A martingál definíciója és a feltételes várható értékre vonatkozó Jensen-egyenlőtlenség szerint

$$\varphi(X_n) = \varphi\left(\mathbf{E}[X_{n+1}|\mathcal{F}_n]\right) \leq \mathbf{E}\left[\varphi(X_{n+1})|\mathcal{F}_n\right],$$

ami éppen az állítás. A (ii) bizonyítása hasonlóan megy. □

3. Exercise. Lássuk be (ii) állítást!

Véletlen változók egy (Z_n) sorozata *előrejelezhető* az $(\mathcal{F}_n)$ filtrációra nézve, ha minden n esetén Z_n mérhető $\mathcal{F}_{n-1}$ szerint.

1. Theorem (Doob-felbontás)**.** *Legyen* $(X_n, \mathcal{F}_n)$ *szubmartingál,* $\mathcal{F}_0 = \{\emptyset, \Omega\}$*. Ekkor létezik* (M_n, Z_n) *sorozat, melyre* $X_n = M_n + Z_n$*, ahol* $(M_n, \mathcal{F}_n)$ *martingál,* $Z_1 = 0$*,* $Z_1 \leq Z_2 \leq \ldots$ *m.b., és* Z_n *előrejelezhető. Továbbá, ez az előállítás egyértelmű.*

Az állítás azt mondja, hogy a szubmartingálban benne levő driftet le tudjuk választani.

Bizonyítás. Legyen $Z_1 = 0$ m.b., és $n \geq 2$ esetén $Z_n = \sum_{k=2}^{n} \mathbf{E}[X_k - X_{k-1} | \mathcal{F}_{k-1}]$. A szubmartingál definíciója szerint Z_n m.b. monoton nemcsökkenő, hiszen

$$Z_n - Z_{n-1} = \mathbf{E}[X_n - X_{n-1} | \mathcal{F}_{n-1}] \geq 0 \text{ m.b.}$$

Az előrejelezhetőség is világos. (Ezzel leválasztottuk a driftet.) Legyen $M_n = X_n - Z_n$. Könnyen ellenőrizhető, hogy ez valóban martingál. Ezzel a létezést beláttuk.

Az egyértelműség bizonyítása a következőképp megy. Legyen $\{M_n^*, Z_n^*\}$ egy, a feltételeknek eleget tevő sorozat. Ekkor a feltétel szerint $Z_1^* = 0 = Z_1$ m.b. Teljes indukcióval bizonyítunk. Tegyük fel, hogy $n-1$ esetén igaz az állítás, azaz $Z_{n-1}^* = Z_{n-1}$ m.b., és $M_{n-1}^* = M_{n-1}$ m.b. Ekkor

$$\begin{aligned}
Z_n^* &= \mathbf{E}[Z_n^* | \mathcal{F}_{n-1}] && \text{előrejelezhetőség} \\
&= \mathbf{E}[X_n - M_n^* | \mathcal{F}_{n-1}] && \text{definíció} \\
&= \mathbf{E}[X_n | \mathcal{F}_{n-1}] - M_{n-1}^* && \text{martingálság} \\
&= \mathbf{E}[X_n | \mathcal{F}_{n-1}] - M_{n-1} && \text{indukciós feltétel} \\
&= \mathbf{E}[X_n - M_n | \mathcal{F}_{n-1}] && \text{martingálság} \\
&= \mathbf{E}|Z_n | \mathcal{F}_{n-1}] && \text{definíció} \\
&= Z_n && \text{előrejelezhetőség.}
\end{aligned}$$

□

A (Z_n) folyamat az $(X_n, \mathcal{F}_n)$ martingál *növekvő folyamata.*

3.2.3. Megállási idő

A $\tau : \Omega \to \mathbf{N}$ nemnegatív egész értékű (kiterjesztett) véletlen változó *megállási idő* az $(\mathcal{F}_n)$ filtrációra nézve, ha minden n esetén $\{\tau \leq n\} \in \mathcal{F}_n$.

Vegyük észre, hogy a definícióban megengedjük, hogy τ pozitív valószínűséggel vegye fel a ∞ értéket.

4. Lemma. *Az alábbiak ekvivalensek:*

(i) τ *megállási idő;*

(ii) $\{\tau > n\} \in \mathcal{F}_n$ *minden* n *esetén;*

(ii) $\{\tau = n\} \in \mathcal{F}_n$ *minden* n *esetén.*

Bizonyítás. A bizonyítás a σ-algebra elemi tulajdonságain múlik. □

4. Exercise. Bizonyítsuk be a lemmát!

Legyen τ megállási idő az $\{\mathcal{F}_n\}$ filtrációra. A τ *előtti események* σ-*algebrája / pre-*τ σ*-algebra* az

$$\mathcal{F}_\tau = \{A \in \mathcal{A} :\ A \cap \{\tau \leq n\} \in \mathcal{F}_n, \forall n\}$$

formulával definiált σ-algebra.

5. Exercise. Mutassuk meg, hogy $\mathcal{F}_\tau$ valóban σ-algebra!

Az $\mathcal{F}_\tau$ σ-algebra azt információt tartalmazza, amit éppen a τ megállási időig gyűjtünk össze.

Legyen $\{X_n\}$ véletlen változók egy sorozata, τ megállási idő. Ekkor X_τ az a véletlen változó, melyre $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$, $\omega \in \Omega$.

2. Proposition. *(i) Ha* τ *megállási idő, akkor* τ *mérhető* $\mathcal{F}_\tau$ *szerint.*

(ii) Ha $\tau \equiv k$*, akkor* $\mathcal{F}_\tau = \mathcal{F}_k$ *(azaz a jelölés konzisztens).*

(iii) Ha σ, τ *megállási idők, akkor* $\min\{\sigma, \tau\} = \sigma \wedge \tau$ *és* $\max\{\sigma, \tau\} = \sigma \vee \tau$ *is megállási idők.*

(iv) Ha $\sigma \leq \tau$ *m.b., akkor* $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$*.*

(v) Ha $\{X_n\}$ *adaptált és* τ *megállási idő, akkor* X_τ *mérhető* $\mathcal{F}_\tau$ *szerint.*

Bizonyítás. A σ-algebra tulajdonságain múlik. □

6. Exercise. Bizonyítsuk be az állítást!

3.3 Martingale measures

A probability measure $\mathbf{Q}$ is called *equivalent martingale measure* (EMM) if $\mathbf{P} \sim \mathbf{Q}$ and $(S_n^i/B_n, \mathcal{F}_n)$ is a $\mathbf{Q}$-martingale for each $i = 1, 2, \ldots, d$.

3.3.1 EMM in binomial markets

In a one-step binomial market the martingale property is easy to check. Indeed, $(S_i/B_i)_{i=0,1}$ is a martingale iff

$$\mathbf{E_Q}\left[\frac{S_1}{B_1}\middle|\mathcal{F}_0\right] = \frac{S_0}{B_0}.$$

We have

$$\begin{aligned}\mathbf{E_Q}\left[\frac{S_1}{B_1}\middle|\mathcal{F}_0\right] &= \mathbf{E_Q}\frac{S_1}{B_1} \\ &= \mathbf{Q}(\rho = a)\frac{(1+a)S_0}{(1+r)B_0} + (1-\mathbf{Q}(\rho = a))\frac{(1+b)S_0}{(1+r)B_0} \\ &= \frac{S_0}{B_0}.\end{aligned}$$

Solving the equation we obtain that

$$\mathbf{Q}(\rho = a) = \frac{b-r}{b-a}, \quad \text{and} \quad \mathbf{Q}(\rho = b) = \frac{r-a}{b-a}.$$

That is $\mathbf{Q}(\{0\}) = (b-r)/(b-a)$, $\mathbf{Q}(\{1\}) = (r-a)/(b-a)$. This is the probability measure $\mathbf{Q}$ we obtained at pricing.

Let us see the general N-step model. Then

$$S_n = \prod_{i=1}^{n}(1+\rho_i)\, S_0,$$

thus the martingale property reads as

$$\mathbf{E_Q}\left[\frac{S_n}{B_n}\middle|\mathcal{F}_{n-1}\right] = \frac{S_{n-1}}{B_{n-1}} \quad n = 0, 1, \ldots N.$$

Using the properties of conditional expectation we have

$$\mathbf{E_Q}\left[\frac{S_n}{B_n}\middle|\mathcal{F}_{n-1}\right] = \frac{S_{n-1}}{B_{n-1}}\frac{1}{1+r_n}\mathbf{E_Q}[1+\rho_n|\mathcal{F}_{n-1}].$$

Therefore S_n/B_n is a $\mathbf{Q}$-martingale iff

$$\mathbf{E_Q}[\rho_n|\mathcal{F}_{n-1}] = r_n.$$

This condition exactly tells that under the new measure $\mathbf{Q}$ the risky asset behaves as the bond on average. Using that $\rho_n \in \{a_n, b_n\}$, we obtain as above

$$\mathbf{Q}(\rho_n = a_n|\mathcal{F}_{n-1}) = \frac{b_n - r_n}{b_n - a_n}, \quad \text{and } \mathbf{Q}(\rho_n = b_n|\mathcal{F}_{n-1}) = \frac{r_n - a_n}{b_n - a_n}.$$

Note the conditioning on $\mathcal{F}_{n-1}$ gives a constant, meaning that ρ_n is independent of $\mathcal{F}_{n-1}$ under the measure $\mathbf{Q}$.

We obtained the following.

{thm:binom-EMM}

Theorem 2. *In the binomial market if $a_n < r_n < b_n$ for each n then there exists a unique EMM $\mathbf{Q}$ given by the formulas above. Moreover, under $\mathbf{Q}$ the random variables $\rho_1, \ldots, \rho_N$ are independent.*

In the proof we used the following simple result.

Exercise 7. Assume that $Y \in \{a, b\}$ and

$$\mathbf{P}(Y = a|\mathcal{F}) = p \text{ a.s.}$$

Show that Y is independent of $\mathcal{F}$.

Note that the original measure $\mathbf{P}$ is irrelevant.

In the special case of the homogeneous binomial market we get that

$$\mathbf{Q}(S_N = S_0(1+b)^k(1+a)^{N-k}) = \binom{N}{k} q^k (1-q)^{N-k}, \quad k = 0, 1, \ldots, N.$$

3.3.2 Pricing with EMM

{prop:Xbar-mtg}

Proposition 3. *If $\mathbf{Q}$ is an EMM then $(\overline{X}_n^\pi = X_n^\pi / B_n)_n$ is a $\mathbf{Q}$-martingale for any SF strategy π.*

Proof. Easily follows from the SF property. Indeed, using that β_n, γ_n are

$\mathcal{F}_{n-1}$-measurable

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}}\left[\frac{X_n^\pi}{B_n}\middle|\mathcal{F}_{n-1}\right] &= \mathbf{E}_{\mathbf{Q}}\left[\beta_n + \gamma_n\frac{S_n}{B_n}\middle|\mathcal{F}_{n-1}\right] \\
&= \beta_n + \gamma_n \mathbf{E}_{\mathbf{Q}}\left[\frac{S_n}{B_n}\middle|\mathcal{F}_{n-1}\right] \\
&= \beta_n + \gamma_n\frac{S_{n-1}}{B_{n-1}} \\
&= \frac{\beta_n B_{n-1} + \gamma_n S_{n-1}}{B_{n-1}} \\
&= \frac{X_{n-1}^\pi}{B_{n-1}},
\end{aligned}$$

where the last equality follow from the self-financing property. □

The following main result is the *first fundamental theorem of asset pricing.*

{thm:emm-arb}

Theorem 3. *There exists an EMM if and only if the market is arbitrage-free.*

Proof. Let $\mathbf{Q}$ be an EMM and π be any strategy with $X_0^\pi = 0$. Then, by the previous statement

$$\mathbf{E}_{\mathbf{Q}}\frac{X_N^\pi}{B_N} = \mathbf{E}_{\mathbf{Q}}\frac{X_0^\pi}{B_0} = 0.$$

Thus $X_N \geq 0$ $\mathbf{P}$-a.s., then also $\mathbf{Q}$-a.s., which implies $X_N \equiv 0$ $\mathbf{Q}$-a.s., thus $\mathbf{P}$-a.s.

We prove the converse later. □

Assume that f_N is a replicable payoff, i.e. there is a prefect hedge π. This means that

$$X_N^\pi = f_N \quad \text{a.s.}$$

Then the fair price for f_N is the initial cost of the portfolio, $X_0^\pi = x$. By the martingale property

$$\mathbf{E}_{\mathbf{Q}}\frac{f_N}{B_N} = \mathbf{E}_{\mathbf{Q}}\frac{X_N^\pi}{B_N} \overset{\text{mtg}}{=} \mathbf{E}_{\mathbf{Q}}\frac{X_0^\pi}{B_0} = \frac{x}{B_0}.$$

That is, the fair price x for a replicable payoff f_N is

$$x = \frac{B_0}{B_N}\mathbf{E}_{\mathbf{Q}}f.$$

In particular, it also follows that for a replicable f, the value $\mathbf{E}_{\mathbf{Q}} f$ is the same for any EMM $\mathbf{Q}$.

Summarizing, we proved the following:

{thm:pricing}

Theorem 4. *Consider an arbitrage-free market and let f be a replicable payoff. Then the fair price of f is*

$$C(f) = C_* = C^* = \frac{B_0}{B_N} \mathbf{E}_{\mathbf{Q}} f,$$

where $\mathbf{Q}$ is any EMM.

3.4 General one-step market

Assume that $B_1 = B_0(1+r)$ with a deterministic interest rate $r > -1$ and

$$S_1 = S_0(1+\rho),$$

where $\rho > -1$ is a random variable, the unique source of randomness in the model. Let

$$F(x) = \mathbf{P}(\rho \leq x), \quad x \in \mathbb{R},$$

be the distribution function of ρ. Then F induces a probability measure (denoted by $\mathbf{P}$) on the Borel sets of $(-1, \infty)$ (or $\mathbb{R}$). If F is concentrated on $\{a, b\}$ then we get back the previous one-step binomial model.

Assume without loss of generality that $B_0 = 1$. Consider a payoff function $f : \mathbb{R} \to \mathbb{R}$ as a function of the stock price S_1. A strategy π is an upper hedge if

$$\beta(1+r) + \gamma S_0(1+\rho) \geq f(S_0(1+\rho)) \quad \text{a.s.} \tag{1}$$ {eq:1step-hedge}

A probability measure on $(\mathbb{R}, \mathcal{B})$ is $\mathbf{Q}$ is EMM if $\mathbf{P} \sim \mathbf{Q}$ (meaning that $\mathbf{P}$ is absolutely continuous to $\mathbf{Q}$ ($\mathbf{P}(A) = 0$ whenever $\mathbf{Q}(A) = 0$) and conversely) if and only if S_n/B_n is $\mathbf{Q}$-martingale, that is

$$\mathbf{E}_{\mathbf{Q}} \frac{S_1}{B_1} = \frac{S_0}{B_0}.$$

This means

$$\mathbf{E}_{\mathbf{Q}} \rho = r.$$

That is a probability measure $\mathbf{Q}$ which is equivalent to $\mathbf{P}$ is EMM iff

$$\int_{\mathbb{R}} \rho \mathrm{d}\mathbf{Q}(\rho) = r.$$

Taking expectation with respect to the EMM $\mathbf{Q}$

$$\beta(1+r) + S_0\gamma(1+r) \geq \mathbf{E}_{\mathbf{Q}} f(S_0(1+\rho)).$$

For the initial cost $\beta + \gamma S_0$ we have

$$\beta + \gamma S_0 \geq \mathbf{E}_{\mathbf{Q}} \frac{f(S_0(1+\rho))}{1+r}.$$

For the class of EMM's put

$$\mathcal{P}(\mathbf{P}) = \{\mathbf{Q} : \mathbf{Q} \text{ probability measure }, \mathbf{Q} \sim \mathbf{P}, (S_n/B_n)_n \text{ is } \mathbf{Q}\text{-martingale}\}.$$

Then

$$\begin{aligned} C^*(f) &= \inf\{\beta + \gamma S_0 \ : (\beta, \gamma) \text{ is an upper hedge}\} \\ &\geq \sup_{\mathbf{Q} \in \mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(S_0(1+\rho))}{1+r}. \end{aligned} \tag{2}$$

{eq:onestep-C*lowe

Similarly, for the lower price

$$\begin{aligned} C_*(f) &= \sup\{\beta + \gamma S_0 \ : (\beta, \gamma) \text{ is a lower hedge}\} \\ &\leq \inf_{\mathbf{Q} \in \mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(S_0(1+\rho))}{1+r}. \end{aligned} \tag{3}$$

{eq:onestep-C_*upp

Assume now that $\rho \in [a, b]$ for some $-1 \leq a < b < \infty$. To ease notation put

$$f(x) = f(S_0(1+x)), \quad x \in [a, b], \tag{4}$$

{eq:onestep-f}

and assume that f is convex and continuous on $[a, b]$. By convexity,

$$\begin{aligned} f(x) &\leq \frac{f(b) - f(a)}{b - a}(1+x) + \frac{(1+b)f(a) - (1+a)f(b)}{b-a} \\ &=: \mu S_0(1+x) + \nu. \end{aligned} \tag{5}$$

{eq:conv-ineq}

Indeed, the left-hand side is a linear function equals to $f(a)$ at a, and $f(b)$ at b. Introduce the strategy,

$$\pi^* = (\beta^*, \gamma^*) := \left(\frac{\nu}{1+r}, \mu \right).$$

Then, by (5)

$$X_1^{\pi^*} = \nu + \mu S_0(1+\rho) \geq f(\rho),$$

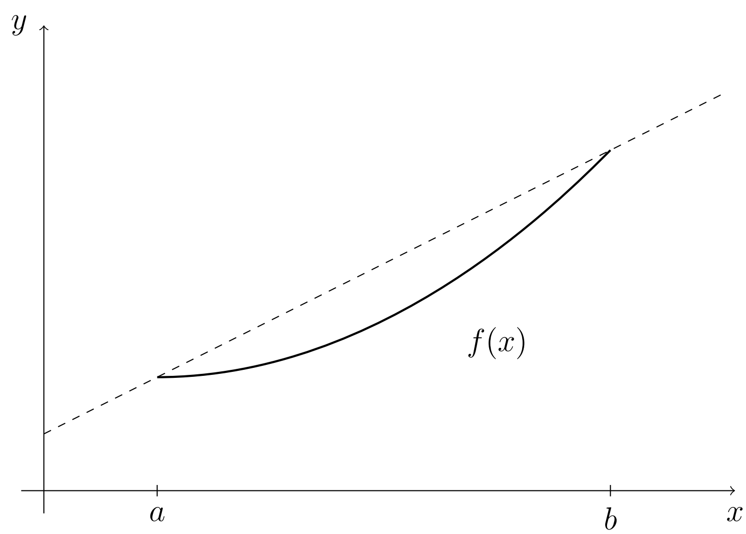

Figure 5: Bounding the upper price C^*

that is, π^* is an upper hedge. Therefore,

$$C^*(f) = \inf_{\pi \text{upper hedge}} X_0^\pi \le \beta^* + S_0\gamma^* = \frac{\nu}{1+r} + \mu S_0. \qquad (6) \quad \texttt{\{eq:C*lower\}}$$

Assumption (weak limit): In the set $\mathcal{P}(\mathbf{P})$ there exists a sequence P_n, such that P_n converges weakly to a measure $\mathbf{Q}^*$ supported on $\{a,b\}$. Since

$$\mathbf{E}_{P_n}\rho = r,$$

the equality holds for the limit

$$\mathbf{E}_{\mathbf{Q}^*}\rho = r.$$

Since $\mathbf{Q}^*$ is supported on $\{a,b\}$

$$\mathbf{E}_{\mathbf{Q}^*}\rho = \mathbf{Q}^*(\{a\})a + (1 - \mathbf{Q}^*(\{a\}))b = r,$$

implying (as in the binomial market setup) that

$$\mathbf{Q}^*(\{a\}) = \frac{b-r}{b-a}, \quad \mathbf{Q}^*(\{b\}) = \frac{r-a}{b-a}.$$

Note that $\mathbf{Q}^*$ is, in general, not equivalent to $\mathbf{P}$. In fact, it is only equivalent in the binomial market setup.

