that the weak limit-2 assumption holds. Then

$$C_*(f) = \inf_{\mathbf{Q} \in \mathcal{P}(\mathbf{P})} \mathbf{E}_{\mathbf{Q}} \frac{f(\rho)}{1+r} = \frac{f(r)}{1+r},$$

and the infimum is attained at the measure $\mathbf{Q}_*$.

3.5 Complete markets

We proved that if EMM exists then we have the fair price for any replicable payoff. A market is *complete* if any payoff is replicable.

We have seen in Theorem 4 that on a complete arbitrage-free market any payoff f has a unique well-defined fair price $B_0 \mathbf{E}_{\mathbf{Q}} f / B_N$.

In section 2.4 we showed that a binomial market is complete.

The second fundamental theorem of asset pricing is the following.

{thm:complete-mark

Theorem 7. *Consider an arbitrage-free market with EMM* $\mathbf{Q}$*. Then the following are equivalent:*

(i) the market is complete;

(ii) $\mathbf{Q}$ *is the unique EMM;*

(iii) for any $\mathbf{Q}$*-martingale* (M_n) *there exists a predictable sequence* γ_n *such that* M_n *can be represented as*

$$M_n = M_0 + \sum_{k=1}^{n} \gamma_k \left(\frac{S_k}{B_k} - \frac{S_{k-1}}{B_{k-1}} \right) = M_0 + \sum_{k=1}^{n} \sum_{i=1}^{d} \gamma_k^i \left(\frac{S_k^i}{B_k} - \frac{S_{k-1}^i}{B_{k-1}} \right).$$

Proof. We prove again the easy parts (i) ⇒ (ii), and (iii) ⇔ (i), and postpone the difficult (ii) ⇒ (i) implication later.

(i) ⇒ (ii): Assume that $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are EMM's. Consider any $A \in \mathcal{F}$. We show that $\mathbf{Q}_1(A) = \mathbf{Q}_2(A)$ implying the uniqueness. Let π be a perfect hedge to $f = I_A$. Then X_n^π / B_n is both $\mathbf{Q}_1$ and $\mathbf{Q}_2$ martingale, so

$$\mathbf{Q}_1(A) = \mathbf{E}_{\mathbf{Q}_1} f = \mathbf{E}_{\mathbf{Q}_1} X_N^\pi = B_N \mathbf{E}_{\mathbf{Q}_1} \underset{\left(\frac{X_n}{B_n}\right)}{\frac{X_N^\pi}{B_N}} = B_N \underset{B_0}{X_0^\pi} = \ldots = \mathbf{Q}_2(A).$$

(i) ⇒ (iii): Consider a $\mathbf{Q}$-martingale M_n. There exists a strategy π_n such that a.s.

$$X_N^\pi = B_N M_N = f$$

$E[Y|\mathcal{F}_n] = M_n$ martingál

$E[M_n|\mathcal{F}_{n-1}] \overset{?}{=} M_{n-1}$

$E[E[Y|\mathcal{F}_n]|\mathcal{F}_{n-1}] = E[Y|\mathcal{F}_{n-1}] = M_{n-1}.$

Using that both M_n and X_n^π/B_n are martingales

$$M_n = \mathbf{E_Q}[M_N|\mathcal{F}_n] = \mathbf{E_Q}\left[\frac{X_N^\pi}{B_N}|\mathcal{F}_n\right] = \frac{X_n^\pi}{B_n} = \beta_n + \gamma_n \frac{S_n}{B_n}.$$

$X_n = \beta_n \cdot B_n + \gamma_n S_n$

Thus, using that π is SF

$$\begin{aligned} M_n - M_{n-1} &= \Delta\beta_n + \gamma_n \frac{S_n}{B_n} - \gamma_{n-1}\frac{S_{n-1}}{B_{n-1}} \\ &= \gamma_n\left(\frac{S_n}{B_n} - \frac{S_{n-1}}{B_{n-1}}\right) + \frac{1}{B_{n-1}}\underbrace{\left(B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n\right)}_{=0 \text{ mert } \pi \text{ önfin}} \\ &= \gamma_n\left(\frac{S_n}{B_n} - \frac{S_{n-1}}{B_{n-1}}\right), \end{aligned}$$

as claimed.

