Regularly varying functions

Exercises 2.

1. Let ℓ be a slowly varying function which is locally bounded on $[0, \infty)$. Assume further that $\int_1^\infty \ell(t)/t \, \mathrm{d}t < \infty$. Show that $\widetilde{\ell}(x) = \int_x^\infty \ell(t)/t \, \mathrm{d}t$ is slowly varying and $\widetilde{\ell}(x)/\ell(x) \to \infty$ as $x \to \infty$.

2. Let $\ell_0(x) \equiv 1$, and let $\ell_{i+1}(x) = \int_1^x \ell_i(t)/t \, \mathrm{d}t$, $i = 0, 1, 2, \ldots$. Find ℓ_i.

3. Let ℓ be slowly varying, locally boundend, and $\alpha < -1$. Show that $\int_x^\infty t^\alpha \ell(t) \mathrm{d}t < \infty$, and

$$\lim_{x \to \infty} \frac{x^{\alpha+1} \ell(x)}{\int_x^\infty t^\alpha \ell(t) \mathrm{d}t} = -\alpha - 1.$$

4. Find an asymptotic inverse of the following functions and prove that it is indeed an asymptotic inverse.

(a) $f_1(x) = x \log x$;

(b) $f_2(x) = x^2 \log \log x$;

(c) $f_3(x) = x^2 (\log x)^3$.

5. Let $f \in \mathcal{RV}_\alpha$, and g is a positive measurable function such that

$$\lim_{x \to \infty} \frac{f(g(x) \lambda^{1/\alpha})}{f(g(\lambda x))} = 1.$$

Show that $g \in \mathcal{RV}_{1/\alpha}$.

6. Let $X \geq 0$, $\alpha > 0$. Show that $\mathbf{E} X^\alpha < \infty$ implies $\lim_{x \to \infty} x^\alpha [1 - F(x)] = 0$. Give a counterexample to show that the converse is not true. (It is almost true, see the next exercise.)

7. Let $X \geq 0$, $\alpha > 0$. Show that $\lim_{x \to \infty} x^\alpha [1 - F(x)] = 0$ implies $\mathbf{E} X^\beta < \infty$ for any $\beta < \alpha$.

8. Determine the Laplace transform of the following distributions.

(a) $X \sim$ Bernoulli(p);

(b) $X \sim$ Binomial(n, p);

(c) $X \sim$ Poisson(λ);

(d) $X \sim$ Uniform(a, b);

(e) $X \sim$ Exp(λ).

9. Show that $\sum_{n=1}^\infty e^{-2^n} 2^{\rho n} < \infty$ for any ρ.