By the convergence of P_n (here we use the continuity of f)

$$\begin{aligned}\sup_{\mathbf{Q}\in\mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(\rho)}{1+r} &\geq \lim_{n\to\infty} \mathbf{E}_{P_n} \frac{f(\rho)}{1+r} \\ &= \mathbf{E}_{\mathbf{Q}^*} \frac{f(\rho)}{1+r} \\ &= \mathbf{Q}^*(\{a\}) \frac{f(a)}{1+r} + (1-\mathbf{Q}^*(\{a\})) \frac{f(b)}{1+r} \\ &= \beta^* + \gamma^* S_0 \geq C^*(f).\end{aligned}$$

Combining with (6) we obtained the following.

Theorem 5. *Assume that the payoff function is convex and continuous on $[a,b]$, and that the weak limit assumption holds. Then*

$$C^*(f) = \sup_{\mathbf{Q}\in\mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(\rho)}{1+r} = \frac{b-r}{b-a}\frac{f(a)}{1+r} + \frac{r-a}{b-a}\frac{f(b)}{1+r},$$

and the supremum is attained on the measure $\mathbf{Q}^*$.

Exercise 8. Let ρ be uniform random variable on $[a,b]$. Show that the weak limit property holds. Construct P_n explicitly!

Try to weaken the condition on the distribution of ρ.

Let's see the lower price $C_*(f)$. Assume again that f in (4) is continuous and convex. Then

$$f(\rho) \geq f(r) + (\rho - r)\lambda(r), \tag{7}$$ {eq:onestep-convex

for some $\lambda(r)$. Here $\lambda(r) = f'(r)$ if f is smooth, but this is not assumed.

If $\mathbf{Q} \in \mathcal{P}(\mathbf{P})$ then taking expectation in (7) and noting that $\mathbf{E}_{\mathbf{Q}}\rho = r$ we have

$$\inf_{\mathbf{Q}\in\mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} f(\rho) \geq f(r).$$

Consider the strategy

$$\beta_* = \frac{f(r)}{1+r} - \lambda(r), \quad \gamma_* = \frac{\lambda(r)}{S_0}.$$

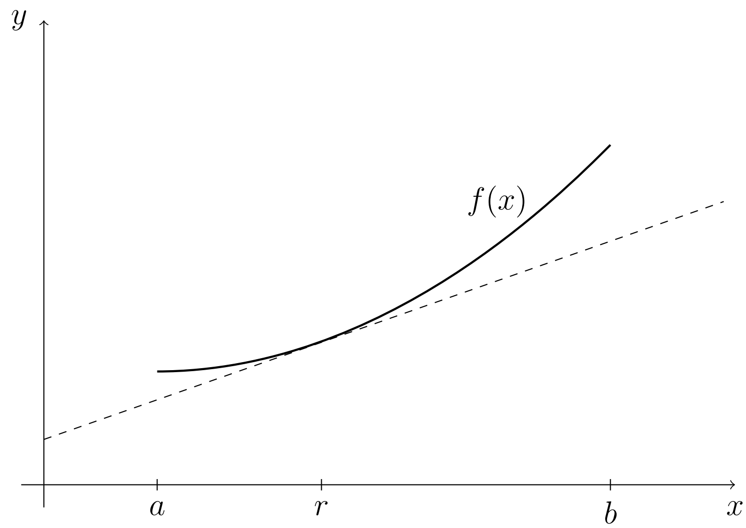

Figure 6: Bounding the lower price C_*

Then, by (7), the value at 1

$$X_1^{\pi_*} = \beta_*(1+r) + \gamma_* S_0(1+\rho) = f(r) + (\rho - r)\lambda(r) \leq f(\rho),$$

that is (β_*, γ_*) is a lower hedge.

Assumption (weak limit-2): In the set $\mathcal{P}(\mathbf{P})$ there exists a sequence P_n, such that P_n converges weakly to a measure $\mathbf{Q}_*$ concentrated at r.

Again note that $\mathbf{Q}_*$ does not belong to $\mathcal{P}(\mathbf{P})$, as it is not equivalent to any nondegenerate measure.

Then by the continuity

$$\begin{aligned}\inf_{\mathbf{Q}\in\mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(\rho)}{1+r} &\leq \lim_{n\to\infty} \mathbf{E}_{P_n} \frac{f(\rho)}{1+r} \\ &= \frac{f(r)}{1+r} = \beta_* + S_0\gamma_* \\ &\leq \sup\{\beta + \gamma S_0 \ : (\beta,\gamma) \text{ lower hedge }\} = C_*.\end{aligned}$$

Combined with (2) we obtain the following.

Theorem 6. *Let f be a convex continuous function on $[a, b]$, and assume* *that the weak limit-2 assumption holds. Then*

$$C_*(f) = \inf_{\mathbf{Q}\in\mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(\rho)}{1+r} = \frac{f(r)}{1+r},$$

and the infimum is attained at the measure $\mathbf{Q}_*$.

3.5 Complete markets

We proved that if EMM exists then we have the fair price for any replicable payoff. A market is *complete* if any payoff is replicable.

We have seen in Theorem 4 that on a complete arbitrage-free market any payoff f has a unique well-defined fair price $B_0\mathbf{E}_{\mathbf{Q}} f/B_N$.

In section 2.4 we showed that a binomial market is complete.

The second fundamental theorem of asset pricing is the following.

{thm:complete-mark

Theorem 7. *Consider an arbitrage-free market with EMM* $\mathbf{Q}$. *Then the following are equivalent:*

(i) the market is complete;

(ii) $\mathbf{Q}$ *is the unique EMM;*

(iii) for any $\mathbf{Q}$*-martingale* (M_n) *there exists a predictable sequence* γ_n *such that* M_n *can be represented as*

$$M_n = M_0 + \sum_{k=1}^{n} \gamma_k \left(\frac{S_k}{B_k} - \frac{S_{k-1}}{B_{k-1}} \right) = M_0 + \sum_{k=1}^{n} \sum_{i=1}^{d} \gamma_k^i \left(\frac{S_k^i}{B_k} - \frac{S_{k-1}^i}{B_{k-1}} \right).$$

Proof. We prove again the easy parts (i) ⇒ (ii), and (iii) ⇔ (i), and postpone the difficult (ii) ⇒ (i) implication later.

(i) ⇒ (ii): Assume that $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are EMM's. Consider any $A \in \mathcal{F}$. We show that $\mathbf{Q}_1(A) = \mathbf{Q}_2(A)$ implying the uniqueness. Let π be a perfect hedge to $f = I_A$. Then X_n^π / B_n is both $\mathbf{Q}_1$ and $\mathbf{Q}_2$ martingale, so

$$\mathbf{Q}_1(A) = \mathbf{E}_{\mathbf{Q}_1} f = \mathbf{E}_{\mathbf{Q}_1} X_N^\pi = B_N \mathbf{E}_{\mathbf{Q}_1} \frac{X_N^\pi}{B_N} = B_N \frac{X_0^\pi}{B_0} = \ldots = \mathbf{Q}_2(A).$$

(i) ⇒ (iii): Consider a $\mathbf{Q}$-martingale M_n. There exists a strategy π_n such that a.s.

$$X_N^\pi = B_N M_N.$$

Using that both M_n and X_n^π/B_n are martingales

$$M_n = \mathbf{E}_{\mathbf{Q}}[M_N|\mathcal{F}_n] = \mathbf{E}_{\mathbf{Q}}\left[\frac{X_N^\pi}{B_N}|\mathcal{F}_n\right] = \frac{X_n^\pi}{B_n} = \beta_n + \gamma_n \frac{S_n}{B_n}.$$

Thus, using that π is SF

$$\begin{aligned} M_n - M_{n-1} &= \Delta\beta_n + \gamma_n \frac{S_n}{B_n} - \gamma_{n-1}\frac{S_{n-1}}{B_{n-1}} \\ &= \gamma_n \left(\frac{S_n}{B_n} - \frac{S_{n-1}}{B_{n-1}}\right) + \frac{1}{B_{n-1}}\left(B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n\right) \\ &= \gamma_n \left(\frac{S_n}{B_n} - \frac{S_{n-1}}{B_{n-1}}\right), \end{aligned}$$

as claimed.

(iii) $\Rightarrow$ (i): Consider a payoff f. We are looking for a strategy π such that $X_N^\pi = f$ $\mathbf{Q}$-a.s. We know that $(X_n^\pi/B_n)_n$ is a martingale, so this should be (M_n). Now the following choice is clear: let

$$M_n = \mathbf{E}_{\mathbf{Q}}\left[\frac{f}{B_N}|\mathcal{F}_n\right].$$

Then M_n is a martingale, therefore by the assumption

$$M_n = M_0 + \sum_{k=1}^{n} \gamma_k \Delta \frac{S_k}{B_k}.$$

Let

$$\beta_n = M_n - \gamma_n \frac{S_n}{B_n},$$

and consider the strategy $\pi_n = (\beta_n, \gamma_n)$. To see that this is indeed a strategy we have to show that it is predictable and SF. The sequence γ_n is predictable by the assumption (iii), and β_n is predictable because all the terms in M_n are $\mathcal{F}_{n-1}$-measurable except $\gamma_n S_n/B_n$, which is subtracted. To see that it is SF note that

$$\begin{aligned} &B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n \\ &= B_{n-1}\left(M_n - M_{n-1} - \gamma_n\frac{S_n}{B_n} + \gamma_{n-1}\frac{S_{n-1}}{B_{n-1}}\right) + S_{n-1}\Delta\gamma_n \\ &= B_{n-1}\left(\gamma_n \Delta\frac{S_n}{B_n} - \gamma_n\frac{S_n}{B_n} + \gamma_{n-1}\frac{S_{n-1}}{B_{n-1}}\right) + S_{n-1}\Delta\gamma_n = 0, \end{aligned}$$

showing that π is SF. It is clearly a perfect hedge since

$$X_N^\pi = \beta_N B_N + \gamma_N S_N = B_N M_N = f,$$

as claimed. $\square$

3.6 Proof of the difficult part of Theorem 3

Here we use strongly that Ω is finite, and let $|\Omega| = k$.

Assume that there is no arbitrage strategy. Let

$$\mathcal{V}_0 = \{X : \Omega \to \mathbb{R} \text{ r.v. } | \exists \pi : X_0^\pi = 0 \text{ and } X_N^\pi = X\},$$

and

$$\mathcal{V}_1 = \{X : \Omega \to \mathbb{R} \text{ r.v. } | X \geq 0, \mathbf{E}X \geq 1\}.$$

We identify a random variable $X : \Omega \to \mathbb{R}$ with a vector in $\mathbb{R}^k$, as $X \leftrightarrow (X(\omega_1), \ldots, X(\omega_k))$. Clearly, $\mathcal{V}_0$ is a linear subspace and $\mathcal{V}_1$ is convex set in $\mathbb{R}^k$.

Since there is no arbitrage strategy, $\mathcal{V}_0 \cap \mathcal{V}_1 = \emptyset$. Therefore, by the Kreps–Yan theorem, there exists a linear functional $\ell : \mathbb{R}^k \to \mathbb{R}$ such that $\ell|_{\mathcal{V}_0} \equiv 0$ and $\ell(v_1) > 0$ for all $v_1 \in \mathcal{V}_1$. A linear function in $\mathbb{R}^k$ (in any Hilbert space) is a inner product, thus there exists $q \in \mathbb{R}^k$ such that

$$\ell(v) = \langle v, q \rangle.$$

Define the random variables

$$X_i(\omega_j) = \delta_{i,j} \frac{1}{\mathbf{P}(\{\omega_i\})}.$$

Then $X_i \geq 0$ and $\mathbf{E}X_i = 1$, so $X_i \in \mathcal{V}_1$. Furthermore

$$\ell(X_i) = \frac{q_i}{\mathbf{P}(\{\omega_i\})} > 0,$$

implying $q_i > 0$ for any i. Define the probability measure $\mathbf{Q}$ as

$$\mathbf{Q}(\{\omega_i\}) = \frac{q_i}{\sum_{i=1}^k q_i}.$$

It is clear that $\mathbf{Q} \sim \mathbf{P}$. We have to check that (S_n/B_n) is a $\mathbf{Q}$-martingale. First we need a lemma.

Lemma 5. *Let* $(X_n)_{n=1}^N$ *be an adapted process. If for any stopping time* $\tau : \Omega \to \{0, \dots, N\}$

$$\mathbf{E}X_\tau = \mathbf{E}X_0,$$

then (X_n) *is martingale.*

Proof. We show that $X_n = \mathbf{E}[X_N | \mathcal{F}_n]$, which implies that X is martingale.

Let $A \in \mathcal{F}_n$ and consider the stopping time

$$\tau_A(\omega) = \begin{cases} n, & \omega \in A, \\ N, & \text{otherwise.} \end{cases}$$

This is indeed a stopping time, since $\{\tau_A \le k\} = \emptyset$ for $k < n$, and A for $k \ge n$, which is $\mathcal{F}_k$-measurable. Then, by the assumption

$$\mathbf{E}X_0 = \mathbf{E}X_{\tau_A} = \mathbf{E}X_n I(A) + \mathbf{E}X_N I(A^c).$$

With $A = \emptyset$ we see that $\mathbf{E}X_0 = \mathbf{E}X_N$, implying

$$\mathbf{E}X_n I(A) = \mathbf{E}X_N I(A).$$

This exactly means that

$$X_n = \mathbf{E}[X_N | \mathcal{F}_n],$$

as claimed. □

We show that (S_n/B_n) satisfies the condition of the lemma above. Let τ be a stopping time and define the strategy

$$\beta_n = \frac{S_\tau}{B_\tau} I(\tau \le n-1) - \frac{S_0}{B_0}, \quad \gamma_n = I(\tau > n-1).$$

Since $\{\tau < n\} = \{\tau \le n-1\} \in \mathcal{F}_{n-1}$, the sequence (β_n, γ_n) is predictable. Furthermore,

$$B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n = \frac{S_\tau}{B_\tau} B_{n-1} I(\tau = n-1) - S_{n-1} I(\tau = n-1) = 0,$$

so it is SF. Finally,

$$X_0^\pi = -\frac{S_0}{B_0} B_0 + S_0 = 0,$$

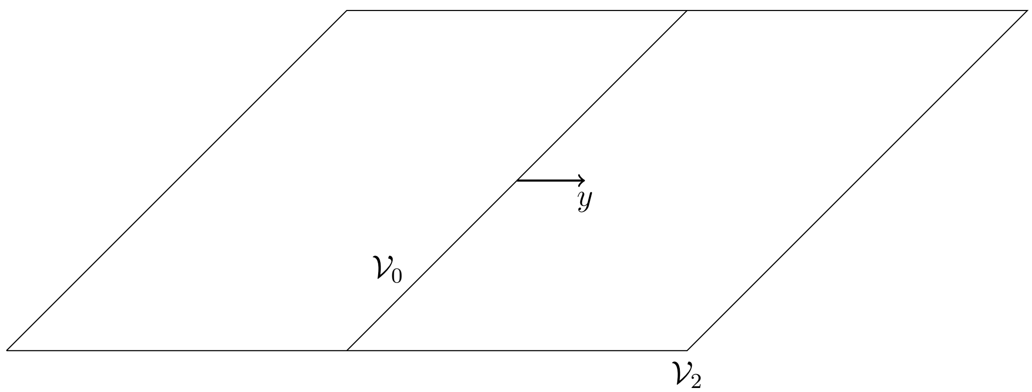

Figure 7: Choice of y

so $X_N^\pi \in \mathcal{V}_0$. Therefore

$$\begin{aligned}
0 = \mathbf{E}_{\mathbf{Q}} X_N^\pi &= \mathbf{E}_{\mathbf{Q}} \beta_N B_N + \gamma_N S_N \\
\mathbf{E}_{\mathbf{Q}} &\left(\left(\frac{S_\tau}{B_\tau} I(\tau \leq N-1) - \frac{S_0}{B_0} \right) B_N + \frac{S_\tau}{B_\tau} I(\tau = N) B_N \right) \\
&= B_N \mathbf{E}_{\mathbf{Q}} \left(\frac{S_\tau}{B_\tau} - \frac{S_0}{B_0} \right).
\end{aligned}$$

That is (S_n/B_n) is indeed a $\mathbf{Q}$-martingale.

3.7 Proof of the difficult part of Theorem 7

Here we prove the implication (ii) $\Rightarrow$ (i).

We use the notation of the previous proof. Let

$$\mathcal{V}_2 = \{X : \Omega \to \mathbb{R} \text{ r.v. } | \mathbf{E}_{\mathbf{Q}} X = 0\}.$$

Then $\mathcal{V}_2$ is a linear subspace in $\mathbb{R}^k$ and we have seen in the previous proof that $\mathcal{V}_0 \subset \mathcal{V}_2$. We claim that equality holds.

Assume first that this is indeed true. Then for any claim X the centered version $X - \mathbf{E}_{\mathbf{Q}} X \in \mathcal{V}_2 = \mathcal{V}_0$, meaning that there is a perfect hedge. Thus the market is complete. So we only have to show that $\mathcal{V}_0 = \mathcal{V}_2$.

Assume on the contrary that $\mathcal{V}_0 \neq \mathcal{V}_2$. Then there is an $y \in \mathcal{V}_2$, which is orthogonal to $\mathcal{V}_0$. Since $q_i > 0$ (see the previous proof) for all $i = 1, \ldots, k$, we may choose $\varepsilon > 0$ small enough such that

$$q_i' = q_i - \varepsilon y_i > 0 \quad \text{for all } i.$$

As both q and y are orthogonal to $\mathcal{V}_0$, q' is also orthogonal. Define the measure

$$\mathbf{Q}'(\{\omega_i\}) = \frac{q_i'}{\sum_{i=1}^k q_i'}.$$

Exactly as in the previous proof we can show that $\mathbf{Q}'$ is EMM. The uniqueness of the EMM implies

$$\frac{q_i'}{\sum_{i=1}^k q_i'} = \frac{q_i}{\sum_{i=1}^k q_i},$$

that is, using also the definition of q',

$$q = \alpha q' = \alpha q - \alpha \varepsilon y,$$

with $\alpha = \sum q_i / \sum q_i'$. Thus

$$(1 - \alpha) q = -\alpha \varepsilon y.$$

But y and q are orthogonal, which is a contradiction. The proof is complete.

4 Girsanov's theorem in discrete time

4.1 Second proof of the difficult part of Theorem 3

Assume that $d = 1$ and first consider the one-step model with $B_0 = B_1 = 1$. The stock price S_0 is known, and the only randomness here is S_1.

Exercise 9. The no arbitrage assumption (in this simple market) is equivalent to

$$\mathbf{P}(\Delta S_1 > 0)\mathbf{P}(\Delta S_1 < 0) > 0.$$

Furthermore, (S_n) is $\mathbf{Q}$-martingale if

$$\mathbf{E}_\mathbf{Q} S_1 = S_0.$$

Therefore we have to construct a measure $\mathbf{Q}$ such that $\mathbf{E}_\mathbf{Q} \Delta S_1 = 0$. This is done in the following lemma.