(iii) $\Rightarrow$ (i): Consider a payoff f. We are looking for a strategy π such that $X_N^\pi = f$ $\mathbf{Q}$-a.s. We know that $(X_n^\pi/B_n)_n$ is a martingale, so this should be (M_n). Now the following choice is clear: let

$$M_n = \mathbf{E_Q}\left[\frac{f}{B_N}|\mathcal{F}_n\right].$$

Then M_n is a martingale, therefore by the assumption

$$M_n = M_0 + \sum_{k=1}^n \gamma_k \Delta\frac{S_k}{B_k}. \quad \ldots + \underbrace{\gamma_n\left(\frac{S_n}{B_n} - \frac{S_{n-1}}{B_{n-1}}\right)}$$

$\beta_n \cdot B_n + \gamma_n \cdot S_n = M_n \cdot B_n$

Let

$$\beta_n = M_n - \gamma_n\frac{S_n}{B_n},$$

and consider the strategy $\pi_n = (\beta_n, \gamma_n)$. To see that this is indeed a strategy we have to show that it is predictable and SF. The sequence γ_n is predictable by the assumption (iii), and β_n is predictable because all the terms in M_n are $\mathcal{F}_{n-1}$-measurable except $\gamma_n S_n/B_n$, which is subtracted. To see that it is SF note that

$$\begin{aligned} &B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n \\ &= B_{n-1}\left(M_n - M_{n-1} - \gamma_n\frac{S_n}{B_n} + \gamma_{n-1}\frac{S_{n-1}}{B_{n-1}}\right) + S_{n-1}\Delta\gamma_n \\ &= B_{n-1}\left(\gamma_n\Delta\frac{S_n}{B_n} - \cancel{\gamma_n\frac{S_n}{B_n}} + \gamma_{n-1}\frac{S_{n-1}}{B_{n-1}}\right) + S_{n-1}\Delta\gamma_n = 0, \checkmark \end{aligned}$$

$\gamma_n\left(\cancel{\frac{S_n}{B_n}} - \frac{S_{n-1}}{B_{n-1}}\right)$ $\qquad$ $S_{n-1}(\gamma_{n-1} - \gamma_n)$

showing that π is SF. It is clearly a perfect hedge since

$$X_N^\pi = \beta_N B_N + \gamma_N S_N = B_N M_N = f,$$

as claimed. $\square$

3.6 Proof of the difficult part of Theorem 3

Here we use strongly that Ω is finite, and let $|\Omega| = k$.

Assume that there is no arbitrage strategy. Let

$$\mathcal{V}_0 = \{X : \Omega \to \mathbb{R} \text{ r.v. } |\underbrace{\exists \pi}_{\text{s.f. str.}} : X_0^\pi = 0 \text{ and } X_N^\pi = X\}, \subseteq \mathbb{R}^k$$

$x \in \mathcal{V}_0 \Rightarrow \alpha x \in \mathcal{V}_0$
$x, y \in \mathcal{V}_0 \Rightarrow x + y \in \mathcal{V}_0$

and

$$\mathcal{V}_1 = \{X : \Omega \to \mathbb{R} \text{ r.v. } |X \geq 0, \mathbf{E}X \geq 1\}. \subseteq \mathbb{R}^k$$

We identify a random variable $X : \Omega \to \mathbb{R}$ with a vector in $\mathbb{R}^k$, as $X \leftrightarrow (X(\omega_1), \dots, X(\omega_k))$. Clearly, $\mathcal{V}_0$ is a linear subspace and $\mathcal{V}_1$ is convex set in $\mathbb{R}^k$.

$\lambda x + (1-\lambda) y$
$\lambda \in [0,1]$

Since there is no arbitrage strategy, $\mathcal{V}_0 \cap \mathcal{V}_1 = \emptyset$. Therefore, by the Kreps–Yan theorem, there exists a linear functional $\ell : \mathbb{R}^k \to \mathbb{R}$ such that $\ell|_{\mathcal{V}_0} \equiv 0$ and $\ell(v_1) > 0$ for all $v_1 \in \mathcal{V}_1$. A linear function in $\mathbb{R}^k$ (in any Hilbert space) is a inner product, thus there exists $q \in \mathbb{R}^k$ such that

$$\ell(v) = \langle v, q \rangle.$$

$E[\lambda X + (1-\lambda) Y]$
$= \lambda E(X) + (1-\lambda) E(Y)$
$\geq \lambda + 1 - \lambda = 1.$ ✓

