Regularly varying functions

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Contents

1 Motivation: maximum of iid random variables

This part is mainly from Feller [4, Chapter VIII.8].

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space, and $X, X_1, X_2, \ldots$ are iid random variables on it such that $F(x) = \mathbf{P}(X \leq x) < 1$ for any $x \in \mathbb{R}$. Let $M_n = \max\{X_1, \ldots, X_n\}$ the partial maximum. In probability theory we are often interested in the following type of question:

What are the necessary and sufficient conditions for the existence of a sequence a_n such that M_n/a_n converges in distribution to a nondegenerate limit?

Recall that converges in distribution to Y, $Y_n \stackrel{\mathcal{D}}{\to} Y$, if $\lim_{n\to\infty} \mathbf{P}(Y_n \leq y) = \mathbf{P}(Y \leq y) = G(y)$ for any $y \in C_G$, where C_G stands for the continuity points of G.

For maximum we can easily calculate the distribution function. Indeed,

$$\mathbf{P}(M_n/a_n \leq x) = \mathbf{P}(M_n \leq a_n x) = F(a_n x)^n.$$

Thus we need that

$$\lim_{n\to\infty} F(a_n x)^n = G(x) \qquad \text{for all } x \in C_G.$$

Taking logarithms, and using that $\log(1+x) \sim x$ as $x \to 0$,

$$\lim_{n\to\infty} n\overline{F}(a_n x) = -\log G(x), \tag{1}$$

where $\overline{F}(x) = 1 - F(x)$. It turns out that the simple limit relation in (1) forces the structure of both the limit function and F. We need the following lemma.

Lemma 1. *Let b_n be a sequence for which $b_{n+1}/b_n \to 1$, $a_n \to \infty$, and U is a monotone function (increasing or decreasing). Assume that*

$$\lim_{n\to\infty} b_n U(a_n x) = h(x) \leq \infty$$

for all $x \in D$, where D is a dense subset, and the limit h is finite and strictly positive on an interval. Then $h(x) = cx^\rho$, for some $c \in \mathbb{R}$, $\rho \in \mathbb{R}$.

Proof. Easy. □

Definition 1. A function $U : [0, \infty) \to [0, \infty)$ is regularly varying with index ρ, $U \in \mathcal{RV}_\rho$, if for each $\lambda > 0$

$$\lim_{x \to \infty} \frac{U(\lambda x)}{U(x)} = \lambda^\rho.$$

For $\rho = 0$, i.e. when for $\lambda > 0$

$$\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1,$$

ℓ is slowly varying, $\ell \in \mathcal{SV}$.

This is regular variation at infinity. Regular variation at zero can be defined similarly, by changing $x \to \infty$ to $x \downarrow 0$.

From the definition we see that if $U \in \mathcal{RV}_\rho$ then $U(x) = x^\rho \ell(x)$, where ℓ is slowly varying.

Example 1. The constant function is trivially slowly varying. Moreover, any function with a strictly positive finite limit is slowly varying. More interesting examples are:

- $\log x$;
- $(\log x)^\alpha$, $\alpha \in \mathbb{R}$;
- $\log \log x$;
- $\exp\{(\log x)^\alpha\}$, $\alpha \in (0, 1)$.

Going back to our maximum process, we see form Lemma 1 and from (1) that the limiting distribution function has to be of the form $G(x) = e^{-cx^\rho}$, for $x > 0$, and 0 otherwise, for some $\rho < 0$. In fact we have the following.

Theorem 1. *Assume the $F(x) < 1$ for all $x \in \mathbb{R}$. There exist a_n such that*

$$\frac{M_n}{a_n} \xrightarrow{\mathcal{D}} Z,$$

with a nondegenerate limit Z, if and only if $\overline{F}$ is regularly varying with index $\rho < 0$. In this case $\mathbf{P}(Z \leq x) = G(x) = e^{-cx^\rho}$ for $x > 0$, and 0 otherwise.

The limiting distribution is the so-called *Fréchet* distribution. There are three type of extreme value distribution; see Exercises 4, 5.

Proof. Choose a_n such that $n\overline{F}(a_n) \to 1$. □

1.1 Exercises

1. Show that $\ell_1(x) = e^{(\log x)^\alpha}$ is slowly varying for $\alpha \in (0,1)$, and not slowly varying for $\alpha \geq 1$.