Lemma 6. *Let X be a random variable on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbf{P})$ such that $\mathbf{P}(X > 0)\mathbf{P}(X < 0) > 0$. Then there exists a probability measure $\mathbf{Q} \sim \mathbf{P}$ such that $\mathbf{E}_{\mathbf{Q}} X = 0$. Furthermore, for any $a \in \mathbb{R}$*

$$\mathbf{E}_{\mathbf{Q}} e^{aX} < \infty.$$

Proof. Define the probability measure

$$P_1(\mathrm{d}x) = ce^{-x^2} F(\mathrm{d}x),$$

where $F(x) = \mathbf{P}(X \le x)$ and $c^{-1} = \int_{\mathbb{R}} e^{-x^2} F(\mathrm{d}x)$. That is

$$P_1(A) = \int_A ce^{-x^2} F(\mathrm{d}x).$$

Then P_1 is equivalent to F. (Recall that μ is absolute continuous with respect to ν, $\mu \ll \nu$ if $\mu(A) = 0$ whenever $\nu(A) = 0$. And μ and ν are equivalent, $\mu \sim \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$.) Let

$$\varphi(a) = \mathbf{E}_{P_1} e^{aX} = \int_{\mathbb{R}} e^{ax} P_1(\mathrm{d}x) = c \int_{\mathbb{R}} e^{ax - x^2} F(\mathrm{d}x).$$

Clearly, $\varphi(a) < \infty$ for any a as the function e^{ax-x^2} is bounded on $\mathbb{R}$. Note that φ is convex, because $\varphi'' > 0$. Put

$$Z_a(x) = \frac{e^{ax}}{\varphi(a)}.$$

Then

$$Q_a(\mathrm{d}x) = Z_a(x) P_1(\mathrm{d}x)$$

is a probability measure for any a, and $Q_a \sim P_1 \sim F$. Again, this means

$$Q_a(A) = \int_A Z_a(x) P_1(\mathrm{d}x) = \frac{c}{\varphi(a)} \int_A e^{ax - x^2} F(\mathrm{d}x).$$

Let

$$\varphi_* = \inf_{a \in \mathbb{R}} \varphi(a).$$

Since $P_1(X > 0) > 0$ and $P_1(X < 0) > 0$ we obtain that

$$\lim_{a \to \pm\infty} \varphi(a) = \infty.$$

Therefore, the infimum is attained, i.e. there is a_* such that $\varphi(a_*) = \varphi_*$. Then $\varphi'(a_*) = 0$, thus

$$0 = \varphi'(a_*) = \mathbf{E}_{P_1} X e^{a_* X} = \varphi(a_*) \mathbf{E}_{P_1} X \frac{e^{a_* X}}{\varphi(a_*)} = \varphi(a_*) \mathbf{E}_{Q_{a_*}} X.$$

Thus the measure Q_{a_*} works. □

Exercise 10. Prove rigorously that

$$\lim_{a \to \pm\infty} \varphi(a) = \infty.$$

Exercise 11. Let $X \sim N(\mu, \sigma^2)$. Determine the measure constructed above explicitly.

Next we extend the previous lemma for a general N-step market.

Exercise 12. The no arbitrage assumption implies that for any n a.s.

$$\mathbf{P}(\Delta S_n > 0 | \mathcal{F}_{n-1}) \mathbf{P}(\Delta S_n < 0 | \mathcal{F}_{n-1}) > 0.$$

As a preliminary result we have to understand how to compute conditional expectation under different measures.

{lemma:condexp-mea

Lemma 7. *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,1,\ldots,N}, \mathbf{P})$ *a filtered probability space, and* Z *a nonnegative random variable* $\mathbf{E}_\mathbf{P} Z = 1$. *Define the new probability measure* $\mathbf{Q}$ *as*

$$\mathrm{d}\mathbf{Q} = Z \mathrm{d}\mathbf{P},$$

that is

$$\mathbf{Q}(A) = \int_A Z \mathrm{d}\mathbf{P}.$$

Put $Z_n = \mathbf{E}_\mathbf{P}[Z | \mathcal{F}_n]$. *For any adapted process* (X_n)

$$Z_{n-1} \mathbf{E}_\mathbf{Q}[X_n | \mathcal{F}_{n-1}] = \mathbf{E}_\mathbf{P}[X_n Z_n | \mathcal{F}_{n-1}].$$

Proof. Both sides are $\mathcal{F}_{n-1}$-measurable. We have to prove that for any $A \in \mathcal{F}_{n-1}$

$$\int_A Z_{n-1} \mathbf{E}_\mathbf{Q}[X_n | \mathcal{F}_{n-1}] \mathrm{d}\mathbf{P} = \int_A X_n Z_n \mathrm{d}\mathbf{P}. \qquad (8)$$ {eq:cemlemma-0}

First note that

$$\mathbf{E}_{\mathbf{P}}[ZX_n|\mathcal{F}_n] = X_n\mathbf{E}_{\mathbf{P}}[Z|\mathcal{F}_n] = X_nZ_n. \tag{9}$$ {eq:cemlemma-1}

Therefore, for an $\mathcal{F}_{n-1}$-measurable Y

$$\mathbf{E}_{\mathbf{P}}[Z_{n-1}Y|\mathcal{F}_{n-1}] = Y\mathbf{E}_{\mathbf{P}}[Z|\mathcal{F}_{n-1}],$$

implying for any $A \in \mathcal{F}_{n-1}$ that

$$\begin{aligned}\int_A Z_{n-1}Y\mathrm{d}\mathbf{P} &= \int_A Y\mathbf{E}_{\mathbf{P}}[Z|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{P} \\ &= \int_A \mathbf{E}_{\mathbf{P}}[ZY|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{P} = \int_A YZ\mathrm{d}\mathbf{P}.\end{aligned}$$

Choosing $Y = \mathbf{E}_{\mathbf{Q}}[X_n|\mathcal{F}_{n-1}]$ we obtain

$$\begin{aligned}\int_A Z_{n-1}\mathbf{E}_{\mathbf{Q}}[X_n|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{P} &= \int_A \mathbf{E}_{\mathbf{Q}}[X_n|\mathcal{F}_{n-1}]Z\mathrm{d}\mathbf{P} \\ &= \int_A \mathbf{E}_{\mathbf{Q}}[X_n|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{Q} && \text{definition of } \mathbf{Q} \\ &= \int_A X_n\mathrm{d}\mathbf{Q} && \text{conditional exp.} \\ &= \int_A X_nZ\mathrm{d}\mathbf{P} && \text{definition of } \mathbf{Q} \\ &= \int_A X_nZ_n\mathrm{d}\mathbf{P}, && \text{by (9)}\end{aligned}$$

which is (8). □

As a simple but useful corollary we obtain the following.

{cor:p-q-mtg}

Corollary 1. *The adapted process (X_n) is $\mathbf{Q}$-martingale if and only if (X_nZ_n) is $\mathbf{P}$-martingale.*

{lemma:existence-e

Lemma 8. *Let $(X_n)_{n=1}^N$ be an adapted process, and assume that*

$$\mathbf{P}(X_n > 0|\mathcal{F}_{n-1})\mathbf{P}(X_n < 0|\mathcal{F}_{n-1}) > 0.$$

Then there exists a probabilty measure $\mathbf{Q} \sim \mathbf{P}$ such that (X_n) is a $\mathbf{Q}$-martingale difference.

Proof. First let

$$P_1(\mathrm{d}\omega) = c \exp\left\{-\sum_{i=0}^{N} X_i^2(\omega)\right\} \mathbf{P}(\mathrm{d}\omega),$$

where c is the normalizing factor, i.e.

$$c^{-1} = \int_\Omega \exp\left\{-\sum_{i=0}^{N} X_i^2\right\} \mathrm{d}\mathbf{P} = \mathbf{E} \exp\left\{-\sum_{i=0}^{N} X_i^2\right\}.$$

This means that for $A \in \mathcal{F}$

$$P_1(A) = c \int_A \exp\left\{-\sum_{i=0}^{N} X_i^2\right\} \mathrm{d}\mathbf{P}.$$

Let

$$\varphi_n(a) = \mathbf{E}_{P_1}[e^{aX_n} | \mathcal{F}_{n-1}].$$

Note that this is an $\mathcal{F}_{n-1}$-measurable random variable. As in the proof of the previous lemma there is a unique finite a_n (random!) such that the infimum of φ_n is attained at a_n. Since φ_n is $\mathcal{F}_{n-1}$-measurable so is a_n.

Let $Z_0 = 1$, and recursively

$$Z_n = Z_{n-1} \frac{e^{a_n X_n}}{\mathbf{E}_{P_1}[e^{a_n X_n} | \mathcal{F}_{n-1}]}.$$

Then (Z_n) is a P_1-martingale, since

$$\mathbf{E}_{P_1}[Z_n | \mathcal{F}_{n-1}] = Z_{n-1}.$$

Then the probability measure

$$\mathbf{Q}(\mathrm{d}\omega) = Z_N(\omega) P_1(\mathrm{d}\omega)$$

works. Indeed,

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}}[X_n | \mathcal{F}_{n-1}] &= \frac{1}{Z_{n-1}} \mathbf{E}_{P_1}[Z_n X_n | \mathcal{F}_{n-1}] && \text{by Lemma 7} \\
&= \frac{1}{Z_{n-1}} \frac{Z_{n-1}}{\mathbf{E}_{P_1}[e^{a_n X_n} | \mathcal{F}_{n-1}]} \mathbf{E}_{P_1}[X_n e^{a_n X_n} | \mathcal{F}_{n-1}] && \text{definition} \\
&= \frac{1}{\mathbf{E}_{P_1}[e^{a_n X_n} | \mathcal{F}_{n-1}]} \cdot 0 = 0. && \text{choice of } a_n
\end{aligned}$$

□

Exercise 13. Show that a_n is $\mathcal{F}_{n-1}$-measurable.

Now we can return to the proof of Theorem 3. The existence of the martingale measure follows from the previous lemma applied to $X_n = \Delta S_n$.

4.2 ARCH processes

Autoregressive conditional heteroscedasticity (ARCH) models were introduced by Robert Engle in 1982 to model log-returns. In 2003 he obtained Nobel prize in economics for this model. The novelty in these models is the stochastic volatility term.

Let

$$R_n = \log \frac{S_n}{S_{n-1}}$$

denote the log-return of the stock, and assume that

$$R_n = \mu_n + \sqrt{\beta + \lambda R_{n-1}^2}\xi_n,$$

where ξ_n's are iid $N(0,1)$ random variables. Then (R_n) is an ARCH(1) process. That is conditionally on $\mathcal{F}_{n-1}$ the log-return R_n is Gaussian with mean μ_n, and variance $\beta + \lambda R_{n-1}^2$. Write $\sigma_n = \beta + \lambda R_{n-1}^2$. Then for S_n we obtain

$$\begin{aligned} S_n = S_{n-1}e^{R_n} &= S_0 \exp\left\{ \sum_{k=1}^{n} \left(\mu_k + \sqrt{\beta + \lambda R_{k-1}^2}\xi_k \right) \right\} \\ &= S_0 \exp\left\{ \sum_{k=1}^{n} (\mu_k + \sigma_k \xi_k) \right\}. \end{aligned}$$

In what follows we only assume that μ_n and σ_n are $\mathcal{F}_{n-1}$-measurable, i.e. the sequence $(\mu_n, \sigma_n)_n$ is predictable, and (ξ_n) is adapted, ξ_n is independent of $\mathcal{F}_{n-1}$, and $N(0,1)$ distributed. Put $h_n = \mu_n + \sigma_n \xi_n$. For simplicity we assume that $B_n \equiv 1$.

We construct a measure $\mathbf{Q}$ such that (S_n) is a $\mathbf{Q}$-martingale. Let

$$Z_N = \prod_{n=1}^{N} z_n := \prod_{n=1}^{N} \frac{e^{a_n h_n}}{\mathbf{E_P}[e^{a_n h_n} | \mathcal{F}_{n-1}]},$$

where

$$a_n = -\frac{\mu_n}{\sigma_n^2} - \frac{1}{2}. \qquad (10) \quad \texttt{\{eq:disc-girs-0\}}$$

Introduce the new measure $\mathbf{Q}$ as

$$\mathrm{d}\mathbf{Q} = Z_N \mathrm{d}\mathbf{P},$$

and let $Z_n = \mathbf{E_P}[Z_N|\mathcal{F}_n] = \prod_{i=1}^n z_i$.

By Corollary 1, to show that S_n is $\mathbf{Q}$-martingale we have to show that $S_n Z_n$ is a $\mathbf{P}$-martingale. We have

$$\mathbf{E_P}[S_n Z_n|\mathcal{F}_{n-1}] = S_{n-1}Z_{n-1}\frac{\mathbf{E_P}[e^{h_n(1+a_n)}|\mathcal{F}_{n-1}]}{\mathbf{E_P}[e^{a_n h_n}|\mathcal{F}_{n-1}]}.$$

Therefore we have to check that

$$\mathbf{E_P}[e^{h_n(1+a_n)}|\mathcal{F}_{n-1}] = \mathbf{E_P}[e^{a_n h_n}|\mathcal{F}_{n-1}]. \qquad (11) \quad \texttt{\{eq:disc-girs-1\}}$$

Recall that for a standard normal ξ

$$\mathbf{E}e^{t\xi} = e^{\frac{t^2}{2}},$$

thus

$$\mathbf{E}e^{\mu+\sigma\xi} = e^{\mu+\frac{\sigma^2}{2}}.$$

Since a_n in (11) is $\mathcal{F}_{n-1}$-measurable and given $\mathcal{F}_{n-1}$ the variable h_n is Gaussian $\mathrm{N}(\mu_n, \sigma_n^2)$, we obtain

$$\mathbf{E_P}[e^{h_n(1+a_n)}|\mathcal{F}_{n-1}] = e^{\mu_n(1+a_n)+\frac{\sigma_n^2(1+a_n)^2}{2}},$$

and

$$\mathbf{E_P}[e^{h_n a_n}|\mathcal{F}_{n-1}] = e^{\mu_n a_n+\frac{\sigma_n^2 a_n^2}{2}},$$

By the choice of a_n in (10)

$$\mu_n(1+a_n) + \frac{\sigma_n^2(1+a_n)^2}{2} = \mu_n a_n + \frac{\sigma_n^2 a_n^2}{2}.$$

Indeed, by (10)

$$\mu_n + \sigma_n^2\left(\frac{1}{2}+a_n\right) = 0.$$

That is, (11) holds.

We proved the following.

Theorem 8 (Discrete Girsanov's theorem)**.** *Let* $(\mu_n, \sigma_n)_n$ *be a predictable sequence and assume that the stock prices are given by*

$$S_n = e^{\sum_{k=1}^n (\mu_k + \sigma_k \xi_k)},$$

where $(\xi_n)_n$ *is a adapted sequence of* $N(0,1)$ *random variables,* ξ_n *is independent of* $\mathcal{F}_{n-1}$*. Further, let* $B_n \equiv 1$*. Then, under the new measure*

$$\mathrm{d}\mathbf{Q} = Z_N \mathrm{d}\mathbf{P}$$

(S_n) *is a martingale.*

5 Pricing and hedging European options

In this section we summarize our findings on pricing and hedging, and consider some special cases in detail.

5.1 Complete markets

Consider an arbitrage-free complete market. The *fair price* of the contingent claim f_N is

$$C(f_N) = \inf\{x : \exists \pi, X_0^\pi = x, \, X_N^\pi = f_N\}.$$

Then, by Theorems 3 and 7 there exists a unique EMM $\mathbf{Q}$. Since (X_n^π / B_n) is $\mathbf{Q}$-martingale

$$\mathbf{E}_\mathbf{Q} \frac{f_N}{B_N} = \mathbf{E}_\mathbf{Q} \frac{X_N^\pi}{B_N} = \mathbf{E}_\mathbf{Q} \frac{x}{B_0} = \frac{x}{B_0},$$

therefore

$$C(f_N) = x = \frac{B_0}{B_N} \mathbf{E}_\mathbf{Q} f_N.$$

Note that x is independent of the hedge π itself, that is for different hedges the initial value is the same.

For a hedge we need to know not only the fair price C, but also the strategy π itself. For the given claim f_N consider the martingale

$$M_n = \mathbf{E}_\mathbf{Q} \left[\frac{f_N}{B_N} \middle| \mathcal{F}_n \right].$$

By Theorem 7 there exists a representation

$$M_n = M_0 + \sum_{k=1}^{n} \gamma_k \Delta \frac{S_k}{B_k},$$

with a predictable sequence (γ_n). Let

$$\beta_n = M_n - \frac{\gamma_n S_n}{B_n}.$$

We proved that $\pi = (\beta_n, \gamma_n)_n$ is an SF strategy and is a perfect hedge for f_N.

Summarizing, we obtained the following.

Theorem 9. *In an arbitrary arbitrage-free complete market the price of the contingent claim f_N is*

$$C(f_N) = B_0 \mathbf{E}_{\mathbf{Q}} \frac{f_N}{B_N}.$$

Moreover, there exists a strategy π which is a perfect hedge of f_N, i.e.

$$X_N^\pi = f_N,$$

where (β_n, γ_n) are given above. The value process is determined by

$$X_n^\pi = B_n \mathbf{E}_{\mathbf{Q}} \left[\frac{f_N}{B_N} \middle| \mathcal{F}_n \right].$$

5.2 Homogeneous binomial market – CRR formula

Consider a homogeneous binomial N-step market with $a < r < b$. That is

$$B_n = (1+r)^n, \quad S_n = S_0 \prod_{k=1}^{n} (1 + \rho_k),$$

where $\rho_k \in \{a, b\}$. We proved that this market is arbitrage-free and complete, and the unique EMM is given by

$$\mathbf{Q}(\rho_i = a) = \frac{b-r}{b-a},$$

and ρ_i's are independent. If the claim f_N only depends on the final price S_N, and not on the whole trajectory, i.e.

$$f_N(\omega) = f_N(S_N(\omega)),$$

then the pricing formula simplifies, and we obtain the Cox–Ross-Rubinstein formula:

$$C(f_N) = \frac{1}{(1+r)^N} \sum_{k=0}^{N} f_N(S_0(1+b)^k(1+a)^{N-k}) \binom{N}{k} q^k (1-q)^{N-k},$$

where $q = \frac{r-a}{b-a}$.

5.3 Incomplete markets

We assume that the market is arbitrage-free, but there are various EMM's. Let $\mathcal{P}(\mathbf{P})$ be the set of EMM's.

In incomplete markets there are contingent claims which are not replicable, that is, there is no perfect hedge. The upper price of a claim f_N is

$$C^*(f_N) = \inf\{x : \pi,\, X_0^\pi = x,\, X_N^\pi \geq f_N\}.$$

We proved the following result in a one-step market. Without a proof we state the general version.

Theorem 10. *The upper price of the claim f_N in an arbitrage-free incomplete market is given by*

$$C^*(f_N) = \sup_{\mathbf{Q} \in \mathcal{P}(\mathbf{P})} B_0 \mathbf{E}_\mathbf{Q} \frac{f_N}{B_N}.$$

6 American options

While European options can be exercised only at the terminal date N, American options can be exercised at any time. Formally, instead of a fixed random payoff function f_N, a sequence of payoffs $(f_n)_{n=0,1,\ldots,N}$ is given, where f_n is $\mathcal{F}_n$-measurable, i.e. $(f_n)_n$ is adapted to $(\mathcal{F}_n)_n$. So f_n is the random payoff if the option is exercised at time n. Clearly, the exercise time has to be a stopping time.

6.1 Reminder: Doob's optional sampling theorem

11. Theorem (Opcionális megállási tétel (Doob))**.** *Legyen* $(X_n, \mathcal{F}_n)$ *szubmartingál,* σ, τ *megállási idők,* $\sigma \leq \tau$ *m.b. Tegyük fel, hogy* $\mathbf{E}|X_\sigma| < \infty$, $\mathbf{E}|X_\tau| < \infty$ *és* $\liminf_{n\to\infty} \int_{\{\tau>n\}} |X_n| \mathrm{d}\mathbf{P} = 0$*. Ekkor* $\mathbf{E}[X_\tau|\mathcal{F}_\sigma] \geq X_\sigma$ *m.b.*

A tétel feltételei mellett, ha $\{X_n, \mathcal{F}_n\}$ martingál, akkor $\mathbf{E}[X_\tau|\mathcal{F}_\sigma] = X_\sigma$ m.b.