Define the random variables

$$X_i(\omega_j) = \delta_{i,j} \frac{1}{\mathbf{P}(\{\omega_i\})}.$$

$\left(0, \dots, 0, \frac{1}{P(\{\omega_i\})}, 0, \dots, 0\right)$

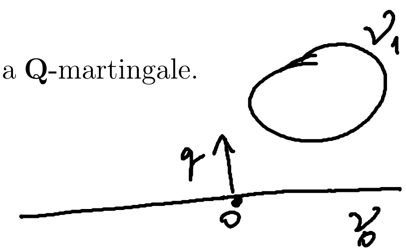
Then $X_i \geq 0$ and $\mathbf{E}X_i = 1$, so $X_i \in \mathcal{V}_1$. Furthermore

$$\ell(X_i) = \frac{q_i}{\mathbf{P}(\{\omega_i\})} > 0,$$

implying $q_i > 0$ for any i. Define the probability measure $\mathbf{Q}$ as

$$\mathbf{Q}(\{\omega_i\}) = \frac{q_i}{\sum_{i=1}^k q_i}.$$

It is clear that $\mathbf{Q} \sim \mathbf{P}$. We have to check that $\underline{(S_n/B_n)}$ is a $\mathbf{Q}$-martingale. First we need a lemma.



$x, y \in \mathbb{R}^k$: $\langle x, y \rangle = \sum_{i=1}^k x_i y_i$
$\exists!\, q \in \mathbb{R}^k$: $\ell(x) = \langle q, x \rangle$

$\ell : \mathbb{R}^k \to \mathbb{R}$ linear
$\ell(x+y) = \ell(x) + \ell(y)$; $\ell(\alpha x) = \alpha \ell(x)$

$\forall n: \quad \{\tau = n\} \in \mathcal{F}_n$

$E[X|\mathcal{G}]: \ - \mathcal{G}\text{-mh}$

$- \int_G E[X|\mathcal{G}]\,dP = \int_G X\,dP$

Lemma 5. *Let $(X_n)_{n=1}^N$ be an adapted process. If for any stopping time $\tau : \Omega \to \{0, \dots, N\}$*

$$\mathbf{E}X_\tau = \mathbf{E}X_0,$$

then (X_n) is martingale.

Proof. We show that $X_n = \mathbf{E}[X_N|\mathcal{F}_n]$, which implies that X is martingale.

Let $A \in \mathcal{F}_n$ and consider the stopping time

$$\tau_A(\omega) = \begin{cases} n, & \omega \in A, \\ N, & \text{otherwise.} \end{cases} \qquad \in \mathcal{F}_k \qquad \mathcal{F}_0 = \{\emptyset, \Omega\}$$

This is indeed a stopping time, since $\{\tau_A \le k\} = \emptyset$ for $k < n$, and A for $k \ge n$, which is $\mathcal{F}_k$-measurable. Then, by the assumption

$$E(X_N) = \mathbf{E}X_0 = \mathbf{E}X_{\tau_A} = \mathbf{E}X_n I(A) + \mathbf{E}X_N I(A^c).$$

With $A = \emptyset$ we see that $\mathbf{E}X_0 = \mathbf{E}X_N$, implying

$$\int_A X_n\,dP = \mathbf{E}X_n I(A) = \mathbf{E}X_N I(A). = \int_A X_N\,dP$$

This exactly means that

$$X_n = \mathbf{E}[X_N|\mathcal{F}_n],$$

as claimed. □

We show that (S_n/B_n) satisfies the condition of the lemma above. Let τ be a stopping time and define the strategy

$$\beta_n = \frac{S_\tau}{B_\tau} I(\tau \le n-1) - \frac{S_0}{B_0}, \quad \gamma_n = I(\tau > n-1).$$

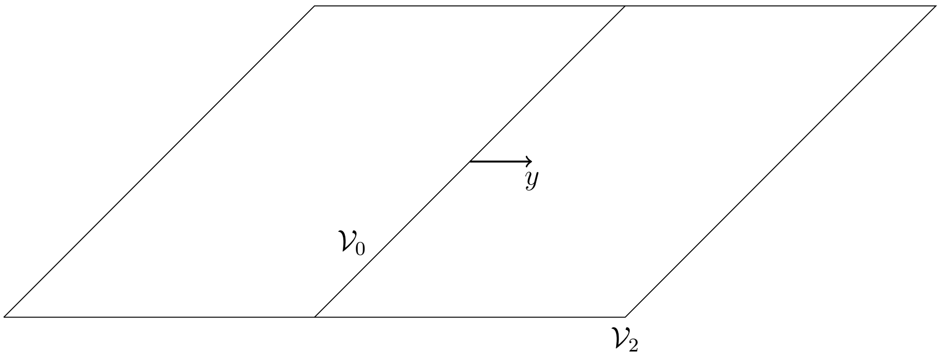
Since $\{\tau < n\} = \{\tau \le n-1\} \in \mathcal{F}_{n-1}$, the sequence (β_n, γ_n) is predictable. Furthermore,

$$B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n = \frac{S_\tau}{B_\tau} B_{n-1} I(\tau = n-1) \overset{-}{+} S_{n-1} I(\tau = n-1) = 0,$$

