Regularly varying functions

Exercises 3.

1. Let X be a nonnegative random variable, F its distribution function, and $\widehat{F}(s) = \int_{[0,\infty)} e^{-sx} \mathrm{d}F(x)$ its Laplace transform. Assume that $\mu_n = \mathbf{E}X^n < \infty$. Define

$$f_n(s) = (-1)^{n+1} \left(\widehat{F}(s) - \sum_{k=0}^{n} \mu_k (-s)^k / k! \right)$$

$$g_n(s) = \frac{\mathrm{d}^n}{\mathrm{d}s^n} f_n(s).$$

Let ℓ be a slowly varying function, $\alpha = n + \beta$ with $\beta \in [0,1]$. Show that $f_n(s) \sim s^\alpha \ell(1/s)$ if and only if $g_n(s) \sim \Gamma(\alpha+1)/\Gamma(\beta+1)\, s^\beta \ell(1/s)$.

2. Show that the Laplace transform of the standard normal distribution is $e^{s^2/2}$.

3. Let X be a random variable with distribution function $F(x) = \mathbf{P}(X \le x)$. Define its quantile function as

$$Q(s) = \inf\{x : F(x) \ge s\}, \; s \in (0,1).$$

Show that $Q(s) \le x$ iff $s \le F(x)$. Show that if U is a Uniform$(0,1)$ random variable then the distribution function of $Q(U)$ is F.

4. Let X be a random variable with continuous distribution function F. Show that $U = F(X)$ is Uniform$(0,1)$.

5. Show that $\int_0^1 n(n-1)u^{n-2}(1-u)\mathrm{d}u = 1$ for any $n \ge 2$.

6. Show that for $\alpha \in (0,1)$, $\lambda > 0$

$$\int_0^1 (u^{-\alpha} - 1)\lambda e^{-\lambda u} \mathrm{d}u = \alpha \int_0^1 \left(1 - e^{-\lambda u}\right) u^{-\alpha-1} \mathrm{d}u.$$

7. Let $X, X_1, X_2, \dots$ be nonnegative iid random variables with finite expectation. Show that $\mathbf{E}M_n/n \to 0$. Show that this implies $M_n/n \xrightarrow{\mathbf{P}} 0$, and that $M_n/S_n \xrightarrow{\mathbf{P}} 0$.

8. Let X be a nonnegative random variable with regularly varying tail function $\overline{F}$ with parameter $\alpha < 0$. Let U be Uniform$(0,1)$ independent of X. Let G be the distribution function of $X + U$. Show that $\overline{G}$ is asymptotically equal to $\overline{F}$; i.e. $\overline{F}(x) \sim \overline{G}(x)$ as $x \to \infty$. In particular it is regularly varying with parameter α.