Bizonyítás. Tegyük fel, hogy τ korlátos, $\tau \leq m$. A szubmartingál és a feltételes várható érték definíciója szerint azt kell megmutatnunk, hogy minden $A \in \mathcal{F}_\sigma$ eseményre

$$\int_A (X_\tau - X_\sigma) \, \mathrm{d}\mathbf{P} \geq 0.$$

Írjuk át az integrandust

$$X_\tau - X_\sigma = \sum_{k=\sigma+1}^{\tau} (X_k - X_{k-1}) = \sum_{k=2}^{m} I(\sigma < k \leq \tau)(X_k - X_{k-1})$$

alakba. Vegyük észre, hogy $A \cap \{\sigma < k \leq \tau\} = (A \cap \{\sigma \leq k-1\}) \cap \{\tau \leq k-1\}^c$. Itt a metszet első tagja $\mathcal{F}_\sigma$ definíciója szerint $\mathcal{F}_{k-1}$-mérhető, a második tagja pedig a megállási idő definíciója szerint, így a metszet maga is elem az $\mathcal{F}_{k-1}$ σ-algebrának. Ezt, feltételes várható érték és a szubmartingál definícióját használva

$$\begin{aligned}
\int_A (X_\tau - X_\sigma) \mathrm{d}\mathbf{P} &= \int_A \sum_{k=2}^{m} I(\sigma < k \leq \tau)(X_k - X_{k-1}) \mathrm{d}\mathbf{P} \\
&= \sum_{k=2}^{m} \int_{A \cap \{\sigma < k \leq \tau\}} (X_k - X_{k-1}) \mathrm{d}\mathbf{P} \\
&= \sum_{k=2}^{m} \int_{A \cap \{\sigma < k \leq \tau\}} (\mathbf{E}[X_k|\mathcal{F}_{k-1}] - X_{k-1}) \, \mathrm{d}\mathbf{P} \geq 0,
\end{aligned}$$

ami éppen a bizonyítandó.

Az általános esetben a τ, σ megállási időkről át kell térni a $\tau \wedge n$, $\sigma \wedge n$ korlátos megállási időkre, és belátni, hogy a kimaradó tagok 0-hoz tartanak egy részsorozat mentén. Ezt nem bizonyítjuk, a részletekért lásd [1]. □

2. Corollary. *Ha* $\mathbf{E}|X_\tau| < \infty$ *és* $\liminf_{n\to\infty} \int_{\{\tau>n\}} |X_n| \mathrm{d}\mathbf{P} = 0$*, akkor ha* *(i)* $\{X_n, \mathcal{F}_n\}$ *szubmartingál, akkor* $\mathbf{E}[X_\tau|\mathcal{F}_1] \geq X_1$ *m.b., és persze* $\mathbf{E}X_\tau \geq \mathbf{E}X_1$*;*

(ii) $\{X_n, \mathcal{F}_n\}$ *martingál, akkor* $\mathbf{E}[X_\tau|\mathcal{F}_1] = X_1$ *m.b., és persze* $\mathbf{E}X_\tau = \mathbf{E}X_1$*.*

Fontos megjegyezni, hogy a tételben szereplő feltételek nem csupán technikai feltételek. Legyen S_n egy egyszerű szimmetrikus bolyongás az egyenesen. Ő martingál a az általa generált természetes filtrációra nézve. Tudjuk, hogy az egydimenziós bolyongás rekurrens, ezért majdnem biztosan eléri az 1-et. Legyen az elérés időpontja τ. Ekkor τ megállási idő, és persze $S_\tau \equiv 1 \neq S_0 = 0$. Csak a $\liminf_{n\to\infty} \int_{\{\tau>n\}} |X_n| \mathrm{d}\mathbf{P} = 0$ feltétellel lehet baj, és valóban, ez nem teljesül.

6.2 Optimal stopping problems

Consider a probability space with a filtration $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,1,\ldots,N}, \mathbf{P})$, and let

$$\mathcal{M}_n^N = \{\tau : \tau \text{ is a stopping time}, \tau \in \{n, \ldots, N\}\}.$$

To ease notation we suppress N in the upper index. Consider a sequence of nonnegative adapted random variables $(X_n)_n$, and define by backward induction its Snell-envelope $(Z_n)_n$ as follows. We are interested in the value

$$Z_N = X_N, \quad Z_n = \max\{X_n, \mathbf{E}[Z_{n+1}|\mathcal{F}_n]\}, \; n < N.$$

For a stopping time τ the stopped process is denoted by Z^τ, i.e.

$$Z_n^\tau = Z_{\tau\wedge n},$$

where $a \wedge b = \min\{a, b\}$.

Proposition 4. *Let* (Z_n) *be the Snell-envelope of* (X_n) *with* $X_n \geq 0$ *a.s.*

(i) Z *is the smallest supermartingale dominating* X*.*

(ii) The random variable $\tau^* = \min\{n : Z_n = X_n\}$ *is a stopping time and the stopped process* $Z_{n\wedge\tau^*} = Z_n^{\tau^*}$ *is martingale.*

Proof. From the definition it is clear that Z is supermatingale and dominates X. Let Y be another supermartingale dominating X. Then $Y_N \geq X_N = Z_N$. Assuming that $Y_n \geq Z_n$ we have

$$Y_{n-1} \geq \max\{\mathbf{E}[Y_n|\mathcal{F}_{n-1}], X_{n-1}\} \geq \max\{\mathbf{E}[Z_n|\mathcal{F}_{n-1}], X_{n-1}\} = Z_{n-1}.$$

Thus the minimality follows.

To see that τ^* is stopping time note that

$$\{\tau^* = n\} = \cap_{k=0}^{n-1}\{Z_k > X_k\} \cap \{Z_n = X_n\}.$$

For the last assertion note that

$$Z_n^{\tau^*} - Z_{n-1}^{\tau^*} = I(\tau^* \geq n)(Z_n - Z_{n-1}).$$

On the event $\{\tau^* \geq n\}$ we have $Z_{n-1} = \mathbf{E}[Z_n|\mathcal{F}_{n-1}]$ therefore

$$\mathbf{E}[I(\tau^* \geq n)(Z_n - Z_{n-1})|\mathcal{F}_{n-1}] = 0.$$

□

A stopping time σ is optimal if

$$\mathbf{E}X_\sigma = \sup_{\tau \in \mathcal{M}_0} \mathbf{E}X_\tau.$$

Proposition 5. *The stopping time τ^* is optimal for X, and*

$$Z_0 = \mathbf{E}X_{\tau^*} = \sup_{\tau \in \mathcal{M}_0} \mathbf{E}X_\tau.$$

Proof. Since Z^{τ^*} is martingale

$$Z_0 = Z_0^{\tau^*} = \mathbf{E}Z_N^{\tau^*} = \mathbf{E}Z_{\tau^*} = \mathbf{E}X_{\tau^*}.$$

On the other hand for any stopping time τ the process Z^τ is supermartingale (by Doob's optional sampling), thus

$$Z_0 = \mathbf{E}Z_0^\tau \geq \mathbf{E}Z_\tau \geq \mathbf{E}X_\tau.$$

□

Proposition 6. *The stopping time σ is optimal iff the following two conditions hold.*

(i) $Z_\sigma = X_\sigma$*;*

(ii) Z^σ *is martingale.*

Proof. If (i) and (ii) hold than σ is optimal. This follows exactly as the optimality of τ^*.

Conversely, assume that σ is optimal. We have seen that $\sup_\tau \mathbf{E} X_\tau = Z_0$ thus

$$Z_0 = \mathbf{E} X_\sigma \leq \mathbf{E} Z_\sigma,$$

by the dominance of Z. By Doob's otional stopping theorem Z^σ is supermartingale, therefore $\mathbf{E} Z_\sigma \leq Z_0$, implying that

$$\mathbf{E} X_\sigma = \mathbf{E} Z_\sigma.$$

Since $Z_n \geq X_n$ this implies $X_\sigma = Z_\sigma$ a.s., proving (i).

By the optimality $\mathbf{E} Z_\sigma = Z_0$, while the supermartingale property implies

$$Z_0 \geq \mathbf{E} Z_{\sigma \wedge n} \geq \mathbf{E} Z_\sigma.$$

Thus

$$E Z_{\sigma \wedge n} = \mathbf{E} Z_\sigma = \mathbf{E}\mathbf{E}[Z_\sigma | \mathcal{F}_n].$$

Furthermore, by Doob's optional stopping

$$Z_{\sigma \wedge n} \geq \mathbf{E}[Z_\sigma | \mathcal{F}_n],$$

implying $Z_{\sigma \wedge n} = \mathbf{E}[Z_\sigma | \mathcal{F}_n]$. Thus (Z_n^σ) is indeed a martingale. □

6.3 Pricing American options

Let us return to our pricing problem. Assume that we have an arbitrage-free complete market, that is the EMM $\mathbf{Q}$ is unique. Let $(f_n)_{n=0,\ldots,N}$ be the payoff of an American option. A hedging strategy now has to fulfil the conditions

$$X_n^\pi \geq f_n, \quad n = 0, 1, \ldots, N,$$

as the option can be exercised at any time. A hedge is *minimal*, if for a stopping time τ^* we have $X_{\tau^*}^\pi = f_{\tau^*}$.

By Doob's optional stopping $(X_0^\pi / B_0, X_\tau^\pi / B_\tau)$ is martingale for any stopping time τ, i.e.

$$\frac{x}{B_0} = \mathbf{E}_\mathbf{Q} \frac{X_0^\pi}{B_0} = \mathbf{E}_\mathbf{Q} \frac{X_\tau^\pi}{B_\tau} \geq \mathbf{E}_\mathbf{Q} \frac{f_\tau}{B_\tau}.$$

Therefore the initial cost of the hedge is at least

$$x \geq B_0 \sup_{\tau \in \mathcal{M}_0^N} \mathbf{E}_\mathbf{Q} \frac{f_\tau}{B_\tau}.$$

At time N we need

$$X_N^\pi \geq f_N.$$

At time $N-1$ the holder either exercise the option or continues to time N, (in that case we discount the price), therefore

$$X_{N-1}^\pi \geq \max\left\{ f_{N-1}, \frac{B_{N-1}}{B_N} \mathbf{E}_{\mathbf{Q}}[f_N | \mathcal{F}_{N-1}] \right\}.$$

Dividing by B_{N-1}

$$\frac{X_{N-1}^\pi}{B_{N-1}} \geq \max\left\{ \frac{f_{N-1}}{B_{N-1}}, \mathbf{E}_{\mathbf{Q}}\left[\frac{f_N}{B_N} \middle| \mathcal{F}_{N-1} \right] \right\}.$$

Thus, we see the connection with the Snell-envelope.

For a hedging strategy π we have that

(i) $(X_n^\pi / B_n)_n$ is a $\mathbf{Q}$-martingale (since $\mathbf{Q}$ is EMM and π is SF), and

(ii) (X_n^π / B_n) dominates (f_n / B_n) (since π is a hedge).

Therefore, the value process of a hedge is larger than the Snell-envelope of (f_n/B_n), i.e.

$$\frac{X_n^\pi}{B_n} \geq Z_n, \quad n = 0, 1, \ldots, N, \tag{12}$$ {eq:di-american-1}

where (Z_n) is the Snell-envelope of (f_n/B_n). The Snell-envelope (Z_n) is a supermartingale, therefore by the Doob-decomposition (that's stated for submartingale, but multiply with -1) we have

$$Z_n = M_n - A_n, \quad n = 0, 1, \ldots, N, \tag{13}$$ {eq:di-american-2}

where M_n is a $\mathbf{Q}$-martingale, and (A_n) is an increasing predictable sequence, $A_0 = 0$. Comparing (12) and (13) we see that

$$\frac{X_n^\pi}{B_n} \geq M_n.$$

On the other hand, the market is complete, which implies (see the easy parts of the proof of Theorem 7) that there exists a strategy π such that

$$\frac{X_n^\pi}{B_n} = M_n, \quad n = 0, 1, \ldots, N.$$

This is a minimal hedging strategy with initial cost

$$\frac{x}{B_0} = \frac{X_0^\pi}{B_0} = M_0 = Z_0.$$

{thm:price-di-amer

Theorem 12. *Consider an aribtrage-free complete market with unique EMM* $\mathbf{Q}$*. Let* (f_n) *be the nonnegative payoff sequence of an American option. Let* (Z_n) *be the Snell-envelope of the discounted payoff sequence* (f_n/B_n)*. The fair price for this option is*

$$C = B_0 Z_0 = B_0 \sup_{\tau \in \mathcal{M}_0^N} \mathbf{E}_{\mathbf{Q}} \frac{f_\tau}{B_\tau} = B_0 \mathbf{E}_{\mathbf{Q}} \frac{f_{\tau^*}}{B_{\tau^*}},$$

where τ^* *is an (not unique in general) optimal exercise time given by*

$$\tau^* = \min \left\{ n : \frac{f_n}{B_n} = Z_n \right\}.$$

Furthermore, there exists a SF strategy π *which is an optimal hedge with inital cost* C *and*

$$X^\pi_{\tau^*} = \frac{f_{\tau^*}}{B_{\tau^*}}.$$

6.4 American vs. European options

Clearly, an American option with payoff sequence $(f_n)_{n=0,1,\ldots,N}$ worth at least as a European option with payoff f_N. However, in some cases the fair prices are equal.

Consider an American call option with strike price K, that is

$$f_n = f(S_n) = (S_n - K)_+.$$

Assume that the deterministic sequence (B_n) is nondecreasing (i.e. the interest rate is nonnegative). Let (Z_n) denote the Snell envelope of (f_n/B_n), that is

$$Z_N = \frac{f_N}{B_N}, \quad Z_n = \max \left\{ \frac{f_n}{B_n}, \mathbf{E}\left[Z_{n+1} | \mathcal{F}_n \right] \right\}, \quad n = 0, 1, \ldots, N-1.$$

Using that (S_n/B_n) is a $\mathbf{Q}$-martingale, by Jensen's inequality

$$\begin{aligned}
\frac{f_{N-1}}{B_{N-1}} &= \frac{(S_{N-1}-K)_+}{B_{N-1}} \\
&= \left(\frac{S_{N-1}}{B_{N-1}} - \frac{K}{B_{N-1}}\right)_+ \\
&\leq \mathbf{E}_{\mathbf{Q}}\left[\left(\frac{S_N}{B_N} - \frac{K}{B_{N-1}}\right)_+ \middle| \mathcal{F}_{N-1}\right] && \text{Jensen's inequality} \\
&\leq \mathbf{E}_{\mathbf{Q}}\left[\left(\frac{S_N}{B_N} - \frac{K}{B_N}\right)_+ \middle| \mathcal{F}_{N-1}\right] && \text{by } B_N \geq B_{N-1} \\
&= \mathbf{E}_{\mathbf{Q}}\left[\frac{(S_N-K)_+}{B_N} \middle| \mathcal{F}_{N-1}\right] \\
&= \mathbf{E}_{\mathbf{Q}}[Z_N|\mathcal{F}_{N-1}].
\end{aligned}$$

This means that at time $N-1$ it is always good to hold the option and continue to step N.

An induction argument shows that at any time it is better to hold the option. Indeed, assume for some n

$$\frac{f_n}{B_n} \leq \mathbf{E}_{\mathbf{Q}}[Z_{n+1}|\mathcal{F}_n].$$

We just proved this for $n = N-1$. The same way as above we have

$$\begin{aligned}
\frac{f_{n-1}}{B_{n-1}} &= \frac{(S_{n-1}-K)_+}{B_{n-1}} \\
&= \left(\frac{S_{n-1}}{B_{n-1}} - \frac{K}{B_{n-1}}\right)_+ \\
&\leq \mathbf{E_Q}\left[\left(\frac{S_n}{B_n} - \frac{K}{B_{n-1}}\right)_+ \middle| \mathcal{F}_{n-1}\right] && \text{Jensen's inequality} \\
&\leq \mathbf{E_Q}\left[\left(\frac{S_n}{B_n} - \frac{K}{B_n}\right)_+ \middle| \mathcal{F}_{n-1}\right] && \text{by } B_n \geq B_{n-1} \\
&= \mathbf{E_Q}\left[\frac{(S_n-K)_+}{B_n} \middle| \mathcal{F}_{n-1}\right] \\
&= \mathbf{E_Q}\left[\frac{f_n}{B_n} \middle| \mathcal{F}_{n-1}\right] \\
&\leq \mathbf{E_Q}\left[\mathbf{E_Q}[Z_{n+1}|\mathcal{F}_n] \middle| \mathcal{F}_{n-1}\right] && \text{induction} \\
&\leq \mathbf{E_Q}[Z_n|\mathcal{F}_{n-1}] && Z \text{ supermartingale}
\end{aligned}$$

Thus $\tau^* \equiv N$ is an optimal stopping time, which means that no matter what happens, we wait until the end. Then the American option behaves as the European, so the prices are equal.

Theorem 13. *Assume that the market is arbitrage free and complete, and the interest rate is nonnegative. Then the price of a European call option equals to the price of the American call option.*

7 Stochastic integration

7.1. Az Itô-formula

Ezek után belátjuk az Itô-formulát.

14. Theorem (Itô-formula (1944))**.** *Legyen* $X_t = X_0 + \int_0^t K_s \mathrm{d}s + \int_0^t H_s \mathrm{d}W_s$ *Itô-folyamat, és* $f \in C^2$ *kétszer folytonosan differenciálható függvény. Ekkor*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)H_s^2 \mathrm{d}s.$$

A következőkben bevezetjük a többdimenziós Itô-folyamatokat.

A $W = (W^1, W^2, \ldots, W^r)$ egy *r-dimenziós Brown-mozgás*, ha a komponensei függetlenek, és minden komponens egy SBM. Az (X_t) egy *d-dimenziós Itô-folyamat*, ha az i-edik komponense

$$X_t^i = X_0^i + \int_0^t K_s^i \mathrm{d}s + \sum_{j=1}^r \int_0^t H_s^{i,j} \mathrm{d}W_s^j, \tag{14}$$ {eq:multid-ito}

ahol $\int_0^T |K_s^i| \mathrm{d}s < \infty$, $\int_0^T (H_s^{i,j})^2 \mathrm{d}s < \infty$ m.b., és $K^i, H^{i,j}$ $\mathcal{F}_t$-adaptált folyamatok, $i = 1, 2, \ldots, d$, $j = 1, 2, \ldots, r$.

15. Theorem (Többdimenziós Itô-formula)**.** *Legyen (X_t) egy többdimenziós Itô-folyamat, (14) formula szerint, és $f : \mathbb{R}^{1+d} \to \mathbb{R}$, $f \in C^{1,2}$. Ekkor*

$$\begin{aligned} f(t, X_t^1, \ldots, X_t^d) = f(0, X_0^1, \ldots, X_0^d) &+ \int_0^t \frac{\partial}{\partial s} f(s, X_s^1, \ldots, X_s^d)\, \mathrm{d}s \\ &+ \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(s, X_s^1, \ldots, X_s^d)\, \mathrm{d}X_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s, X_s^1, \ldots, X_s^d) \sum_{k=1}^r H_s^{i,k} H_s^{j,k}\, \mathrm{d}s. \end{aligned}$$

7.2. Alkalmazások

Az Itô-formulára nézünk néhány alkalmazást.

2. Example. Parciális integrálás I. Legyen (X, Y) kétdimenziós Itô-folyamat

$$\begin{aligned} X_t &= X_0 + \int_0^t K_s\, \mathrm{d}s + \int_0^t H_s\, \mathrm{d}W_s \\ Y_t &= Y_0 + \int_0^t L_s\, \mathrm{d}s + \int_0^t G_s\, \mathrm{d}W_s, \end{aligned}$$

ahol K, L, H, G olyanok amilyennek lenniük kell. Ekkor

$$\int_0^t X_s \mathrm{d}Y_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s \mathrm{d}X_s - \int_0^t H_s G_s \mathrm{d}s.$$

Vegyük észre, hogy a hagyományos parciális integrálási formulában (amikor tehát X, Y *determinisztikus, korlátos változású* függvények) nem szerepel az utolsó tag.

