we may choose $\varepsilon > 0$ small enough such that

$$q_i' = q_i - \varepsilon y_i > 0 \quad \text{for all } i.$$

$q' = q - \varepsilon \cdot y$

As both q and y are orthogonal to $\mathcal{V}_0$, q' is also orthogonal. Define the measure

$$\mathbf{Q}'(\{\omega_i\}) = \frac{q_i'}{\sum_{i=1}^k q_i'}.$$

Exactly as in the previous proof we can show that $\mathbf{Q}'$ is EMM. The uniqueness of the EMM implies

$$\frac{q_i'}{\sum_{i=1}^k q_i'} = \frac{q_i}{\sum_{i=1}^k q_i},$$

that is, using also the definition of q',

$\alpha = \frac{\Sigma q_i}{\Sigma q_i'}$

$$q = \alpha q' = \alpha q - \alpha \varepsilon y,$$

with $\alpha = \sum q_i / \sum q_i'$. Thus

$$(1 - \alpha) q = -\alpha \varepsilon y.$$

But y and q are orthogonal, which is a contradiction. The proof is complete.

4 Girsanov's theorem in discrete time

4.1 Second proof of the difficult part of Theorem 3

Assume that $d = 1$ and first consider the one-step model with $B_0 = B_1 = 1$. The stock price S_0 is known, and the only randomness here is S_1.

Exercise 9. The no arbitrage assumption (in this simple market) is equivalent to

$$\mathbf{P}(\Delta S_1 > 0)\mathbf{P}(\Delta S_1 < 0) > 0.$$

$\Delta S_1 = S_1 - S_0$

Furthermore, (S_n) is martingale if

$\left(\frac{S_n}{B_n}\right)_n$ mtg $\hat{Q}$

$$\mathbf{E}_\mathbf{Q} S_1 = S_0.$$

$E_Q[S_1 | \mathcal{F}_0] = E_Q[S_1]$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$

Therefore we have to construct a measure $\mathbf{Q}$ such that $\mathbf{E}_\mathbf{Q} \Delta S_1 = 0$. This is done in the following lemma.

$X = \Delta S_1$

Lemma 6. *Let X be a random variable on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbf{P})$ such that $\mathbf{P}(X > 0)\mathbf{P}(X < 0) > 0$. Then there exists a probability measure $\mathbf{Q} \sim \mathbf{P}$ such that $\mathbf{E}_\mathbf{Q} X = 0$. Furthermore, for any $a \in \mathbb{R}$*

$$\mathbf{E}_\mathbf{Q} e^{aX} < \infty.$$

$(\Omega, \mathcal{F}, P) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), P)$

Proof. Define the probability measure

$$P_1(\mathrm{d}x) = ce^{-x^2} F(\mathrm{d}x),$$

F által indukált L-S mérték

where $F(x) = \mathbf{P}(X \leq x)$ and $c^{-1} = \int_\mathbb{R} e^{-x^2} F(\mathrm{d}x)$. That is

$\mu_F((a,b]) = F(b) - F(a)$

$d\mu_F =: dF$

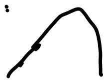
$$P_1(A) = \int_A ce^{-x^2} F(\mathrm{d}x).$$

Then P_1 is equivalent to F. (Recall that μ is absolute continuous with respect to ν, $\mu \ll \nu$ if $\mu(A) = 0$ whenever $\nu(A) = 0$. And μ and ν are equivalent, $\mu \sim \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$.) Let

$$\varphi(a) = \mathbf{E}_{P_1} e^{aX} = \int_\mathbb{R} e^{ax} P_1(\mathrm{d}x) = c \int_\mathbb{R} e^{ax-x^2} F(\mathrm{d}x).$$

$e^{ax-x^2} : \mathbb{R} \to \mathbb{R}$

$ax - x^2$:



Clearly, $\varphi(a) < \infty$ for any a as the function e^{ax-x^2} is bounded on $\mathbb{R}$. Note that φ is convex, because $\varphi'' > 0$. Put

$$Z_a(x) = \frac{e^{ax}}{\varphi(a)}.$$

$\varphi'(a) = E_{P_1} X e^{aX}$

$\varphi''(a) = E_{P_1} \underbrace{X^2 e^{aX}}_{\geq 0} > 0$

Then

$$Q_a(\mathrm{d}x) = Z_a(x) P_1(\mathrm{d}x)$$

is a probability measure for any a, and $Q_a \sim P_1 \sim F$. Again, this means

$$Q_a(A) = \int_A Z_a(x) P_1(\mathrm{d}x) = \frac{c}{\varphi(a)} \int_A e^{ax-x^2} F(\mathrm{d}x).$$

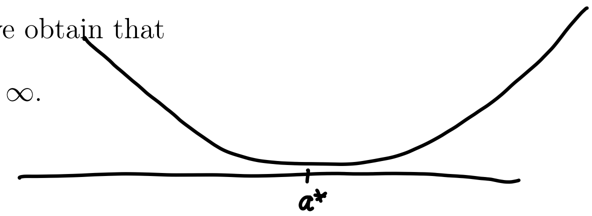
spec: $A = \mathbb{R}$ $\quad \mathbb{R} \quad \mathbb{R} \quad \mathbb{R} \quad = 1$

Let

$$\varphi_* = \inf_{a \in \mathbb{R}} \varphi(a).$$

Since $P_1(X > 0) > 0$ and $P_1(X < 0) > 0$ we obtain that

$$\lim_{a \to \pm\infty} \varphi(a) = \infty.$$



Therefore, the infimum is attained, i.e. there is a_* such that $\varphi(a_*) = \varphi_*$. Then $\varphi'(a_*) = 0$, thus

$$0 = \varphi'(a_*) = \mathbf{E}_{P_1} X e^{a_* X} = \varphi(a_*) \mathbf{E}_{P_1} X \frac{e^{a_* X}}{\varphi(a_*)} = \varphi(a_*) \mathbf{E}_{Q_{a_*}} X.$$

Thus the measure Q_{a_*} works. □

Exercise 10. Prove rigorously that

$$\lim_{a \to \pm\infty} \varphi(a) = \infty.$$

Exercise 11. Let $X \sim N(\mu, \sigma^2)$. Determine the measure constructed above explicitly.

Next we extend the previous lemma for a general N-step market.

$B_0 = B_1 = \dots = B_N = 1.$

Exercise 12. The no arbitrage assumption implies that for any n a.s.

$$\mathbf{P}(\Delta S_n > 0 | \mathcal{F}_{n-1}) \mathbf{P}(\Delta S_n < 0 | \mathcal{F}_{n-1}) > 0.$$

As a preliminary result we have to understand how to compute conditional expectation under different measures.

{lemma:condexp-mea

Lemma 7. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0,1,\dots,N}, \mathbf{P})$ a filtered probability space, and Z a nonnegative random variable $\mathbf{E_P} Z = 1$. Define the new probability measure $\mathbf{Q}$ as*

$$\mathrm{d}\mathbf{Q} = Z \mathrm{d}\mathbf{P},$$

$(\Omega, \mathcal{F}, \mathbf{P})$, $\mathcal{G} \subseteq \mathcal{F}$ sub σ-alg.

that is

$$\mathbf{Q}(A) = \int_A Z \mathrm{d}\mathbf{P}.$$

$E_{\mathbf{P}}[X|\mathcal{G}]$: – $\mathcal{G}$-mb.

$\mathbf{Q}$

– $\int_G E_P[X|\mathcal{G}]\, dP = \int_G X\, dP$

$\mathbf{Q}$ $\mathbf{Q}$ $\mathbf{Q}$

$\forall G \in \mathcal{G}$.