2. Show that $f(x) = 2 + \sin x$ is not slowly varying.

3. Show that $\ell_2(x) = \exp\left\{(\log x)^{1/3} \cos\left((\log x)^{1/3}\right)\right\}$ is slowly varying, and $\liminf_{x\to\infty} \ell_2(x) = 0$, $\limsup_{x\to\infty} \ell_2(x) = \infty$.

4. Let $X, X_1, X_2, \dots$ be iid Exponential(1) random variables, and let $M_n = \max\{X_1, \dots, X_n\}$ denote the partial maxima. Find a sequence a_n such that $M_n - a_n$ converges in distribution to a nondegenerate limit. The limiting distribution is the Gumbel distribution.

5. Let $X, X_1, X_2, \dots$ be iid Uniform$(0,1)$ random variables, and let $M_n = \max\{X_1, \dots, X_n\}$ denote the partial maxima. Find a sequence a_n, b_n such that $a_n(M_n - b_n)$ converges in distribution to a nondegenerate limit. Determine the limit distribution.

2 Steinhaus theory and the Cauchy functional equation

Main theory on regular variation follows Bingham et al.[1].

Theorem 2. *Let $A \subset \mathbb{R}$ be a measurable set with positive Lebesgue measure. Then $A - A$ contains in interval.*

Theorem 3. *Let $A, B \subset \mathbb{R}$ be measurable sets with positive Lebesgue measure. Then $A - B$ contains in interval.*

Corollary 1. *Let $A \subset \mathbb{R}$ be a measurable set with positive Lebesgue measure. Then $A + A$ contains in interval.*

Corollary 2. *(i) If $S \subset \mathbb{R}$ is and additive subgroup, and S contains a set of positive measure, then $S = \mathbb{R}$. (ii) If $S \subset (0, \infty)$ is and additive semigroup, and S contains a set of positive measure, then there exists $b > 0$ such that $S \supset (b, \infty)$.*

Definition 2. A function $k : \mathbb{R} \to \mathbb{R}$ is additive if $k(x + y) = k(x) + k(y)$ for all x, y.

Lemma 2. *If k is additive and bounded above on a set A with positive measure, then k is bounded in the neighborhood of the origin.*

Theorem 4. *Let k be additive and bounded above on a set A with positive measure. Then $k(x) = cx$ for some $c \in \mathbb{R}$.*

Corollary 3. *If k is additive and measurable then $k(x) = cx$.*

There are pathological solutions to the Cauchy functional equations. Consider $\mathbb{R}$ as a vector space above $\mathbb{Q}$, and let B be a Hamel base. This exist by the Zorn lemma, and the cardinality of B is continuum. For $b_0 \in B$ fixed let $k(b_0) = b_0$, and $k(b) = 0$ for $b \in B$, $b \neq b_0$. Define

$$k(x) = \sum_{i=1}^{n} r_i k(b_i), \qquad \text{if } x = \sum_{i=1}^{n} r_i b_i.$$

Then k is additive, but not of the form $k(x) = cx$.

2.1 Exercises

6. (i) If $S \subset \mathbb{R}$ is and additive subgroup, and S contains a set of positive measure, then $S = \mathbb{R}$. (ii) If $S \subset (0, \infty)$ is and additive semigroup, and S contains a set of positive measure, then there exists $b > 0$ such that $S \supset (b, \infty)$.

3 Slowly varying functions

Definition 3. A nonnegative measurable function $\ell : [a, \infty) \to [0, \infty)$, $a \geq 0$, is slowly varying, if

$$\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \qquad \text{for each} \quad \lambda > 0. \tag{2}$$

For simplicity, we assume that $a = 0$.

Theorem 5. *Uniform convergence theorem. Let ℓ be a slowly varying function. Then (2) holds uniformly on each compact set of $(0, \infty)$; that is for each $\varepsilon > 0$, $K < \infty$*

$$\sup_{\lambda \in [\varepsilon, K]} \left| \frac{\ell(\lambda x)}{\ell(x)} - 1 \right| = 0.$$

Proof. I. Direct proof

II: Indirect proof by Erdős and Csiszár. □

Theorem 6. *Representation theorem. Let ℓ be a nonnegative measurable function. It is slowly varying if and only if*

$$\ell(x) = c(x) \exp\left\{ \int_a^x \frac{\varepsilon(u)}{u} \mathrm{d}u \right\}, \quad x > a,$$

where $a \geq 0$, $\lim_{x\to\infty} c(x) = c \in (0, \infty)$, $\lim_{x\to\infty} \varepsilon(x) = 0$.