A bizonyításhoz alkalmazzuk az Itô-formulát az (X, Y) folyamatra, és az $f(x, y) = xy$ függvényre. Ekkor a (14) formula szerinti szereposztás:

$$r = 1,\ d = 2,\ K_s^1 = K_s,\ K_s^2 = L_s,\ H_s^{1,1} = H_s,\ H_s^{2,1} = G_s.$$

Mivel $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$, $\frac{\partial^2 f}{\partial^2 x} = \frac{\partial^2 f}{\partial^2 y} = 0$, és $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 1$, így

$$X_t Y_t = X_0 Y_0 + \int_0^t Y_s \mathrm{d}X_s + \int_0^t X_s \mathrm{d}Y_s + \frac{1}{2} 2 \int_0^t H_s G_s \mathrm{d}s,$$

ami rendezés után éppen az állítás.

3. Example. Parciális integrálás II. Egy kicsit módosítjuk az előző példát. Legyen $\widetilde{W}$ egy W-től független SBM, és (X, Y) kétdimenziós Itô-folyamat

$$X_t = X_0 + \int_0^t K_s \,\mathrm{d}s + \int_0^t H_s \,\mathrm{d}W_s$$
$$Y_t = Y_0 + \int_0^t L_s \,\mathrm{d}s + \int_0^t G_s \,\mathrm{d}\widetilde{W}_s,$$

ahol K, L, H, G olyanok amilyennek lenniük kell. Ekkor

$$\int_0^t X_s \mathrm{d}Y_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s \mathrm{d}X_s.$$

Ennek bizonyítása ugyanúgy megy, mint az előbb. Vegyük észre, hogy itt $d = r = 2$, és a két folyamatot különböző Wiener-folyamat hajtja meg, ezért nem jelenik meg az extra tag.

4. Example. Korábban már meghatároztuk az $\int W_s \mathrm{d}W_s$ sztochasztikus integrál értékét. Most meghatározzuk az Itô-formula segítségével.

A Wiener-folyamat Itô-folyamatként való reprezentációja $K_s \equiv 0$, $H_s \equiv 1$. Legyen $f(x) = x^2$. Az Itô-formula szerint

$$W_t^2 = W_0^2 + \int_0^t 2 W_s \mathrm{d}W_s + \frac{1}{2} \int_0^t 2 \mathrm{d}s.$$

Ezt átrendezve kapjuk a már ismert

$$\int_0^t W_s \mathrm{d}W_s = \frac{W_t^2 - t}{2}$$

formulát. Innen azt is rögtön látjuk, hogy $W_t^2 - t$ martingál, hiszen minden sztochasztikus integrál martingál (na nem mintha a direkt bizonyítás bonyolult lett volna).

14. Exercise. Az Itô-formula alkalmazásával igazoljuk, hogy $Y(t) = e^{t/2} \cos W_t$ martingál!

15. Exercise. Mutassuk meg, hogy

$$\int_0^t W_s^2 \mathrm{d}W_s = \frac{1}{3} W_t^3 - \int_0^t W_s \mathrm{d}s,$$

és

$$\int_0^t W_s^3 \mathrm{d}W_s = \frac{1}{4} W_t^4 - \frac{3}{2} \int_0^t W_s^2 \mathrm{d}s.$$

16. Exercise. Legyen $\mathbf{W} = (W^1, \ldots, W^r)$ r-dimenziós SBM, $r \geq 2$, és legyen R a $\mathbf{W}$ hossza, azaz

$$R_t = \sqrt{\sum_{i=1}^r (W^i)^2}.$$

Igazoljuk, hogy R_t teljesíti a

$$\mathrm{d}R_t = \frac{r-1}{2R_t} \mathrm{d}t + \sum_{i=1}^r \frac{W_t^i}{R_t} \mathrm{d}W_t^i$$

differenciálegyenletet! Ez a *sztochasztikus Bessel-egyenlet*, és R_t a *Bessel-folyamat.*

8. Folytonos idejű piacok

A sztochasztikus integrálelmélettel felvértezve rátérünk a folytonos idejű piaci modellek tárgyalására.

8.1. Piacok általában

Az alapfogalmak a diszkrét időben már megismert fogalmak természetes folytonos idejű megfelelői.

A továbbiakban a $[0,T]$ *véges* időhorizonton dolgozunk, $T<\infty$. Adott egy $(\Omega,\mathcal{A},\mathbf{P})$ valószínűségi mező, azon egy $(\mathcal{F}_t)$ filtráció. A piacon két termék adott, egy kockázatmentes és egy kockázatos. A kötvény a kockázatmentes, az árfolyamata (B_t) egy determinisztikus folyamat, a részvény a kockázatos, árfolyamata (S_t) egy pozitív véletlen sztochasztikus folyamat, ami adaptált az $(\mathcal{F}_t)$ filtrációhoz. Továbbá azt is feltesszük, hogy (S_t) az $(\mathcal{F}_t)$ filtrációhoz adaptált Itô-folyamat.

A *stratégia / portfólió* egy $(\pi_t=(\beta_t,\gamma_t))$ folyamat, ami adaptált (hát persze, hiszen nem látunk a jövőbe), és

$$\int_0^T|\beta_t|\mathrm{d}t<\infty,\quad \int_0^T\gamma_t^2\mathrm{d}t<\infty,\ \text{m.b.}$$

A (β_t) folyamat jelenti a t-ben birtokunkban levő kötvény, (γ_t) pedig a részvény mennyiségét. Természetesen mindkét folyamat lehet negatív is.

A (π) portfólió értéke t-ben

$$X_t^\pi=\beta_tB_t+\gamma_tS_t, \tag{15}$$ {eq:ertekfoly}

ez a portfólió *értékfolyamata*.

Az önfinanszírozó stratégiát szeretnénk definiálni a diszkrét idő analogonjaként. Az, hogy nem fektetek be plusz pénzt a portfóliómba, és nem is veszek ki belőle, azt jelenti, hogy amikor az n-edik napon este átrendezem a portfóliómat, akkor az összérték meg kell egyezzen az n-edik napon a portfólióm értékével, azaz

$$\beta_{n+1}B_n+\gamma_{n+1}S_n=\beta_nB_n+\gamma_nS_n,$$

és a β_{n+1},γ_{n+1} változók $\mathcal{F}_n$-mérhetőek. Felírva, hogy $X_{n+1}=\beta_{n+1}B_{n+1}+\gamma_{n+1}S_{n+1}$, azt kapom, hogy a portfólióm értékének megváltozása

$$X_{n+1}-X_n=\beta_{n+1}(B_{n+1}-B_n)+\gamma_{n+1}(S_{n+1}-S_n).$$

Ez azt jelenti, hogy az értékfolyamat megváltozása a kötvényár és a részvényár megváltozásából tevődik össze, külső forrást nem veszünk igénybe. Ennek az egyenletnek a folytonos megfelelője a

$$\mathrm{d}X_t^\pi=\beta_t\mathrm{d}B_t+\gamma_t\mathrm{d}S_t$$

sztochasztikus differenciálegyenlet. Ez lesz az önfinanszírozóság definíciója.

Azt mondjuk, hogy a $(\pi_t = (\beta_t, \gamma_t))$ stratégia *önfinanszírozó*, ha teljesül a

$$\mathrm{d}X_t^\pi = \beta_t \mathrm{d}B_t + \gamma_t \mathrm{d}S_t \tag{16}$$ {eq:onfin}

sztochasztikus differenciálegyenlet.

Az $(\overline{S}_t = S_t B_0/B_t)$ folyamat a *diszkontált részvényárfolyamat*, az $(\overline{X}_t^\pi = X_t^\pi B_0/B_t)$ folyamat pedig a *diszkontált értékfolyamat*.

Mostantól feltesszük, hogy a folytonos kamatráta $r > 0$, azaz

$$B_t = e^{rt},\ t \geq 0.$$

Ekkor

$$\overline{S}_t = e^{-rt} S_t,\ \text{és } \overline{X}_t^\pi = e^{-rt} X_t^\pi.$$

{all:onfin-ekv}

7. Proposition. *A $(\pi_t = (\beta_t, \gamma_t))$ stratégia pontosan akkor önfinanszírozó, ha*

$$\overline{X}_t^\pi = X_0^\pi + \int_0^t \gamma_s \mathrm{d}\overline{S}_s, \quad t \in [0, T].$$

Bizonyítás. Tegyük fel, hogy π önfinanszírozó. Az Itô-formula alapján

$$\begin{aligned}
\mathrm{d}\overline{X}_t^\pi &= \mathrm{d}\left(e^{-rt} X_t^\pi\right) = -re^{-rt} X_t^\pi \mathrm{d}t + e^{-rt} \mathrm{d}X_t \\
&= -re^{-rt}(\beta_t e^{rt} + \gamma_t S_t)\mathrm{d}t + e^{-rt}\left(\beta_t \mathrm{d}e^{rt} + \gamma_t \mathrm{d}S_t\right) \\
&= -re^{-rt}\gamma_t S_t \mathrm{d}t + e^{-rt}\gamma_t \mathrm{d}S_t \\
&= \gamma_t \mathrm{d}\left(e^{-rt} S_t\right),
\end{aligned}$$

amint állítottuk.

Megfordítva, tegyük fel, hogy

$$\mathrm{d}\overline{X}_t^\pi = \gamma_t \mathrm{d}\overline{S}_t.$$

Mivel $X_t^\pi = \beta_t e^{rt} + \gamma_t S_t$, így a bal oldal

$$\mathrm{d}\overline{X}_t^\pi = -re^{-rt} X_t^\pi \mathrm{d}t + e^{-rt}\mathrm{d}X_t^\pi = -e^{-rt}\beta_t \mathrm{d}B_t - re^{-rt}\gamma_t S_t \mathrm{d}t + e^{-rt}\mathrm{d}X_t^\pi.$$

A jobb oldal

$$\gamma_t \mathrm{d}\overline{S}_t = -re^{-rt}\gamma_t S_t \mathrm{d}t + \gamma_t e^{-rt}\mathrm{d}S_t.$$

A két oldal egyenlőségéből adódik, hogy

$$\mathrm{d}X_t^\pi = \beta_t \mathrm{d}B_t + \gamma_t \mathrm{d}S_t,$$

ami éppen az önfinanszírozóság definíciója. □

Bevezetjük az arbitrázs fogalmát. A π önfinanszírozó stratégia *arbitrázsstratégia*, ha $X_0^\pi = 0$ m.b., $X_T \geq 0$ m.b., és $\mathbf{P}\{X_T^\pi > 0\} > 0$. A piac *arbitrázsmentes*, ha nincs arbitrázsstratégia.

Ez a fogalom fejezi ki azt, hogy 0 kezdőtőkével indulva, biztosan nyerünk, azaz *ingyen ebédhez* jutunk. Természetes feltenni, hogy a valóságban arbitrázs nem létezik a piacon, hiszen ha létezne, akkor mindenki ezt a stratégiát játszaná meg, ezzel módosítva az árakat, és így nagyon gyorsan megszűnne az arbitrázslehetőség. Diszkrét idejű piacon láttuk, hogy (bizonyos feltételek mellett) az arbitrázsmentesség ekvivalens azzal, hogy létezik piacon olyan, az eredeti $\mathbf{P}$ mértékkel ekvivalens mérték, melyre nézve a diszkontált részvényárfolyamat martingál. Ez bizonyos feltételek mellett a folytonos esetben is igaz, ráadásul az egyik irányú implikáció most is nagyon egyszerű.

Tegyük fel, hogy van a piacon egy olyan $\mathbf{Q}$ valószínűségi mérték, melyre $\mathbf{P} \sim \mathbf{Q}$ (azaz a két mérték ekvivalens, azaz $\mathbf{P} \ll \mathbf{Q}$ és $\mathbf{Q} \ll \mathbf{P}$), és az $(\overline{S}_t)$ folyamat martingál. Az ilyen mértéket *ekvivalens martingálmértéknek* (EMM) nevezzük. Legyen π egy tetszőleges önfinanszírozó stratégia. A 7 Állítás szerint ekkor az értékfolyamat

$$\overline{X}_t^\pi = X_0^\pi + \int_0^t \gamma_s \mathrm{d}\overline{S}_s.$$

Mivel $(\overline{S}_t)$ $\mathbf{Q}$-martingál, és $\overline{X}_t^\pi$ e szerinti sztochasztikus integrál, ezért az $(\overline{X}_t^\pi)$ folyamat is $\mathbf{Q}$-martingál. (Vegyük észre, hogy ugyanezt az állítást beláttuk a diszkrét piacok esetén is.) Eszerint

$$\mathbf{E}_\mathbf{Q}\overline{X}_T^\pi = \mathbf{E}_\mathbf{Q} X_0^\pi.$$

Mivel $\mathbf{P} \sim \mathbf{Q}$, ezért ha $X_0^\pi = 0$, $X_T^\pi \geq 0$ $\mathbf{P}$-m.b., akkor $\mathbf{Q}$-m.b. is. Na de $\mathbf{E}_\mathbf{Q}\overline{X}_T^\pi = \mathbf{E}_\mathbf{Q} X_0^\pi = 0$, amiből következik, hogy $X_T^\pi \equiv 0$ $\mathbf{Q}$-m.b., de így $\mathbf{P}$-m.b. is.

Ezzel beláttuk az alábbit.

16. Theorem. *Ha az* $(\Omega, \mathcal{A}, \mathbf{P}, (S_t), (B_t = e^{rt}), (\mathcal{F}_t))$ *folytonos idejű piacon létezik* $\mathbf{Q}$ *EMM, akkor a piac arbitrázsmentes.*

Természetesen folytonos idejű piacon is tekinthetünk opciókat, ill. tetszőleges követeléseket. Az egyik célunk az ilyen követelések igazságos árának definiálása, meghatározása, ill. fedezeti portfólió összeállítása. Igazságos árat és fedezeti stratégiát csak speciális esetben adunk meg a következő fejezetben, azonban a definíciót kimondjuk és néhány tulajdonságot bebizonyítunk az általános esetben.

Az f_T egy *véletlen követelés*, ha $\mathcal{F}_T$-mérhető. A π egy *fedezeti stratégia* f_T-re x kezdőtőkével, röviden (f_T, x)-fedezet, ha

$$X_T^\pi \geq f_T \text{ m.b., és } X_0^\pi = x.$$

Az f_T követelés *igazságos ára* a legkisebb olyan x érték, melyre létezik (f_T, x)-fedezet, azaz

$$C_T(f_T) = \inf\{x \geq 0 : \text{ létezik } (f_T, x)\text{-fedezet }\}.$$

Tegyük fel, hogy a piacon létezik EMM, legyen $\mathbf{Q}$ egy ilyen. Ekkor tetszőleges π (f_T, x)-fedezetre

$$x = \mathbf{E_Q} X_0^\pi = \mathbf{E_Q} \overline{X}_t^\pi = \mathbf{E_Q} e^{-rT} X_T^\pi \geq \mathbf{E_Q} e^{-rT} f_T.$$

Ezzel beláttuk, hogy

$$C(T, f_T) \geq \mathbf{E_Q}(e^{-rT} f_T). \tag{17}$$ {eq:ia-ineq}

9. Mértékváltás

A diszkrét idejű piacok elméletében láttuk mennyire fontos az ekvivalens martingálmérték, ugyanis ez alapján tudtunk árazni. Jelen fejezet célja, hogy megértsük hogy változik egyes folyamatok dinamikája ha megváltoztatjuk a mértéket, vagy másképpen, hogyan vezessünk be olyan mértéket, ami szerint a folyamatunk martingál lesz.

9.1. Girsanov-tétel

17. Theorem (Lévy tétele a Wiener-folyamat karakterizációjáról)**.** *Legyen M_t folytonos martingál, melyre $M_0 = 0$. Ha $M_t^2 - t$ martingál, akkor M_t Wiener-folyamat.*

Megjegyezzük, hogy a folytonossági feltétel nélkül nem igaz az állítás. Hiszen ha N_t 1 intenzitású Poisson-folyamat, akkor $N_t - t$ és $(N_t - t)^2 - t$ is martingál.

Az új mérték bevezetése diszkrét modell esetén nem jelentett nehézséget. Folytonos modelleknél a dolog nem ilyen egyszerű.

Legyen $(\Omega, \mathcal{A}, \mathbf{P})$ egy valószínűségi mező, $(\mathcal{F}_t)$ egy filtráció, és $\mathbf{Q}$ egy másik valószínűségi mérték $(\Omega, \mathcal{A})$-n, ami abszolút folytonos $\mathbf{P}$-re, jelben $\mathbf{Q} \ll \mathbf{P}$. Legyen M_∞ a $\mathbf{Q}$ Radon–Nikodym-deriváltja,

$$M_\infty = \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}},$$

ami azt jelenti, hogy

$$\mathbf{Q}(A) = \int_A M_\infty \mathrm{d}\mathbf{P}.$$

Mivel a továbbiakban általában több mértékkel dolgozunk, ezért a várható érték alsó indexében jelöljük, hogy melyik szerint vesszük a várható értéket; azaz $\mathbf{E}_\mathbf{P} X = \int_\Omega X \mathrm{d}\mathbf{P}$ és $\mathbf{E}_\mathbf{Q} X = \int_\Omega X \mathrm{d}\mathbf{Q}$. Továbbá a $\mathbf{P}$ mérték szerinti martingálokat röviden $\mathbf{P}$-martingálnak, a $\mathbf{Q}$ mérték szerintieket $\mathbf{Q}$-martingálnak nevezzük.

Definiáljuk az

$$M_t = \mathbf{E}_\mathbf{P}[M_\infty | \mathcal{F}_t]$$

$\mathbf{P}$-martingált. A következő lemma megadja a $\mathbf{P}$- és $\mathbf{Q}$-martingálok közti kapcsolatot a Radon–Nikodym-derivált segítségével.

{lemma:p-q-mtg}

9. Lemma. *Az (X_t) $\mathcal{F}_t$-adaptált sztochasztikus folyamat pontosan akkor $\mathbf{Q}$-martingál, ha az $(M_t X_t)$ folyamat $\mathbf{P}$-martingál.*

Bizonyítás. Mivel

$$\mathbf{E}_\mathbf{P}[M_\infty X_t | \mathcal{F}_t] = X_t M_t,$$

így minden $A \in \mathcal{F}_t$ eseményre

$$\int_A X_t M_\infty \mathrm{d}\mathbf{P} = \int_A X_t M_t \mathrm{d}\mathbf{P}.$$

Ezért, ha $A \in \mathcal{F}_s \subset \mathcal{F}_t$, akkor

$$\int_A X_t \mathrm{d}\mathbf{Q} = \int_A X_t M_\infty \mathrm{d}\mathbf{P} = \int_A X_t M_t \mathrm{d}\mathbf{P}$$
$$\int_A X_s \mathrm{d}\mathbf{Q} = \int_A X_s M_\infty \mathrm{d}\mathbf{P} = \int_A X_s M_s \mathrm{d}\mathbf{P}.$$

Az (X_t) folyamat pontosan akkor $\mathbf{Q}$-martingál, ha a bal oldalak egyenlőek minden $A \in \mathcal{F}_s$ halmazra, és $s < t$ esetén, ami persze pontosan akkor teljesül, ha a jobb oldalak egyenlőek, ami azt jelenti, hogy $(M_t X_t)$ $\mathbf{P}$-martingál. □

Legyen

$$\zeta_t^s = \int_s^t X_u \mathrm{d}W_u - \frac{1}{2}\int_s^t X_u^2 \mathrm{d}u, \quad \zeta_t = \zeta_t^0,$$

ahol X_t adaptált folyamat. Ekkor $Z_t = e^{\zeta_t}$ kielégíti a

$$Z_t = 1 + \int_0^t Z_s X_s \mathrm{d}W_s$$

sztochasztikus differenciálegyenletet. (Ezt a formulát használni fogjuk a Girsanov-tétel bizonyításánál.)