Put $Z_n = \mathbf{E_P}[Z|\mathcal{F}_n]$. For any adapted process (X_n)

$$Z_{n-1} \mathbf{E_Q}[X_n | \mathcal{F}_{n-1}] = \mathbf{E_P}[X_n Z_n | \mathcal{F}_{n-1}].$$

Proof. Both sides are $\mathcal{F}_{n-1}$-measurable. We have to prove that for any $A \in \mathcal{F}_{n-1}$

$$\int_A Z_{n-1} \underbrace{\mathbf{E_Q}[X_n|\mathcal{F}_{n-1}]}_{=Y} \mathrm{d}\mathbf{P} = \int_A X_n Z_n \mathrm{d}\mathbf{P}. \qquad (8)$$ {eq:cemlemma-0}

(X_n) Q-mtg $\Leftrightarrow$ $(X_n Z_n)$ P-mtg

Lemma: $Z_{n-1} E_Q[X_n | \mathcal{F}_{n-1}] = E_P[X_n Z_n | \mathcal{F}_{n-1}]$ P/Q - m.b.

Köv: (X_n) Q-mtg $\iff$ $(X_n Z_n)$ P-mtg

Biz: $\Rightarrow$: Tfh.: (X_n) Q-mtg:

$$E_Q[X_n | \mathcal{F}_{n-1}] = X_{n-1}$$

$$Z_{n-1} \cdot E_Q[X_n | \mathcal{F}_{n-1}] = X_{n-1} \cdot Z_{n-1} \overset{\text{Lemma}}{=} E_P[X_n Z_n | \mathcal{F}_{n-1}]$$

$(X_n Z_n)$ P-mtg

$\Leftarrow$: uaz.

↯

First note that

$$\mathbf{E}_{\mathbf{P}}[ZX_n|\mathcal{F}_n] = X_n\mathbf{E}_{\mathbf{P}}[Z|\mathcal{F}_n] = X_nZ_n. \qquad (9)$$ {eq:cemlemma-1}

Therefore, for an $\mathcal{F}_{n-1}$-measurable Y

$$\mathbf{E}_{\mathbf{P}}[Z_{n-1}Y|\mathcal{F}_{n-1}] = Y\overbrace{\mathbf{E}_{\mathbf{P}}[Z|\mathcal{F}_{n-1}]}^{= Z_{n-1}},$$

implying for any $A \in \mathcal{F}_{n-1}$ that

$$\begin{aligned}\int_A Z_{n-1}Y\mathrm{d}\mathbf{P} &= \int_A Y\mathbf{E}_{\mathbf{P}}[Z|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{P} \\ &= \int_A \mathbf{E}_{\mathbf{P}}[ZY|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{P} = \int_A YZ\mathrm{d}\mathbf{P}.\end{aligned}$$

Choosing $Y = \mathbf{E}_{\mathbf{Q}}[X_n|\mathcal{F}_{n-1}]$ we obtain

$$\begin{aligned}\int_A Z_{n-1}\mathbf{E}_{\mathbf{Q}}[X_n|\mathcal{F}_{n-1}]\mathrm{d}\mathbf{P} &= \int_A \mathbf{E}_{\mathbf{Q}}[X_n|\mathcal{F}_{n-1}]Z\mathrm{d}\mathbf{P} \\ &= \int_A \mathbf{E}_{\underline{\mathbf{Q}}}[X_n|\mathcal{F}_{n-1}]\mathrm{d}\underline{\mathbf{Q}} && \text{definition of } \mathbf{Q} \\ &= \int_A X_n\mathrm{d}\mathbf{Q} && \text{conditional exp.} \\ &= \int_A X_nZ\mathrm{d}\mathbf{P} && \text{definition of } \mathbf{Q} \\ &= \int_A X_nZ_n\mathrm{d}\mathbf{P}, && \text{by (9)}\end{aligned}$$

which is (8). □

As a simple but useful corollary we obtain the following.