Changing to the additive notation $h(x) = \log \ell(e^x)$, we have

$$h(x) = c_1(e^x) + \int_{\log a}^x \varepsilon(e^x)\mathrm{d}x =: d(x) + \int_b^x e(x)\mathrm{d}x. \tag{3}$$

Proof. Sufficiency is clear.

For the necessity, write

$$h(x) = \int_x^{x+1} [h(x) - h(t)]\,\mathrm{d}t + \int_{x_0}^x [h(t+1) - h(t)]\,\mathrm{d}t + \int_{x_0}^{x_0+1} h(t)\mathrm{d}t.$$

The last term is constant. In the second term is integrand $e(t) = h(t+1) - h(t) \to 0$ as $t \to \infty$. While the first

$$\int_x^{x+1} [h(x) - h(t)]\,\mathrm{d}t = \int_0^1 [h(x) - h(x+u)]\mathrm{d}u,$$

and here the integrand tends to 0 uniformly by the UCT. □

We use the following lemma without explicitly mentioning.

Lemma 3. *If $\ell \in \mathcal{SV}$ then ℓ is locally bounded far enough to the right; i.e. there exists $a > 0$ such that $\sup_{x\in[a,a+n]} \ell(x) < \infty$ for each n.*

Proposition 1. *Let ℓ, ℓ_1, ℓ_2 be slowly varying functions. Then*

1. $(\log \ell(x))/\log x \to 0$;
2. $(\ell(x))^\alpha$ *is slowly varying for each* $\alpha \in \mathbb{R}$;
3. $\ell_1\ell_2$, $\ell_1 + \ell_2$ *are slowly varying;*
4. *for each* $\varepsilon > 0$ $\lim_{x\to\infty} x^\varepsilon \ell(x) = \infty$, $\lim_{x\to\infty} x^{-\varepsilon}\ell(x) = 0$.

3.1 Exercises

7. Show that the representation theorem implies the UCT.

8. Let ℓ, ℓ_1, ℓ_2 be slowly varying functions. Then

1. $(\log \ell(x))/\log x \to 0$;
2. $(\ell(x))^\alpha$ is slowly varying for each $\alpha \in \mathbb{R}$;
3. $\ell_1\ell_2$, $\ell_1 + \ell_2$ are slowly varying;
4. for each $\varepsilon > 0$ $\lim_{x\to\infty} x^\varepsilon \ell(x) = \infty$, $\lim_{x\to\infty} x^{-\varepsilon}\ell(x) = 0$.

4 The limit function

Let $f : [0,\infty) \to (0,\infty)$ be a measurable function, and assume that

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = g(\lambda) \in (0,\infty), \qquad \lambda \in S, \tag{4}$$

for some set S. Then $\lambda, \mu \in S$ implies $\lambda\mu \in S$ and $g(\lambda\mu) = g(\lambda)g(\mu)$. Also $\lambda \in S$ implies $1/\lambda \in S$ and $g(1/\lambda) = 1/g(\lambda)$. Thus S is a multiplicative subgroup of $(0,\infty)$.

Changing to the additive notation $h(x) = \log f(e^x)$, $k(x) = \log g(e^x)$, we have that $k(u+v) = k(u)+k(v)$ for $u, v \in T$, where T is an additive subgroup of $\mathbb{R}$.

Theorem 7 (Characterization theorem)**.** *Assume that* (4) *holds and S has positive measure. Then*

(i) $\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)}$ *exists for all* $\lambda > 0$.

(ii) $g(\lambda) = \lambda^\rho$ *for some* $\rho \in \mathbb{R}$.

(iii) $f(x) = x^\rho \ell(x)$ *for some* $\ell \in \mathcal{SV}$.