A fenti sztochasztikus differenciálegyenletet differenciálegyenletes jelöléssel

$$\mathrm{d}Z_t = Z_t X_t \mathrm{d}W_t, \quad Z_0 = 1,$$

alakba írható.

A ζ folyamatot Itô-folyamatként felírva

$$\zeta_t = \int_0^t -\frac{1}{2}X_u^2 \mathrm{d}u + \int_0^t X_u \mathrm{d}W_u.$$

Legyen $f(x) = e^x$, ekkor az Itô-formula szerint

$$\begin{aligned}
Z_t = e^{\zeta_t} &= 1 + \int_0^t e^{\zeta_s} \mathrm{d}\zeta_s + \frac{1}{2}\int_0^t e^{\zeta_s} X_s^2 \mathrm{d}s \\
&= 1 + \int_0^t e^{\zeta_s}\left(-\frac{1}{2}X_s^2 \mathrm{d}s + X_s \mathrm{d}W_s\right) + \frac{1}{2}\int_0^t e^{\zeta_s} X_s^2 \mathrm{d}s \\
&= 1 + \int_0^t e^{\zeta_s} X_s \mathrm{d}W_s \\
&= 1 + \int_0^t Z_s X_s \mathrm{d}W_s,
\end{aligned}$$

amint állítottuk. Azt is rögtön látjuk, hogy Z_t martingál.

17. Exercise. Legyen ζ_t mint fent. Mutassuk meg, hogy a $Y_t = e^{-\zeta_t}$ folyamat kielégíti a

$$\mathrm{d}Y_t = Y_t X_t^2 \mathrm{d}t - X_t Y_t \mathrm{d}W_t, \quad Y_0 = 1,$$

sztochasztikus differenciálegyenletet!

18. Theorem (Girsanov-tétel)**.** *Legyen* (θ_t) *adaptált folyamat, melyre* $\int_0^T \theta_s^2 \mathrm{d}s < \infty$ *m.b., és tegyük föl, hogy*

$$\Lambda_t = \exp\left\{-\int_0^t \theta_s \mathrm{d}W_s - \frac{1}{2}\int_0^t \theta_s^2 \mathrm{d}s\right\} \tag{18}$$ {eq:Lambda}

$\mathbf{P}$*-martingál, ahol* (W_t) *SBM a* $\mathbf{P}$ *mérték szerint. Definiáljuk a* $\mathbf{Q}_\theta = \mathbf{Q}$ *mértéket a*

$$\left.\frac{\mathrm{d}\mathbf{Q}_\theta}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T} = \Lambda_T$$

formulával. Ekkor a $\tilde{W}_t = W_t + \int_0^t \theta_s \mathrm{d}s$ *SBM a* $\mathbf{Q}$*-mérték szerint.*

1. *Remark.* Tanultuk, hogy a Λ_t folyamat martingál. Akkor meg miért tesszük föl a Girsanov-tételben, hogy martingál? A helyzet az, hogy a martingálsághoz kell integrálhatóság, ami nem feltétlenül igaz, ha a θ_t folyamat nagy lehet. Ha bizonyos momentumfeltétel teljesül, akkor már Λ_t tényleg martingál. Lényegében a „tegyük föl, hogy" helyett gondolhatunk „legyen"-t is.

Bizonyítás. Először azt kell belátni, hogy $\mathbf{Q}$ tényleg valószínűségi mérték. Láttuk, hogy

$$\Lambda_t = 1 - \int_0^t \Lambda_s \theta_s \mathrm{d}W_s,$$

ami martingál, így

$$\mathbf{E}_\mathbf{P}\Lambda_T = \mathbf{E}_\mathbf{P}\Lambda_0 = 1,$$

és mivel $\Lambda_T > 0$ ezért $\mathbf{Q}$ tényleg valószínűségi mérték.

Most megmutatjuk, hogy a $\tilde{W}$ folyamat teljesíti a Lévy-féle karakterizációs tétel feltételeit a $\mathbf{Q}$ mérték szerint.

A folytonosság nyilvánvaló, hiszen W folytonos, és $\mathbf{Q} << \mathbf{P}$. Mivel Λ_t martingál, ezért a 9 Lemma szerint $(\tilde{W}_t)$ pontosan akkor $\mathbf{Q}$-martingál, ha $(\tilde{W}_t\Lambda_t)$ $\mathbf{P}$-martingál. Írjuk fel az Itô-formulát az $f(x,y) = xy$ függvényre a

$$\tilde{W}_t = \int_0^t \theta_s \mathrm{d}s + \int_0^t 1 \mathrm{d}W_s$$

$$\Lambda_t = 1 - \int_0^t \Lambda_s \theta_s \mathrm{d}W_s,$$

kétdimenziós Itô-folyamattal. Eszerint

$$\begin{aligned}\Lambda_t \tilde{W}_t &= \int_0^t \tilde{W}_s \mathrm{d}\Lambda_s + \int_0^t \Lambda_s \mathrm{d}\tilde{W}_s + \int_0^t -\Lambda_s \theta_s \mathrm{d}s \\ &= -\int_0^t \tilde{W}_s \Lambda_s \theta_s \mathrm{d}W_s + \int_0^t \Lambda_s \left(\theta_s \mathrm{d}s + \mathrm{d}W_s\right) - \int_0^t \Lambda_s \theta_s \mathrm{d}s \\ &= \int_0^t \Lambda_s (1 - \theta_s \tilde{W}_s) \mathrm{d}W_s,\end{aligned}$$

ami **P**-martingál. Tehát $(\tilde{W}_t)$ valóban **Q**-martingál.

Ahhoz, hogy a $(\tilde{W}_t^2 - t)$ folyamat **Q**-martingál, megint azt mutatjuk meg, hogy $(\tilde{W}_t^2 - t)\Lambda_t$ **P**-martingál. Először felírjuk $(\tilde{W}_t^2 - t)$ Itô-folyamatos reprezentációját. Az Itô-formulát az x^2 függvényre felírva

$$\tilde{W}_t^2 = 2\int_0^t \tilde{W}_s \mathrm{d}\tilde{W}_s + \frac{1}{2}\int_0^t 2\mathrm{d}t,$$

amit rendezve, és beírva $\tilde{W}_t$ előállítását

$$\tilde{W}_t^2 - t = 2\int_0^t \tilde{W}_s \left(\theta_s \mathrm{d}s + \mathrm{d}W_s\right).$$

A kétváltozós Itô-formulát felírva, mint az előbb

$$\begin{aligned}\Lambda_t(\tilde{W}_t^2 - t) &= \int_0^t \Lambda_s 2\tilde{W}_s \left(\theta_s \mathrm{d}s + \mathrm{d}W_s\right) + \int_0^t (\tilde{W}_s^2 - s)\mathrm{d}\Lambda_s - \int_0^t \Lambda_s \theta_s 2\tilde{W}_s \mathrm{d}s \\ &= \int_0^t \left[2\Lambda_s \tilde{W}_s - (\tilde{W}_s^2 - s)\Lambda_s \theta_s\right] \mathrm{d}W_s\end{aligned}$$

adódik, ami **P**-martingál. Tehát $(\tilde{W}_t^2 - t)$ **Q**-martingál, és ezzel az állítást beláttuk. □

Végül, bizonyítás (és precíz állítás) nélkül megemlítjük, hogy minden folytonos martingál előállítható, mint egy megfelelő adaptált folyamat Wiener-folyamat szerinti sztochasztikus integrálja. Vagyis minden szemimartingál Itô-folyamat.

19. Theorem (Martingál reprezentációs tétel)**.** *Legyen* (W_t) *SBM az* $(\Omega, \mathcal{A}, \mathbf{P})$ *valószínűségi mezőn, és legyen* $(\mathcal{F}_t)$ *a hozzá tartozó filtráció, azaz a* (W_t) *által* *generált filtráció, amihez hozzávesszük a* **P***-null halmazokat. Ha* (M_t) *folytonos, négyzetintegrálható martingál,* $M_0 = 0$ *m.b., akkor létezik olyan* (Y_t) *adaptált folyamat, melyre*

$$M_t = \int_0^t Y_s \mathrm{d}W_s.$$

9.2. Black–Scholes modell

Ebben a részben egy speciális folytonos modellben kiszámítjuk a követelések igazságos árát, és megadunk egy tökéletes replikáló portfóliót. Speciális esetként levezetjük a híres Black–Scholes-formulát, ami az európai call opció igazságos árát adja meg.

Legyen $r > 0, \mu \in \mathbb{R}$ és $\sigma > 0$. Legyen $(\Omega, \mathcal{A}, \mathbf{P})$ valószínűségi mező, (W_t) SBM a $[0, T]$ intervallumon, $T < \infty$, és $\mathcal{F}_t$ a (W_t)-hez tartozó filtráció. A *Black–Scholes-modellben* a kötvényárfolyamatot és a részvényárfolyamatot a

$$\begin{aligned} \mathrm{d}B_t &= rB_t\,\mathrm{d}t, \quad B_0 = 1, \\ \mathrm{d}S_t &= \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \quad S_0 = S_0, \end{aligned} \tag{19}$$

{eq:black-shcholes

differenciálegyenletek határozzák meg.

A kötvényárra $B_t = e^{rt}$ adódik, amit már a korábbiakban is feltettünk.

Az S_t Itô-folyamatként való felírása

$$S_t = S_0 + \int_0^t \mu S_s \mathrm{d}s + \int_0^t \sigma S_s \mathrm{d}W_s.$$

Az $f(x) = \log x$ függvénnyel felírva az Itô-formulát

$$\begin{aligned} \log S_t &= \log S_0 + \int_0^t \frac{1}{S_s}\left(\mu S_s \mathrm{d}s + \sigma S_s \mathrm{d}W_s\right) + \frac{1}{2}\int_0^t -\frac{1}{S_s^2}\sigma^2 S_s^2 \mathrm{d}s \\ &= \log S_0 + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t. \end{aligned}$$

Innen kapjuk, hogy

$$S_t = S_0 \cdot e^{\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t}. \tag{20}$$

{eq:exp-BM}

Ez a folyamat az *exponenciális Brown-mozgás.*

A differenciálegyenletes alakból azt is látjuk, hogy ez pontosan akkor martingál, ha a korlátos változású rész $\equiv 0$, azaz $\mu = 0$.

Vegyük észre, hogy ez a megoldás nem teljes, hiszen a logaritmus függvény nem kétszer folytonosan deriválható, a 0-ban nem definiált. Így az előbbi gondolatmenet csak segít megtalálni a megoldást.

18. Exercise. Igazoljuk az Itô-formula segítségével, hogy (20) valóban megoldása a differenciálegyenletnek!

(A feladat egy konstruktívabb megoldása az, hogy felírjuk az Itô-formulát egy általános f függvénnyel, majd megválasztjuk úgy az f-et, hogy minél egyszerűbb egyenletet kapjunk. Az $f(x) = \log x$ választás esetén a martingál részben az integrandus a konstans függvény lesz.)

9.2.1. Ekvivalens martingálmérték és az igazságos ár

Olyan mértéket szeretnénk megadni, mely szerint $(\overline{S}_t)$, a diszkontált részvényár martingál. Ezt a Girsanov-tétel segítségével adjuk meg. A (19) egyenlet alapján rövid számolás után kapjuk, hogy

$$\mathrm{d}\overline{S}_t = \overline{S}_t\left((\mu - r)\mathrm{d}t + \sigma \mathrm{d}W_t\right) = \overline{S}_t \sigma \mathrm{d}\widetilde{W}_t^{\mu}, \tag{21}$$ {eq:tildeS}

ahol

$$\widetilde{W}_t^{\mu} = W_t + \frac{\mu - r}{\sigma} t. \tag{22}$$ {eq:tildeW}

Ha találunk egy olyan $\mathbf{P}_\mu$ mértéket mely szerint a $\widetilde{W}_t^{\mu}$ folyamat SBM, akkor a (21) differenciálegyenlet szerint az $(\overline{S}_t)$ folyamat $\mathbf{P}_\mu$-martingál. A Girsanov-tétel éppen ilyen $\mathbf{P}_\mu$ mértéket definiál. Legyen $\theta_t \equiv \theta = \frac{\mu - r}{\sigma}$, és

$$\left.\frac{\mathrm{d}\mathbf{P}_\mu}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T} = \Lambda_T = \exp\left\{-\int_0^T \theta \mathrm{d}W_s - \frac{1}{2}\int_0^T \theta^2 \mathrm{d}s\right\} = e^{-\theta W_T - \frac{\theta^2 T}{2}}.$$

A Girsanov-tétel szerint $(\widetilde{W}_t^{\mu})$ éppen $\mathbf{P}_\mu$-SBM, és így $(\overline{S}_t)$ $\mathbf{P}_\mu$-martingál. Mivel $\Lambda_T > 0$ m.b., így $\mathbf{P} \sim \mathbf{P}_\mu$, tehát $\mathbf{P}_\mu$ EMM. Sőt, meg is határozhatjuk az $(\overline{S}_t)$ dinamikáját $\mathbf{P}_\mu$ szerint. Az (21) egyenletet megoldva

$$\overline{S}_t = S_0 \cdot e^{\sigma \widetilde{W}_t^{\mu} - \frac{\sigma^2}{2} t}. \tag{23}$$ {eq:S-mu}

Megmutatjuk, hogy a Black–Scholes-modellben az igazságos ár a (17) formulában szereplő alsó becslés. Legyen f_T egy tetszőleges olyan követelés, melyre $\mathbf{E} f_T^2 < \infty$. Tekintsük az

$$N_t = \mathbf{E}_{\mathbf{P}_\mu}\left[e^{-rT} f_T | \mathcal{F}_t\right],\ 0 \leq t \leq T,$$

$\mathbf{P}_\mu$-martingált. A martingál reprezentációs tétel szerint létezik olyan Y_t adaptált folyamat, hogy

$$N_t = N_0 + \int_0^t Y_s \mathrm{d}\widetilde{W}_s^\mu, \qquad (24)$$ {eq:N-def}

ahol persze $N_0 = \mathbf{E}_{\mathbf{P}_\mu} e^{-rT} f_T$. Definiáljuk a $\pi_t = (\beta_t, \gamma_t)$ stratégiát a

$$\beta_t = N_t - \frac{Y_t}{\sigma}, \quad \gamma_t = \frac{Y_t e^{rt}}{\sigma S_t}$$

formulával.

10. Lemma. *A $(\pi_t = (\beta_t, \gamma_t))$ stratégia önfinanszírozó, és $\overline{X}_t^\pi = N_t$.*

Bizonyítás. A definíció alapján

$$X_t^\pi = \beta_t B_t + \gamma_t S_t = \left(N_t - \frac{Y_t}{\sigma} \right) e^{rt} + \frac{Y_t}{\sigma} e^{rt} = e^{rt} N_t,$$

azaz $\overline{X}_t^\pi = N_t$.

Ahhoz, hogy megmutassuk, hogy π önfinanszírozó, a 7 Állítás szerint azt kell belátni, hogy $\mathrm{d}\overline{X}_t^\pi = \gamma_t \mathrm{d}\overline{S}_t$. Mivel $\overline{X}_t^\pi = N_t$, így (24) alapján

$$\mathrm{d}\overline{X}_t^\pi = \mathrm{d}N_t = Y_t \mathrm{d}\widetilde{W}_t^\mu.$$

Ugyanakkor (21) szerint

$$\gamma_t \mathrm{d}\overline{S}_t = \gamma_t \overline{S}_t \sigma \mathrm{d}\widetilde{W}_t^\mu = Y_t \mathrm{d}\widetilde{W}_t^\mu,$$

ahol az utolsó egyenlőségnél használtuk π definícióját. Ezzel az állítást beláttuk. □

Mivel

$$X_T^\pi = e^{rT} N_T = e^{rT} \mathbf{E}_{\mathbf{P}_\mu} \left[e^{-rT} f_T | \mathcal{F}_T \right] = f_T,$$

így a lemma szerint π egy tökéletes f_T-fedezet $X_0^\pi = N_0 = \mathbf{E}_{\mathbf{P}_\mu} e^{-rT} f_T$ kezdeti tőkével. Ezzel beláttuk az alábbit.

{tetel:bs-arazas}

20. Theorem. *A Black–Scholes-modellben egy f_T követelés igazságos ára*

$$C_T(f_T) = \mathbf{E}_{\mathbf{P}_\mu} e^{-rT} f_T.$$

Továbbá a $\pi_t = (\beta_t, \gamma_t)$,

$$\beta_t = N_t - \frac{Y_t}{\sigma}, \quad \gamma_t = \frac{Y_t e^{rt}}{\sigma S_t},$$

egy tökéletes fedezeti stratégia, ahol $N_t = \mathbf{E}_{\mathbf{P}_\mu}[e^{-rT} f_T | \mathcal{F}_t]$, és $N_t = N_0 + \int_0^t Y_s \mathrm{d}\widetilde{W}_s^\mu$.

9.2.2. A Black–Scholes-formula

A Black–Scholes-formula az európai call opció árára vonatkozik. Egy K kötési árú európai call opció kifizetési függvénye $f_T = (S_T - K)_+$. A 20 Tétel szerint az igazságos ár

$$C_T(K) = \mathbf{E}_{\mathbf{P}_\mu}\left(e^{-rT}(S_T - K)_+\right).$$

A (23) formula alapján

$$S_T = S_0 e^{rT} e^{\sigma \widetilde{W}_T^\mu - \frac{\sigma^2}{2}T},$$

ahol $\widetilde{W}_T^\mu \sim \mathrm{N}(0,T)$ a $\mathbf{P}_\mu$ mérték szerint. Tehát, bevezetve egy Z standard normális véletlen változót

$$\begin{aligned}
C_T(K) &= \mathbf{E}_{\mathbf{P}_\mu}\left(e^{-rT}(S_T - K)_+\right) \\
&= \mathbf{E}_{\mathbf{P}_\mu}\left(S_0 e^{\sigma \widetilde{W}_T^\mu - \frac{\sigma^2}{2}T} - e^{-rT}K\right)_+ \\
&= \mathbf{E}\left(S_0 e^{\sigma\sqrt{T}Z - \frac{\sigma^2}{2}T} - e^{-rT}K\right)_+ \\
&= \frac{1}{\sqrt{2\pi}}\int_\gamma^\infty \left(S_0 e^{\sigma\sqrt{T}x - \frac{\sigma^2}{2}T} - e^{-rT}K\right) e^{-\frac{x^2}{2}}\,\mathrm{d}x \\
&= S_0 \frac{1}{\sqrt{2\pi}}\int_\gamma^\infty e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}\,\mathrm{d}x - e^{-rT}K(1-\Phi(\gamma)) \\
&= S_0\left(1 - \Phi(\gamma - \sigma\sqrt{T})\right) - e^{-rT}K(1-\Phi(\gamma)),
\end{aligned}$$

ahol

$$\gamma = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{K}{S_0} + \left(\frac{\sigma^2}{2} - r\right)T\right].$$

A

$$C_T(K) = S_0\left(1 - \Phi(\gamma - \sigma\sqrt{T})\right) - e^{-rT}K(1-\Phi(\gamma))$$

árazási formula a híres *Black–Scholes-formula*, melyet 1973-ban publikált Fischer Black és Myron Scholes. A mögöttes elméletet később Merton általánosította. Munkájukért 1997-ben Scholes és Merton közgazdasági Nobel-díjat kapott, Black azért maradt ki, mert 1995-ben meghalt.