{cor:p-q-mtg}

Corollary 1. *The adapted process (X_n) is $\mathbf{Q}$-martingale if and only if (X_nZ_n) is $\mathbf{P}$-martingale.*

{lemma:existence-e

$X_n = \Delta S_n = S_n - S_{n-1}$

Lemma 8. *Let $(X_n)_{n=1}^N$ be an adapted process, and assume that*

$$\mathbf{P}(X_n > 0|\mathcal{F}_{n-1})\mathbf{P}(X_n < 0|\mathcal{F}_{n-1}) > 0.$$

Then there exists a probabilty measure $\mathbf{Q} \sim \mathbf{P}$ such that (X_n) is a $\mathbf{Q}$-martingale difference.

$\mathbb{E}_Q[X_n | \mathcal{F}_{n-1}] = 0$

$(\Omega, \mathcal{F}, P)$ $(X_n), (\mathcal{F}_n)$

Proof. First let

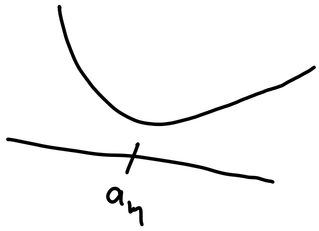
$$P_1(\mathrm{d}\omega) = c \overbrace{\exp\left\{-\sum_{i=0}^{N} X_i^2(\omega)\right\}}^{\geq 0} \mathbf{P}(\mathrm{d}\omega),$$

where c is the normalizing factor, i.e.

$$c^{-1} = \int_\Omega \exp\left\{-\sum_{i=0}^{N} X_i^2\right\} \mathrm{d}\mathbf{P} = \mathbf{E} \overbrace{\exp\left\{-\sum_{i=0}^{N} X_i^2\right\}}^{\leq 1}.$$

This means that for $A \in \mathcal{F}$

$$P_1(A) = c \int_A \exp\left\{-\sum_{i=0}^{N} X_i^2\right\} \mathrm{d}\mathbf{P}.$$



Let

$$\varphi_n(a) = \mathbf{E}_{P_1}[e^{aX_n} | \mathcal{F}_{n-1}].$$

Note that this is an $\mathcal{F}_{n-1}$-measurable random variable. As in the proof of the previous lemma there is a unique finite a_n (random!) such that the infimum of φ_n is attained at a_n. Since φ_n is $\mathcal{F}_{n-1}$-measurable so is a_n.

Let $Z_0 = 1$, and recursively

$$Z_n = Z_{n-1} \frac{e^{a_n X_n}}{\mathbf{E}_{P_1}[e^{a_n X_n} | \mathcal{F}_{n-1}]}.$$

$E[Z_n] = Z_0 = 1$

Then (Z_n) is a P_1-martingale, since

$Z = Z_N$ utolsó elem

$$\mathbf{E}_{P_1}[Z_n | \mathcal{F}_{n-1}] = Z_{n-1}.$$

$Z_n = E[Z_N | \mathcal{F}_n]$

Then the probability measure

$$\mathbf{Q}(\mathrm{d}\omega) = Z_N(\omega) P_1(\mathrm{d}\omega)$$

$Q(A) = \int_A Z_N \, dP_1 \qquad A \in \mathcal{F}$

works. Indeed,

$$\begin{aligned}
\mathbf{E}_{\mathbf{Q}}[X_n | \mathcal{F}_{n-1}] &= \frac{1}{Z_{n-1}} \mathbf{E}_{P_1}[Z_n X_n | \mathcal{F}_{n-1}] && \text{by Lemma 7} \\
&= \frac{1}{\cancel{Z_{n-1}}} \frac{\cancel{Z_{n-1}}}{\mathbf{E}_{P_1}[e^{a_n X_n} | \mathcal{F}_{n-1}]} \mathbf{E}_{P_1}[X_n e^{a_n X_n} | \mathcal{F}_{n-1}] && \text{definition} \\
&= \frac{1}{\mathbf{E}_{P_1}[e^{a_n X_n} | \mathcal{F}_{n-1}]} \cdot 0 = 0. && \text{choice of } a_n
\end{aligned}$$

□

Exercise 13. Show that a_n is $\mathcal{F}_{n-1}$-measurable.

