Regularly varying functions

Exercises 1.

1. Let $X, X_1, X_2, \dots$ be iid Exponential(1) random variables, and let $M_n = \max\{X_1, \dots, X_n\}$ denote the partial maxima. Find a sequence a_n such that $M_n - a_n$ converges in distribution to a nondegenerate limit. The limiting distribution is the Gumbel distribution.

2. Let $X, X_1, X_2, \dots$ be iid Uniform$(0,1)$ random variables, and let $M_n = \max\{X_1, \dots, X_n\}$ denote the partial maxima. Find sequences a_n, b_n such that $a_n(M_n - b_n)$ converges in distribution to a nondegenerate limit. Determine the limit distribution.

3. Show that $\ell_1(x) = e^{(\log x)^\alpha}$ is slowly varying for $\alpha \in (0,1)$, and not slowly varying for $\alpha \geq 1$.

4. Show that $f(x) = 2 + \sin x$ is not slowly varying.

5. Show that $\ell_2(x) = \exp\left\{(\log x)^{1/3} \cos\left((\log x)^{1/3}\right)\right\}$ is slowly varying, and $\liminf_{x\to\infty} \ell_2(x) = 0$, $\limsup_{x\to\infty} \ell_2(x) = \infty$.

6. (i) If $S \subset \mathbb{R}$ is and additive subgroup, and S contains a set of positive measure, then $S = \mathbb{R}$. (ii) If $S \subset (0,\infty)$ is and additive semigroup, and S contains a set of positive measure, then there exists $b > 0$ such that $S \supset (b, \infty)$.

7. Show that the representation theorem implies the uniform convergence theorem.

8. Let ℓ, ℓ_1, ℓ_2 be slowly varying functions. Then

(i) $(\log \ell(x))/\log x \to 0$;

(ii) $(\ell(x))^\alpha$ is slowly varying for each $\alpha \in \mathbb{R}$;

(iii) $\ell_1\ell_2$, $\ell_1 + \ell_2$ are slowly varying;

(iv) for each $\varepsilon > 0$ $\lim_{x\to\infty} x^\varepsilon \ell(x) = \infty$, $\lim_{x\to\infty} x^{-\varepsilon}\ell(x) = 0$.

9. (i) If $f \in \mathcal{RV}_\rho$ then $f^\alpha \in \mathcal{RV}_{\rho\alpha}$.
(ii) If $f_i \in \mathcal{RV}_{\rho_i}$, $i = 1, 2$, and $f_2(x) \to \infty$, then $f_1(f_2(x)) \in \mathcal{RV}_{\rho_1\rho_2}$.
(iii) If $f_i \in \mathcal{RV}_{\rho_i}$, $i = 1, 2$, then $f_1 + f_2 \in \mathcal{RV}_{\max\{\rho_1,\rho_2\}}$.