9.3. A CRR-formulától a Black–Scholes-formuláig

Ebben a részben megmutatjuk, hogy a Black–Scholes árazási formulát megkaphatjuk úgy, mint a homogén binomiális piacon a Cox–Ross–Rubinstein árazási formula határértékét. Ez a rész a [2] jegyzet 2.6 fejezetén alapul.

A folytonos modellt a $[0, T]$ intervallumon tekintjük. A folytonosan számított kamatláb $r > 0$, és $\sigma > 0$ rögzített paraméter, a volatilitás. A diszkrét modellben legyen

$$0 = \tau_0 < \tau_1 < \ldots < \tau_N = T, \quad \tau_i = \frac{i}{N}T.$$

Ezek a lehetséges kereskedési időpontok az N-lépéses binomiális modellben. Vezessük be a $T/N = h$ jelölést. Majd az $N \to \infty$ határátmenetet vizsgáljuk. Jelölje $B^N_{\tau_n}, S^N_{\tau_n}$ a kötvény, ill. a részvény árát a τ_n időpontban az N-edik piacon. Az N-lépéses diszkrét idejű homogén binomiális piac paraméterei legyenek r_N, a_N, és b_N.

Most megválasztjuk az r_N, a_N, b_N paramétereket. Legyen $B_0 = 1$. Folytonos időben a kötvényár t-ben $B_t = e^{rt}$. Diszkrét időben a t-hez tartozó osztópont $\tau_{\lfloor tN/T \rfloor}$, ahol $\lfloor x \rfloor$ az x egészrészét jelöli, ezt a későbbiekben elhagyjuk. Tehát

$$e^{rt} = B_t \approx B^N_{\tau_{\frac{tN}{T}}} = (1 + r_N)^{\lfloor \frac{tN}{T} \rfloor}.$$

Ha $r_N = rT/N = rh$, akkor a jobb oldal $N \to \infty$ esetén konvergál a bal oldalhoz. Legyen

$$r_N = r\frac{T}{N} = rh. \tag{25}$$ {eq:r-valaszt}

(A későbbiekben említés nélkül többször felhasználjuk, hogy $h = T/N$.) Hasonló okoskodással megmutatható, hogy ahhoz, hogy $\mathbf{Var} S^N_{\tau_N}$ határértéke $N \to \infty$ esetén létezzen, nagyjából az kell, hogy

$$\log \frac{1 + b_N}{1 + r_N} = \sigma\sqrt{h}, \quad \log \frac{1 + a_N}{1 + r_N} = -\sigma\sqrt{h} \tag{26}$$ {eq:ab-valaszt}

teljesüljön. Az N-edik modellben így választjuk a paramétereket. Belátjuk, hogy ilyen választás mellett a K kötési árú európai call opció binomiális modell alapján számolt igazságos ára $N \to \infty$ esetén a Black–Scholes-árhoz konvergál.

A binomiális modellben meghatároztuk az egyértelmű ekvivalens martingálmértéket. Ez az volt, mely szerint a részvényár

$$p^*_N = \frac{r_N - a_N}{b_N - a_N}$$

valószínűséggel $(1+b_N)$-szeresére nő, $1-p_N^*$ valószínűséggel $(1+a_N)$-szeresére, és az N-lépés során ezek egymástól függetlenül történnek. Vagyis a részvényár eloszlása a $\mathbf{P}_N^*$ EMM szerint

$$S_{\tau_N}^N = S_0(1+b_N)^{Y_N}(1+a_N)^{N-Y_N} = S_0\left(\frac{1+b_N}{1+a_N}\right)^{Y_N}(1+a_N)^N,$$

ahol $Y_N \sim \text{Binom}(N, p_N^*)$. A CRR árazási formula szerint a K kötési árú európai call igazságos ára

$$C_N(K) = \mathbf{E}_N^* \frac{(S_{\tau_N}^N - K)_+}{B_{\tau_N}^N}. \tag{27}$$ {eq:crr-ar}

Most meghatározzuk ennek a határértékét. A centrális határeloszlás-tétel szerint

$$\frac{Y_N - Np_N^*}{\sqrt{Np_N^*(1-p_N^*)}} \xrightarrow{\mathcal{D}} \text{N}(0,1),\ N\to\infty, \tag{28}$$ {eq:Y_N-conv}

ha $0 < \liminf_{N\to\infty} p_N^* \le \limsup_{N\to\infty} p_N^* < 1$, de majd megmutatjuk, hogy ez teljesül, sőt $\lim_{N\to\infty} p_N^* = 1/2$. A fenti formula bal oldalát kialakítva az $S_{\tau_N}^N$-ben,

$$\begin{aligned}\left(\frac{1+b_N}{1+a_N}\right)^{Y_N}(1+a_N)^N &= \exp\left\{Y_N \log\frac{1+b_N}{1+a_N} + N\log(1+a_N)\right\}\\ &= \exp\left\{\frac{Y_N - Np_N^*}{\sqrt{Np_N^*(1-p_N^*)}}\sqrt{Np_N^*(1-p_N^*)}\log\frac{1+b_N}{1+a_N}\right.\\ &\qquad \left. + N\left(p_N^*\log\frac{1+b_N}{1+a_N} + \log(1+a_N)\right)\right\}.\end{aligned}$$

Látjuk, hogy (28) alapján a

$$\lim_{N\to\infty}\sqrt{Np_N^*(1-p_N^*)}\log\frac{1+b_N}{1+a_N},\ \text{és}\quad \lim_{N\to\infty} N\left(p_N^*\log\frac{1+b_N}{1+a_N} + \log(1+a_N)\right)$$

határértékeket kell meghatároznunk. A (26) formula és a Taylor-sorfejtés szerint

$$\begin{aligned}1+b_N = e^{\sigma\sqrt{h}}(1+r_N) &= \left(1+\sigma\sqrt{h}+\frac{\sigma^2}{2}h + O(h^{3/2})\right)(1+rh)\\ &= 1+\sigma\sqrt{h}+\left(\frac{\sigma^2}{2}+r\right)h + O(h^{3/2}),\end{aligned}$$

így

$$b_N = \sigma\sqrt{h} + \left(\frac{\sigma^2}{2} + r\right) h + O(h^{3/2}),$$

és ugyanígy

$$a_N = -\sigma\sqrt{h} + \left(\frac{\sigma^2}{2} + r\right) h + O(h^{3/2}).$$

Innen kapjuk, hogy

$$\begin{aligned} p_N^* = \frac{r_N - a_N}{b_N - a_N} &= \frac{\sigma\sqrt{h} - \frac{\sigma^2}{2}h + O(h^{3/2})}{2\sigma\sqrt{h} + O(h^{3/2})} \\ &= \frac{1}{2 + O(h)} - \frac{\sigma\sqrt{h} + O(h)}{4 + O(h)} \\ &= \frac{1}{2} - \frac{\sigma}{4}\sqrt{h} + O(h). \end{aligned}$$

Rögtön látjuk, hogy $p_N^* \to 1/2$, tehát (28) valóban teljesül. A kapott aszimptotikákat visszaírva a kérdéses limeszekbe ($h = T/N$), és felhasználva a $\log(1 + x) = x - x^2/2 + O(x^3)$, $x \to 0$, sorfejtést (az elsőrendű sorfejtés nem elég!), kapjuk hogy

$$\lim_{N\to\infty} \sqrt{N p_N^*(1 - p_N^*)} \log \frac{1 + b_N}{1 + a_N} = \lim_{N\to\infty} \sqrt{p_N^*(1 - p_N^*)} 2\sigma\sqrt{T} = \sigma\sqrt{T},$$

és

$$\begin{aligned} &\lim_{N\to\infty} N \left(p_N^* \log \frac{1 + b_N}{1 + a_N} + \log(1 + a_N) \right) \\ &= \lim_{N\to\infty} N \left(\left[\frac{1}{2} - \frac{\sigma}{4}\sqrt{\frac{T}{N}} + O(N^{-1}) \right] 2\sigma\sqrt{\frac{T}{N}} - \sigma\sqrt{\frac{T}{N}} + r\frac{T}{N} + O(N^{-3/2}) \right) \\ &= \left(r - \frac{\sigma^2}{2} \right) T. \end{aligned}$$

Mindezt visszaírva (27)-be

$$\begin{aligned} \lim_{N\to\infty} C_N(K) &= e^{-rT} \mathbf{E}^* \left(S_0 e^{\sigma\sqrt{T}Z + T\left(r - \frac{\sigma^2}{2}\right)} - K \right)_+ \\ &= \mathbf{E}^* \left(S_0 e^{\sigma\sqrt{T}Z - \frac{\sigma^2}{2}T} - e^{-rT} K \right)_+, \end{aligned}$$

ami éppen a Black–Scholes-formulában kapott ár. Ezzel beláttuk, amit akartunk.

Itt persze a határátmenet jogosságáról hallgattunk. Valójában van egy eloszlásbeli konvergenciánk, mert (28)-ból következik a részvényár eloszlásbeli konvergenciája. Innen a momentumkonvergencia tétel alapján akkor következik a várható értékek konvergenciája, ha megmutatjuk az egyenletes integrálhatóságot. Mint már sokszor, ezt nem bizonyítjuk.

Azt is fontos megemlíteni, hogy nemcsak a részvényár lejáratkori eloszlása konvergál a Black–Scholes-modellben szereplő lejáratkori eloszláshoz, hanem az egész folyamat is (tehát mint a $[0, T]$ intervallumon értelmezett folytonos függvény) eloszlásban konvergál az exponenciális Brown-mozgáshoz. Ennek igazolása azonban már kifinomultabb technikát igényel.

10 Interest rate models

10.1 The general setup

In what follows we are interested in options on bonds instead of stocks. Therefore, we assume that the stock price B_t is also random. The bond price is given by

$$B_t = \exp\left\{\int_0^t r_u \mathrm{d}u\right\}, \tag{29}$$ {eq:bond}

where r_t, *the interest rate* is an adapted stochastic process. The time interval is $[0, \mathcal{T}]$. The stock price is given by

$$S_t = S_0 + \int_0^t \mu(u) S_u \mathrm{d}u + \int_0^t \sigma_u S_u \mathrm{d}W_u, \tag{30}$$ {eq:stock}

with some adapted process μ and σ. Note that the bond price B_t is a stochastic process too, but it is much smoother than the stock price S_t, as it is the exponential of the Lebesgue integral of a stochastic process. In particular, the path of B_t are of bounded variation, while the path of S_t are not. (Recall that an Itô process is of bounded variation if and only if the stochastic integral part vanishes.)

We want to find an equivalent martingale measure. For the discounted stock price $\overline{S}_t = S_t/B_t$

$$\begin{aligned} \mathrm{d}\frac{S_t}{B_t} &= \mathrm{d}\left(S_t e^{-\int_0^t r_u \mathrm{d}u}\right) \\ &= e^{-\int_0^t r_u} \mathrm{d}S_t + S_t(-r_t)e^{-\int_0^t r_u \mathrm{d}u}\mathrm{d}t \\ &= \overline{S}_t \left((\mu_t - r_t)\mathrm{d}t + \sigma_t \mathrm{d}W_t\right) \\ &= \overline{S}_t \sigma_t \mathrm{d}\widetilde{W}_t, \end{aligned}$$

where

$$\widetilde{W}_t = \int_0^t \theta_s \mathrm{d}s + W_t,$$

with $\theta_s = \frac{\mu_s - r_s}{\sigma_s}$. Applying Girsanov's theorem $\widetilde{W}_t$ is SBM under the measure Q_θ, where

$$\frac{\mathrm{d}Q_\theta}{\mathrm{d}\mathbf{P}} = \exp\left\{-\int_0^T \theta_s \mathrm{d}W_s - \frac{1}{2}\int_0^T \theta_s^2 \mathrm{d}s\right\}.$$

Therefore, under Q_θ the discounted stock price $\overline{S}_t$ is a martingale, i.e. Q_θ is an equivalent martingale measure.

We are not interested in the specific form of the underlying risky asset (S_t) in (30), but we assume that there exists a unique equivalent martingale measure (that is (S_t/B_t) is martingale). This will be the only measure on the probability space, therefore it is denoted by $\mathbf{P}$ (instead of Q_θ).

Formally, let $(\Omega, \mathcal{A}, (\mathcal{F}_t), \mathbf{P})$ be a filtered probability space, (r_u) an adapted stochastic process, and (B_t) is given by (29). We assume that the risky asset (S_t) is an adapted stochastic process, such that $(S_t/B_t)_t$ is a martingale under $\mathbf{P}$, and $\mathbf{P}$ is the unique such measure.

A *zero coupon bond (elemi kötvény)* maturing at time T is a claim that pays 1 at time T. Its value at time $t \in [0, T]$ is denoted by $P(t, T)$, $0 \leq t \leq T \leq \mathcal{T}$.

From the pricing theorem we see that the fair price of the zero coupon bond at time 0 is

$$P(0, T) = \mathbf{E}\left[\frac{1}{B_T}\right],$$

thus at time $0 \leq t \leq T$

$$P(t, T) = B_t \mathbf{E}\left[\frac{1}{B_T}\Big|\mathcal{F}_t\right] = \mathbf{E}\left[\exp\left\{-\int_t^T r_u \mathrm{d}u\right\}\Big|\mathcal{F}_t\right]. \qquad (31) \quad \texttt{\{eq:P(tT)\}}$$

A *term structure model (hozamgörbe modell)* is a mathematical model for the prices $P(t,T)$.

We are interested in pricing bond options. The fair price at time 0 of a *European call option* with strike price K at expiry date T_1 for a zero coupon bound with expiry date T_2, where $T_2 > T_1$, is given by

$$\mathbf{E}e^{-\int_0^{T_1} r_u \mathrm{d}u} \left(P(T_1,T_2) - K\right)_+ . \tag{32}$$ {eq:bond-calleu}

10.2 Short rate diffusion models

In short rate diffusion models the interest rate r_t is given as a solution of a stochastic differential equation.

10.2.1. Ornstein–Uhlenbeck-folyamat

Tekintsük az ún. *Langevin-egyenletet*

$$\mathrm{d}Y_t = -\mu Y_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t, \quad Y_0 \text{ független az } \sigma(W_s : s \geq 0)\ \sigma\text{-algebrától},$$

ahol $\mu > 0$, $\sigma > 0$. A homogén egyenlet megoldása $e^{-\mu t}$, és így differenciálegyenletek elméletéből ismert módszer szerint $e^{\mu t} Y_t$ differenciálját tekintjük. Ez

$$\mathrm{d}\left(e^{\mu t} Y_t\right) = e^{\mu t}\,\mathrm{d}Y_t + \mu\, e^{\mu t} Y_t \,\mathrm{d}t = e^{\mu t}\,\sigma\,\mathrm{d}W_t,$$

amit integrálva kapjuk a Langevin-egyenlet megoldását

$$Y_t = e^{-\mu t}\left(Y_0 + \int_0^t e^{\mu s}\,\sigma\,\mathrm{d}W_s\right).$$

Mivel determinisztikus függvény Wiener-folyamat szerinti sztochasztikus integrálja normális eloszlású, így

$$Y_t - e^{-\mu t} Y_0$$

normális eloszlású, várható értéke és szórásnégyzete

$$\mathbf{E}Y_t = e^{-\mu t}\,\mathbf{E}Y_0,$$
$$\mathbf{E}Y_t^2 = e^{-2\mu t}\,\mathbf{E}Y_0^2 + e^{-2\mu t}\int_0^t \sigma^2\, e^{2\mu s}\,\mathrm{d}s = e^{-2\mu t}\,\mathbf{E}Y_0^2 + \frac{\sigma^2}{2\mu}\,(1 - e^{-2\mu t}).$$

Innen látjuk, hogy Y_t eloszlásban konvergál egy $\mathrm{N}(0, \sigma^2/(2\mu))$ eloszláshoz, amint $t \to \infty$. Ez adja az ötletet, hogy válasszuk az Y_0 kezdeti értéket ilyen eloszlásúnak. Ezzel a kezdeti eloszlással

$$Y_t \sim \mathrm{N}\left(0, \frac{\sigma^2}{2\mu}\right),$$

és (Y_t) egy Gauss-folyamat. Meghatározzuk a kovarianciafüggvényét, ahonnan látjuk, hogy (Y_t) egy stacionárius Gauss-folyamat.

Vegyük észre, hogy az

$$Y_t = e^{-\mu t}\left(Y_0 + \int_0^t \sigma\, e^{\mu u}\, \mathrm{d}W_u\right)$$

előállítás alapján

$$Y_t - e^{-\mu(t-s)}Y_s = e^{-\mu t}\int_s^t \sigma\, e^{\mu u}\, \mathrm{d}W_u,\ t > s, \tag{33}$$ {eq:ou-fgt}

ami független a $\sigma(W_u : u \leq s)$ σ-algebrától. Ezért

$$\begin{aligned}\mathbf{Cov}(Y_t, Y_s) &= \mathbf{E}Y_tY_s = \mathbf{E}\left(Y_t - e^{-\mu(t-s)}Y_s + e^{-\mu(t-s)}Y_s\right)Y_s \\ &= e^{-\mu(t-s)}\,\mathbf{E}Y_s^2 = \frac{\sigma^2}{2\mu}e^{-\mu(t-s)},\end{aligned}$$

ami csak a $t-s$ különbségtől függ, azaz az (Y_t) folyamat valóban stacionárius.

A (33) formulából levezetjük, hogy az (Y_t) folyamat Markov-folyamat, majd meghatározzuk az átmenetsűrűségeket is. Valóban, ha $A \in \mathcal{B}(\mathbb{R})$, akkor

$$\begin{aligned}&\mathbf{P}\{Y_t \in A | Y_u : u \leq s, Y_s = x\} \\ &= \mathbf{P}\{Y_t - e^{-\mu(t-s)}Y_s \in A - e^{-\mu(t-s)}x | Y_u : u \leq s, Y_s = x\} \\ &= \mathbf{P}\{Y_t - e^{-\mu(t-s)}Y_s \in A - e^{-\mu(t-s)}x\}.\end{aligned}$$

Az $Y_t - e^{-\mu(t-s)}Y_s$ változó 0 várható értékű normális eloszlású, melynek szórásnégyzete (33) alapján

$$\mathbf{E}\left(Y_t - e^{-\mu(t-s)}Y_s\right)^2 = e^{-2\mu t}\int_s^t \sigma^2 e^{2\mu u}\mathrm{d}u = \frac{\sigma^2}{2\mu}\left(1 - e^{-2\mu(t-s)}\right).$$

Ebből következik s helyére 0-t helyettesítve, hogy

$$p_t(\cdot|x) \sim \mathrm{N}\left(e^{-\mu t}x, \frac{\sigma^2}{2\mu}\left(1-e^{-2\mu t}\right)\right),$$

vagyis az átmenetsűrűségek

$$\rho_t(y|x) = \sqrt{\frac{\mu}{\pi\sigma^2(1-e^{-2\mu t})}}\exp\left\{-\frac{\mu(y-e^{-\mu t}x)^2}{\sigma^2(1-e^{-2\mu t})}\right\}.$$

Az (Y_t) folytonos trajektóriájú stacionárius Markov-folyamatot Ornstein–Uhlenbeck-folyamatnak (OU) nevezzük. Megmutatható, hogy az OU-folyamat az egyetlen ilyen tulajdonságú folyamat.