Now we can return to the proof of Theorem 3. The existence of the martingale measure follows from the previous lemma applied to $X_n = \Delta S_n$.

4.2 ARCH processes

Autoregressive conditional heteroscedasticity (ARCH) models were introduced by Robert Engle in 1982 to model log-returns. In 2003 he obtained Nobel prize in economics for this model. The novelty in these models is the stochastic volatility term.

Let

$$R_n = \log \frac{S_n}{S_{n-1}}$$

log return

denote the log-return of the stock, and assume that

$$R_n = \mu_n + \sqrt{\beta + \lambda R_{n-1}^2}\, Z_n,$$

where Z_n's are iid $N(0,1)$ random variables. Then (R_n) is an ARCH(1) process. That is conditionally on $\mathcal{F}_{n-1}$ the log-return R_n is Gaussian with mean μ_n, and variance $\beta + \lambda R_{n-1}^2$. Write $\sigma_n^2 = \beta + \lambda R_{n-1}^2$. Then for S_n we obtain

$$\begin{aligned} S_n = S_{n-1} e^{R_n} &= S_0 \exp\left\{ \sum_{k=1}^{n} \left(\mu_k + \sqrt{\beta + \lambda R_{k-1}^2}\, Z_k \right) \right\} \\ &= S_0 \exp\left\{ \sum_{k=1}^{n} (\mu_k + \sigma_k Z_k) \right\}. \end{aligned}$$

In what follows we only assume that μ_n and σ_n are $\mathcal{F}_{n-1}$-measurable, i.e. the sequence $(\mu_n, \sigma_n)_n$ is predictable, and (Z_n) is adapted, Z_n is independent of $\mathcal{F}_{n-1}$, and $N(0,1)$ distributed. Put $h_n = \mu_n + \sigma_n Z_n$. For simplicity we assume that $B_n \equiv 1$.

We construct a measure $\mathbf{Q}$ such that (S_n) is a $\mathbf{Q}$-martingale. Let

$$Z_N = \prod_{n=1}^{N} z_n := \prod_{n=1}^{N} \frac{e^{a_n h_n}}{\mathbf{E_P}[e^{a_n h_n} | \mathcal{F}_{n-1}]},$$

$S_n = S_{n-1} \cdot e^{\mu_n + \sigma_n Z_n}$
$= S_{n-1} \cdot e^{h_n}$

$$E_P[Z_N|\mathcal{F}_n] = E_P[\underbrace{z_1 \cdots z_n}_{\mathcal{F}_n\text{-mb.}} \cdot z_{n+1} \cdots z_N | \mathcal{F}_n]$$

$$= z_1 \cdots z_n \cdot \underbrace{E_P[z_{n+1} \cdots z_N | \mathcal{F}_n]}_{=1}$$

where

$$a_n = -\frac{\mu_n}{\sigma_n^2} - \frac{1}{2}. \qquad (10) \quad \texttt{\{eq:disc-girs-0\}}$$

Introduce the new measure $\mathbf{Q}$ as

$$\mathrm{d}\mathbf{Q} = Z_N \mathrm{d}\mathbf{P},$$

and let $Z_n = \mathbf{E}_\mathbf{P}[Z_N|\mathcal{F}_n] = \prod_{i=1}^n z_i$.

By Corollary 1, to show that S_n is $\mathbf{Q}$-martingale we have to show that $S_n Z_n$ is a $\mathbf{P}$-martingale. We have

$$\mathbf{E}_\mathbf{P}[S_n Z_n|\mathcal{F}_{n-1}] = S_{n-1} Z_{n-1} \frac{\mathbf{E}_\mathbf{P}[e^{h_n(1+a_n)}|\mathcal{F}_{n-1}]}{\mathbf{E}_\mathbf{P}[e^{a_n h_n}|\mathcal{F}_{n-1}]}$$

rell. ? $= 1$

$S_n = S_{n-1} e^{h_n}$

Therefore we have to check that

$$\mathbf{E}_\mathbf{P}[e^{h_n(1+a_n)}|\mathcal{F}_{n-1}] = \mathbf{E}_\mathbf{P}[e^{a_n h_n}|\mathcal{F}_{n-1}]. \qquad (11) \quad \texttt{\{eq:disc-girs-1\}}$$