so it is SF. Finally,

$$X_0^\pi = -\frac{S_0}{B_0} B_0 + S_0 = 0,$$

$\Delta\gamma_n = I(\tau > n-1) - I(\tau > n-2) = -I(\tau = n-1)$



$\ell \setminus \mathcal{V}_0 \equiv 0$

Figure 7: Choice of y

$E_Q X = \frac{\sum_{i=1}^{k} x_i \cdot q_i}{\sum_{i=1}^{k} q_i} = \underbrace{\langle X, q \rangle}_{=0} \frac{1}{\sum q_i}$

so $X_N^\pi \in \mathcal{V}_0$. Therefore

$$
\begin{aligned}
0 = \mathbf{E_Q} X_N^\pi &= \mathbf{E_Q}\left(\beta_N B_N + \gamma_N S_N\right) \\
&= \mathbf{E_Q}\left(\left(\frac{S_\tau}{B_\tau} I(\tau \le N-1) - \frac{S_0}{B_0}\right) B_N + \frac{S_\tau}{B_\tau} I(\tau = N) B_N\right) \\
&= B_N \mathbf{E_Q}\left(\frac{S_\tau}{B_\tau} - \frac{S_0}{B_0}\right).
\end{aligned}
$$

$\Rightarrow$ $\forall \tau$ megállási idő $E_Q \frac{S_\tau}{B_\tau} = E_Q \frac{S_0}{B_0}$

+Lemma $\Rightarrow \left(\frac{S_n}{B_n}\right)_n$ Q-mtg.

That is (S_n/B_n) is indeed a $\mathbf{Q}$-martingale. $\square$

3.7 Proof of the difficult part of Theorem 7

Here we prove the implication (ii) $\Rightarrow$ (i).

We use the notation of the previous proof. Let

$$\mathcal{V}_2 = \{X : \Omega \to \mathbb{R} \text{ r.v. } | \mathbf{E_Q} X = 0\}.$$

Then $\mathcal{V}_2$ is a linear subspace in $\mathbb{R}^k$ and we have seen in the previous proof that $\mathcal{V}_0 \subset \mathcal{V}_2$. We claim that equality holds.

Assume first that this is indeed true. Then for any claim X the centered version $X - \mathbf{E_Q} X \in \mathcal{V}_2 = \mathcal{V}_0$, meaning that there is a perfect hedge. Thus the market is complete. So we only have to show that $\mathcal{V}_0 = \mathcal{V}_2$.

Assume on the contrary that $\mathcal{V}_0 \neq \mathcal{V}_2$. Then there is an $y \in \mathcal{V}_2$, which is orthogonal to $\mathcal{V}_0$. Since $q_i > 0$ (see the previous proof) for all $i = 1, \ldots, k$, we may choose $\varepsilon > 0$ small enough such that

$$q'_i = q_i - \varepsilon y_i > 0 \quad \text{for all } i. \qquad q' = q - \varepsilon \cdot y$$

As both q and y are orthogonal to $\mathcal{V}_0$, q' is also orthogonal. Define the measure

$$\mathbf{Q}'(\{\omega_i\}) = \frac{q'_i}{\sum_{i=1}^k q'_i}.$$

Exactly as in the previous proof we can show that $\mathbf{Q}'$ is EMM. The uniqueness of the EMM implies

$$\frac{q'_i}{\sum_{i=1}^k q'_i} = \frac{q_i}{\sum_{i=1}^k q_i},$$

that is, using also the definition of q',

$$\alpha = \frac{\sum q_i}{\sum q'_i}$$

$$q = \alpha q' = \alpha q - \alpha \varepsilon y,$$

with $\alpha = \sum q_i / \sum q'_i$. Thus

$$(1 - \alpha) q = -\alpha \varepsilon y.$$

But y and q are orthogonal, which is a contradiction. The proof is complete.

4 Girsanov's theorem in discrete time

4.1 Second proof of the difficult part of Theorem 3

Assume that $d = 1$ and first consider the one-step model with $B_0 = B_1 = 1$. The stock price S_0 is known, and the only randomness here is S_1.

Exercise 9. The no arbitrage assumption (in this simple market) is equivalent to

$$\mathbf{P}(\Delta S_1 > 0)\mathbf{P}(\Delta S_1 < 0) > 0.$$

Furthermore, (S_n) is martingale if

$$\mathbf{E}_{\mathbf{Q}} S_1 = S_0.$$

Therefore we have to construct a measure $\mathbf{Q}$ such that $\mathbf{E}_{\mathbf{Q}} \Delta S_1 = 0$. This is done in the following lemma.