Végül felírjuk a Kolmogorov-egyenleteket az átmenetsűrűségekre. A Kolmogorov-hátra egyenlet

$$\frac{\partial}{\partial t}\rho_t(y|x) = -\mu x\frac{\partial}{\partial x}\rho_t(y|x) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\rho_t(y|x),$$

alakú. Ezt az egyenletet *Fokker–Planck-egyenletnek* nevezik. Az előre egyenlet pedig

$$\frac{\partial}{\partial t}\rho_t(y|x) = -\frac{\partial}{\partial y}\left(-\mu y\rho_t(y|x)\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\rho_t(y|x).$$

10.2.2 Vasicek model

For r_0, a, b, σ given positive numbers let r_t is given by the stochastic differential equation

$$\mathrm{d}r_t = a(b - r_t)\mathrm{d}t + \sigma\mathrm{d}W_t, \tag{34}$$ {eq:vasicek}

where W_t is a standard Brownian motion. Thus r_t is a translated Ornstein–Uhlenbeck process. Indeed, $X_t = r_t - b$ satisfies

$$\mathrm{d}X_t = \mathrm{d}r_t = -aX_t\mathrm{d}t + \sigma\mathrm{d}W_t,$$

thus

$$X_t = e^{-at}\left(X_0 + \int_0^t e^{as}\sigma\mathrm{d}W_s\right),$$

from which

$$r_t = b + e^{-at}\left(r_0 - b + \int_0^t e^{as}\sigma\mathrm{d}W_s\right).$$

Thus r_t is normally distributed for any fixed t with mean

$$\mathbf{E} r_t = b + e^{-at}(r_0 - b)$$

and variance

$$\mathbf{Var}(r_t) = \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

This implies that r_t can take arbitrarily large negative values, which is not very realistic.

Now we determine the distribution of $P(t,T)$. By (31)

$$\begin{aligned} P(t,T) &= \mathbf{E}\left[\exp\left\{ - \int_t^T r_u \mathrm{d}u \right\} \middle| \mathcal{F}_t \right] \\ &= e^{-b(T-t)} \mathbf{E}\left[\exp\left\{ - \int_t^T X_u \mathrm{d}u \right\} \middle| \mathcal{F}_t \right], \end{aligned}$$

where $X_t = r_t - b$ as above. Since X_t is a Markov process, we have that

$$P(t,T) = e^{-b(T-t)} \mathbf{E} \exp\left\{ - \int_0^{T-t} \widetilde{X}_u \mathrm{d}u \right\}, \tag{35}$$ {eq:vasicek-pt}

where $\widetilde{X}$ is the solution to the Langevin equation

$$\mathrm{d}\widetilde{X}_s = -a\widetilde{X}_s + \sigma \mathrm{d}W_s, \qquad \widetilde{X}_0 = x_0 = r_t - b. \tag{36}$$ {eq:vasicek-initia

Therefore, we need to determine the distribution of

$$\int_0^t \widetilde{X}_u \mathrm{d}u.$$

We have seen that (X_u) is a continuous Gaussian process, therefore its integral is Gaussian too. Since $\mathbf{E}\widetilde{X}_u = e^{-at}x_0$, we have

$$\mathbf{E} \int_0^t \widetilde{X}_u \mathrm{d}u = x_0 \int_0^t e^{-au} \mathrm{d}u = \frac{x_0}{a}(1 - e^{-at}).$$

Furthermore, for $t \geq s$

$$\begin{aligned}\mathbf{Cov}(\widetilde{X}_t, \widetilde{X}_s) &= \mathbf{E}e^{-at}\int_0^t \sigma e^{au}\mathrm{d}W_u e^{-as}\int_0^s \sigma e^{au}\mathrm{d}W_u \\ &= \sigma^2 e^{-a(t+s)}\mathbf{E}\left(\int_0^s e^{au}\mathrm{d}W_u\right)^2 \\ &= \sigma^2 e^{-a(t+s)}\int_0^s e^{2au}\mathrm{d}u \\ &= \frac{\sigma^2}{2a}e^{-a(t+s)}\left(e^{2as}-1\right).\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{Var}\left(\int_0^t \widetilde{X}_u\mathrm{d}u\right) &= \mathbf{Cov}\left(\int_0^t \widetilde{X}_u\mathrm{d}u, \int_0^t \widetilde{X}_u\mathrm{d}u\right) \\ &= \mathbf{E}\int_0^t (\widetilde{X}_u - \mathbf{E}\widetilde{X}_u)\mathrm{d}v\int_0^t (\widetilde{X}_v - \mathbf{E}\widetilde{X}_v)\mathrm{d}v \\ &= \int_0^t\int_0^t \mathbf{E}(\widetilde{X}_u - \mathbf{E}\widetilde{X}_u)(\widetilde{X}_v - \mathbf{E}\widetilde{X}_v)\mathrm{d}u\mathrm{d}v \\ &= \int_0^t\int_0^t \mathbf{Cov}(\widetilde{X}_u, \widetilde{X}_v)\mathrm{d}u\mathrm{d}v \\ &= 2\int_0^t\int_0^v \mathbf{Cov}(\widetilde{X}_u, \widetilde{X}_v)\mathrm{d}u\mathrm{d}v \\ &= 2\int_0^t\int_0^v \frac{\sigma^2}{2a}e^{-a(u+v)}\left(e^{2au}-1\right)\mathrm{d}u\mathrm{d}v \\ &= \frac{\sigma^2}{2a^3}\left(at - 3 + 4e^{-at} - e^{-2at}\right).\end{aligned}$$

Thus we have the expectation and variance of the Gaussian random variable $\int_0^t \widetilde{X}_u\mathrm{d}u$. Since $\mathbf{E}e^{t(aZ+b)} = e^{a^2t^2/2+bt}$ for $Z \sim N(0,1)$, we have

$$\mathbf{E}\exp\left\{\int_0^t \widetilde{X}_u\mathrm{d}u\right\} = \exp\left\{-\frac{x_0}{a}(1-e^{-at}) + \frac{\sigma^2}{4a^3}\left(at - 3 + 4e^{-at} - e^{-2at}\right)\right\}.$$

Substituting back into (35) and using the initial condition (36), we obtain

$$\begin{aligned}P(t,T) = \exp\Big\{ &- b(T-t) - \frac{r_t - b}{a}(1 - e^{-a(T-t)}) \\ &+ \frac{\sigma^2}{4a^3}\left(a(T-t) - 3 + 4e^{-a(T-t)} - e^{-2a(T-t)}\right)\Big\}.\end{aligned}$$

The fair price of a European call option with strike price K at T_1 for a zero coupon bond with expiry $T_2 > T_1$ is

$$C(K; T_1, T_2) = \mathbf{E} e^{-\int_0^{T_1} r_t \mathrm{d}t} \left(P(T_1, T_2) - K\right)_+ . \tag{37}$$ {eq:vasicek-eucall

Since $P(T_1, T_2)$ is determined by r_{T_1}, to evaluate the latter integral we need the joint distribution of $\int_0^{T_1} r_t dt$ and r_{T_1}. They are jointly Gaussian, and their covariance is

$$\begin{aligned} \mathbf{Cov}\left(\int_0^t r_u \mathrm{d}u, r_t\right) &= \int_0^t \mathbf{Cov}(r_u, r_t)\mathrm{d}u \\ &= \int_0^t \frac{\sigma^2}{2a} e^{-a(t+u)}(e^{2au} - 1)\mathrm{d}u \\ &= \frac{\sigma^2}{2a^2}(1 - 2e^{-at} + e^{-2at}). \end{aligned}$$

Therefore, the fair price in (37) is

$$\begin{aligned} &C(K; , T_1, T_2) \\ &= \mathbf{E} e^{-U} \Bigg(\exp\Big\{ - b(T_2 - T_1) - \frac{V - b}{a}(1 - e^{-a(T_2 - T_1)}) \\ &\quad + \frac{\sigma^2}{4a^3}\left(a(T_2 - T_1) - 3 + 4e^{-a(T_2 - T_1)} - e^{-2a(T_2 - T_1)}\right) \Big\} - K \Bigg)_+ , \end{aligned}$$

where (U, V) is a two dimensional normal random vector with covariance matrix

$$\begin{pmatrix} \frac{\sigma^2}{2a^3}\left(aT_1 - 3 + 4e^{-aT_1} - e^{-2aT_1}\right) & \frac{\sigma^2}{2a^2}(1 - 2e^{-aT_1} + e^{-2aT_1}) \\ \frac{\sigma^2}{2a^2}(1 - 2e^{-aT_1} + e^{-2aT_1}) & \frac{\sigma^2}{2a}(1 - e^{-2aT_1}). \end{pmatrix}$$

The main point here is that there exists an explicit formula, which can be computed numerically easily.

10.2.3 Hull–White model

This is a simple generalization of the Vasicek model, where we allow the parameters to be time dependent. Assume that for some deterministic functions a, b, and σ

$$\mathrm{d}r_t = (a(t) - b(t)r_t)\mathrm{d}t + \sigma(t)\mathrm{d}W_t, \qquad r_0 = r_0 > 0. \tag{38}$$ {eq:HW}

Then the solution is

$$r(t) = e^{-\beta(t)} \left(r_0 + \int_0^t e^{\beta(u)} a(u) \mathrm{d}u + \int_0^t e^{\beta(u)} \sigma(u) \mathrm{d}W_u \right).$$

where $\beta(t) = \int_0^t b(u)\mathrm{d}u$.

10.2.4 Cox–Ingersoll–Ross model

In the Vasicek and the Hull–White model the distribution of r_t is normal for any t, therefore it can take any large negative number, which is not so realistic. In the following model r_t is nonnegative.

First consider n independent Ornstein–Uhlenbeck processes, that is

$$\mathrm{d}X_i(t) = -\frac{1}{2}\alpha X_i(t)\mathrm{d}t + \frac{\sigma}{2}\mathrm{d}W_i(t), \quad i = 1, 2, \ldots, n,$$

where $W_1, \ldots, W_n$ are independent standard Brownian motions. Then

$$X_i(t) = e^{-\frac{\alpha}{2}t} \left(X_i(0) + \frac{\sigma}{2} \int_0^t e^{\frac{\alpha}{2}s} \mathrm{d}W_i(s) \right).$$

Put

$$r_t = X_1^2(t) + \ldots + X_n^2(t).$$

Using the multivariate version of Itô's formula

$$\begin{aligned}
\mathrm{d}r_t &= \sum_{i=1}^n 2X_i(t)\mathrm{d}X_i(t) + \sum_{i=1}^n \frac{\sigma^2}{4}\mathrm{d}t \\
&= -\alpha r_t \mathrm{d}t + \sigma \sum_{i=1}^n X_i(t)\mathrm{d}W_i(t) + \frac{n\sigma^2}{4}\mathrm{d}t \\
&= \left(\frac{n\sigma^2}{4} - \alpha r_t \right) \mathrm{d}t + \sigma\sqrt{r_t} \sum_{i=1}^n \frac{X_i(t)}{\sqrt{r_t}} \mathrm{d}W_i(t).
\end{aligned}$$

The process

$$W_t = \sum_{i=1}^n \int_0^t \frac{X_i(u)}{\sqrt{r_u}} \mathrm{d}W_i(u)$$

is a continuous martingale, such that

$$W_t^2 = 2\int_0^t W_u \mathrm{d}W_u + \sum_{i=1}^n \int_0^t \frac{X_i(u)^2}{r_u}\mathrm{d}u$$
$$= 2\int_0^t W_u \mathrm{d}W_u + t,$$

which means that $W_t^2 - t$ is a martingale too. Therefore, by Lévy's characterization of the Wiener process we obtain that W_t is a SBM. Substituting back we have

$$\mathrm{d}r_t = \left(\frac{n\sigma^2}{4} - \alpha r_t\right)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t$$

with W_t SBM. This is the definition of the Cox–Ingersoll–Ross (CIR) process.

The *CIR process* with parameters $a > 0$, $b > 0$, $\sigma > 0$ is the solution of the stochastic differential equation

$$\mathrm{d}r_t = (a - br_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t. \tag{39}$$

Note that existence and uniqueness result for SDE's does not apply here, because the function $\sqrt{x}$ is not Lipschitz at 0. However, it can be shown that a unique strictly positive solution exist for $a \geq \sigma^2/2$. We have seen this for $a = n\sigma^2/4$.

We have seen that at determining the fair price of a European call we need the joint distribution of $(r_t, \int_0^t r_u \mathrm{d}u)$. The joint Laplace transform of the vector can be determined explicitly. We state the following result without proof.

Theorem 21. *For any $u \geq 0$, $v \geq 0$*

$$\mathbf{E}\exp\left\{-ur_t - v\int_0^t r_s \mathrm{d}s\right\} = e^{-a\phi_{u,v}(t) - r_0\psi_{u,v}(t)},$$

where

$$\phi_{u,v}(t) = -\frac{2}{\sigma^2}\log\left(\frac{2\gamma e^{t(b+\gamma)/2}}{\sigma^2 u(e^{\gamma t}-1) + \gamma - b + e^{\gamma t}(\gamma + b)}\right)$$
$$\psi_{u,v}(t) = \frac{u(\gamma+b) + e^{\gamma t}(\gamma - b) + 2v(e^{\gamma t}-1)}{\sigma^2 u(e^{\gamma t}-1) + \gamma - b + e^{\gamma t}(\gamma + b)},$$

where $\gamma = \sqrt{b^2 + 2\sigma^2 v}$.

Therefore, using the result above and the Markov property the value of the zero coupon bound

$$\begin{aligned}P(t,T) &= \mathbf{E}\left[e^{-\int_t^T r_u \mathrm{d}u}\middle|\mathcal{F}_t\right] \\ &= \mathbf{E}\left[e^{-\int_0^{T-t} r_u \mathrm{d}u}\middle|r_0 = r_t\right] \\ &= \exp\left\{-a\phi_{0,1}(T-t) - r_t\psi_{0,1}(T-t)\right\}\end{aligned}$$

The price of a European call option with strike price K at T_1 for a zero coupon bound with expiry $T_2 > T_1$

$$\begin{aligned}&C(K;T_1,T_2) \\ &= \mathbf{E}\left[e^{-\int_0^{T_1} r_u \mathrm{d}u}\left(\exp\left\{-a\phi_{0,1}(T_2-T_1) - r_{T_1}\psi_{0,1}(T_2-T_1)\right\} - K\right)_+\right].\end{aligned}$$

This is not an explicit formula, but we now the joint Laplace transform of the vector $(\int_0^{T_1} r_u \mathrm{d}u, r_{T_1})$, therefore it is numerically computable.

10.3 The Heath–Jarrow–Morton model

10.3.1 Forward rate

Assume that at time t we buy one zero coupon bond with expiry T and short sell $P(t,T)/P(t,T+\varepsilon)$ unit zero coupon bond with expiry $T+\varepsilon$. The value of this portfolio at t

$$P(t,T) - \frac{P(t,T)}{P(t,T+\varepsilon)}P(t,T+\varepsilon) = 0,$$

so it costs nothing. What happens is that at time T we borrow 1 dollar, and we have to pay $P(t,T)/P(t,T+\varepsilon)$ at time $T+\varepsilon$. Therefore the interest rate we pay at time T is $R(t,T,T+\varepsilon)$

$$\frac{P(t,T)}{P(t,T+\varepsilon)} = e^{\varepsilon R(t,T,T+\varepsilon)},$$

that is

$$R(t,T,T+\varepsilon) = -\frac{1}{\varepsilon}\left(\log P(t,T+\varepsilon) - \log P(t,T)\right).$$

Thus the instantaneous forward interest rate at time T calculated at time t, called *forward rate* is

$$f(t,T) = \lim_{\varepsilon\downarrow 0} R(t,T,T+\varepsilon) = -\frac{\partial}{\partial T}\log P(t,T). \tag{40}$$

Intuitively, it is clear that at time t we predict the interest at time t to equal the short rate r_t, that is $r_t = f(t,t)$. In what follows we prove this statement.

{lemma:forward-sho

Lemma 11. *For any* $t \in [0, \mathcal{T}]$

$$f(t,t) = r_t.$$

Proof. As $\log P(t,t) = 0$

$$\log P(t,T) = \int_t^T \frac{\partial}{\partial T} \log P(t,u) \mathrm{d}u = -\int_t^T f(t,u) \mathrm{d}u,$$

we obtain

$$P(t,T) = e^{-\int_t^T f(t,u)\mathrm{d}u}. \tag{41}$$ {eq:P-f}

Differentiating

$$\frac{\partial}{\partial T} P(t,T) = -f(t,T)P(t,T),$$

which at $t = T$

$$\left.\frac{\partial}{\partial T} P(t,T)\right|_{T=t} = -f(t,t),$$

On the other hand, differentiating

$$P(t,T) = \mathbf{E}\left[e^{-\int_t^T r_u \mathrm{d}u} \middle| \mathcal{F}_t \right]$$

we obtain

$$\frac{\partial}{\partial T} P(t,T) = \mathbf{E}\left[-r_T e^{-\int_t^T r_u \mathrm{d}u} \middle| \mathcal{F}_t \right]$$

which at $T = t$

$$\left.\frac{\partial}{\partial T} P(t,T)\right|_{T=t} = -r_t,$$

and the statement follows. □

10.3.2 The Heath–Jarrow–Morton model

The *Heath–Jarrow–Morton (HJM) model* describes the dynamic of the forward rate $f(t,T)$ with the SDE

$$\mathrm{d}f(t,T) = \alpha(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}W_t, \tag{42}$$ {eq:HJM}

which holds for every $0 \le t \le T \le \mathcal{T}$, where α and σ are adapted processes.

Note that the model has two time scales. In the followings we determine the necessary conditions on α and σ. We have

$$\begin{aligned}
\mathrm{d}\left(-\int_t^T f(t,u)\mathrm{d}u\right) &= f(t,t)\mathrm{d}t - \int_t^T (\mathrm{d}f(t,u))\mathrm{d}u \\
&= r_t\mathrm{d}t - \int_t^T (\alpha(t,u)\mathrm{d}t + \sigma(t,u)\mathrm{d}W_t)\mathrm{d}u \\
&= r_t\mathrm{d}t - \alpha^*(t,T)\mathrm{d}t - \sigma^*(t,T)\mathrm{d}W_t,
\end{aligned}$$

where

$$\alpha^*(t,T) = \int_t^T \alpha(t,u)\mathrm{d}u, \quad \sigma^*(t,T) = \int_t^T \sigma(t,u)\mathrm{d}u.$$

Here we use a stochastic version of Fubini's theorem, which we did not even formulate. Put

$$X_t = \log P(t,t) = -\int_t^T f(t,u)\mathrm{d}u.$$

Then the above calculation gives

$$\mathrm{d}X_t = (r_t - \alpha^*(t,T))\mathrm{d}t - \sigma^*(t,T)\mathrm{d}W_t.$$

Thus

$$\begin{aligned}
\mathrm{d}P(t,T) &= e^{X_t}\left(r_t - \alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\right)\mathrm{d}t - e^{X_t}\sigma^*(t,T)\mathrm{d}W_t \\
&= P(t,T)\left[\left(r_t - \alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\right)\mathrm{d}t - \sigma^*(t,T)\mathrm{d}W_t\right].
\end{aligned}$$

Under the equivalent martingale measure the discounted value process of a zero coupon bond

$$e^{-\int_0^t r_u \mathrm{d}u}P(t,T)$$

is a martingale. Since

$$\begin{aligned}
\mathrm{d}\left(e^{-\int_0^t r_u \mathrm{d}u}P(t,T)\right) &= e^{-\int_0^t r_u \mathrm{d}u}\mathrm{d}P(t,T) - r_t e^{-\int_0^t r_u \mathrm{d}u}P(t,T)\mathrm{d}t \\
&= e^{-\int_0^t r_u \mathrm{d}u}P(t,T)\left[\left(-\alpha^*(t,T) + \frac{1}{2}\sigma^*(t,T)^2\right)\mathrm{d}t - \sigma^*(t,T)\mathrm{d}W_t\right],
\end{aligned}$$

which is martingale if and only if for any $0 \le t \le T \le \mathcal{T}$

$$\alpha^*(t,T) = \frac{1}{2}\sigma^*(t,T)^2.$$

Substituting back the definition of α^* and σ^*, after differentiation we obtain that

$$\alpha(t,T) = \sigma(t,T)\int_0^T \sigma(t,u)\mathrm{d}u. \tag{43}$$ {eq:HJM-alpha-sigm

We proved the following.

Theorem 22. *If the HJM model is determined by the SDE* (42) *then necessarily* (43) *holds.*

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