Recall that for a standard normal Z

$$\mathbf{E} e^{tZ} = e^{\frac{t^2}{2}},$$

thus

$$\mathbf{E} e^{\mu + \sigma Z} = e^{\mu + \frac{\sigma^2}{2}}.$$

Since a_n in (11) is $\mathcal{F}_{n-1}$-measurable and given $\mathcal{F}_{n-1}$ the variable h_n is Gaussian $\mathrm{N}(\mu_n, \sigma_n^2)$, we obtain

$$\mathbf{E}_\mathbf{P}[e^{h_n(1+a_n)}|\mathcal{F}_{n-1}] = e^{\mu_n(1+a_n) + \frac{\sigma_n^2(1+a_n)^2}{2}},$$

$$h_n \cdot (1+a_n) = (\mu_n + \sigma_n Z_n)(1+a_n)$$

$$= \underbrace{\mu_n(1+a_n)}_{=\mu} + \underbrace{\sigma_n(1+a_n)}_{=\sigma} \cdot Z_n$$

and

$$\mathbf{E}_\mathbf{P}[e^{h_n a_n}|\mathcal{F}_{n-1}] = e^{\mu_n a_n + \frac{\sigma_n^2 a_n^2}{2}},$$

By the choice of a_n in (10)

$$\mu_n(1+a_n) + \frac{\sigma_n^2(1+a_n)^2}{2} = \mu_n a_n + \frac{\sigma_n^2 a_n^2}{2}.$$

Indeed, by (10)

$$\mu_n + \sigma_n^2 \left(\frac{1}{2} + a_n \right) = 0.$$

$$a_n = -\frac{1}{2} - \frac{\mu_n}{\sigma_n^2}$$

That is, (11) holds.

We proved the following.

Theorem 8 (Discrete Girsanov's theorem)**.** *Let $(\mu_n, \sigma_n)_n$ be a predictable sequence and assume that the stock prices are given by*

$$S_n = e^{\sum_{k=1}^{n}(\mu_k + \sigma_k Z_k)},$$

where $(Z_n)_n$ is a adapted sequence of $N(0,1)$ random variables, Z_n is independent of $\mathcal{F}_{n-1}$. Further, let $B_n \equiv 1$. Then, under the new measure

$$\mathrm{d}\mathbf{Q} = Z_N \mathrm{d}\mathbf{P}$$

(S_n) is a martingale.

5 Pricing and hedging European options

In this section we summarize our findings on pricing and hedging, and consider some special cases in detail.

5.1 Complete markets

Consider an arbitrage-free complete market. The *fair price* of the contingent claim f_N is

$$C(f_N) = \inf\{x : \exists \pi, X_0^\pi = x,\, X_N^\pi = f_N\}.$$

Then, by Theorems 3 and 7 there exists a unique EMM $\mathbf{Q}$. Since (X_n^π / B_n) is $\mathbf{Q}$-martingale

$$\mathbf{E}_\mathbf{Q} \frac{f_N}{B_N} = \mathbf{E}_\mathbf{Q} \frac{X_N^\pi}{B_N} = \mathbf{E}_\mathbf{Q} \frac{x}{B_0} = \frac{x}{B_0},$$

therefore

$$C(f_N) = x = \frac{B_0}{B_N} \mathbf{E}_\mathbf{Q} f_N.$$

Note that x is independent of the hedge π itself, that is for different hedges the initial value is the same.

For a hedge we need to know not only the fair price C, but also the strategy π itself. For the given claim f_N consider the martingale

$$M_n = \mathbf{E}_\mathbf{Q} \left[\frac{f_N}{B_N} \middle| \mathcal{F}_n \right].$$