Proof. This follows from Corollary 2. □

Definition 4. A positive measurable function f is regularly varying with index $\rho \in \mathbb{R}$ if

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho \qquad \text{for all } \lambda > 0.$$

Regular variation at 0 defined similarly, but $x \downarrow 0$ instead of $x \to \infty$. Simply $f(x)$ is regularly varying at 0 if and only if $f(1/x)$ is regularly varying at infinity.

There are more general characterization therorems.

Theorem 8. *Let f be positive measurable function and assume that for $g^*(\lambda) = \limsup_{x\to\infty} f(\lambda x)/f(x)$, we have $\limsup_{\lambda\downarrow 1} g^*(\lambda) \le 1$. Then the following are equivalent.*

(i) There is a $\rho \in \mathbb{R}$ such that $\lim_{x\to\infty} f(\lambda x)/f(x) = \lambda^\rho$ for all $\lambda > 0$.

(ii) $\lim_{x\to\infty} f(\lambda x)/f(x)$ exists and finite on a set of positive measure.

(iii) $\lim_{x\to\infty} f(\lambda x)/f(x)$ exists and finite on a dense subset of $(0, \infty)$.

(iv) $\lim_{x\to\infty} f(\lambda x)/f(x)$ exists and finite for $\lambda = \lambda_1, \lambda_2$, where $\log \lambda_1 / \log \lambda_2$ is irrational.

5 Regularly varying functions: first properties

An immediate consequence of Proposition 1 is the following.

Proposition 2. *For $f \in \mathcal{RV}_\rho$, as $x \to \infty$*

$$f(x) \to \begin{cases} \infty, & \rho > 0, \\ 0, & \rho < 0. \end{cases}$$

Theorem 9 (Uniform convergence theorem for regularly varying functions)**.** *Let $f \in \mathcal{RV}_\rho$ locally bounded on $[0, \infty)$. Then $f(\lambda x)/f(x) \to \lambda^\rho$ uniformly in λ*

- *on each $[a, b] \subset (0, \infty)$ for $\rho = 0$;*
- *on each $(0, b] \subset (0, \infty)$ for $\rho > 0$;*
- *on each $[a, \infty) \subset (0, \infty)$ for $\rho < 0$.*

Proof. The case $\rho = 0$ is the UCT for slowly varying functions. We only prove the statement for $\rho > 0$, the other case is similar.

By the UCT for slowly varying functions it is enough to prove on $(0, 1]$. By the representation theorem

$$f(x) = x^\rho \ell(x) = x^\rho c(x) \exp\left\{ \int_0^x \varepsilon(u)/u \mathrm{d}u \right\}.$$

There exists $x_0 > 0$ such that for $x \geq x_0$ $c(x) \in (c/2, 2c)$ and $|\varepsilon(x)| < 1$. Thus, whenever $\lambda x \geq x_0$

$$\frac{f(\lambda x)}{f(x)} \leq \lambda^\rho \frac{2c}{c/2} e^{\log \lambda} = 4\lambda^{\rho+1}.$$

Let $\varepsilon > 0$ be fix. If $\lambda \leq \varepsilon^{1/(\rho+1)}$ then for $\lambda x \geq x_0$

$$\frac{f(\lambda x)}{f(x)} \leq 4\varepsilon.$$

Therefore, if $\lambda \leq \varepsilon^{1/(\rho+1)}$ and $\lambda x \geq x_0$

$$\left| \frac{f(\lambda x)}{f(x)} - \lambda^\rho \right| \leq 4\varepsilon + \epsilon^{\rho/(\rho+1)}.$$

On the other hand, if $\lambda x \leq x_0$ then

$$\left| \frac{f(\lambda x)}{f(x)} - \lambda^\rho \right| \leq \frac{\sup_{y \in (0, x_0]} f(y)}{f(x)} + \left(\frac{x_0}{x} \right)^\rho.$$

The latter bound goes to 0 as $x \to \infty$ (uniformly in λ, since it does not contain any λ).

Finally, for $\lambda \in [\varepsilon^{1/(\rho+1)}, 1]$ the UCT works. $\square$

As a consequence we obtain that a regularly varying function with index $\rho \neq 0$ is asymptotically equivalent to a monotone function.

Theorem 10. *Let $f \in \mathcal{RV}_\rho$ locally bounded on $[a, \infty)$. If $\rho > 0$ then*

(i) $\overline{f}(x) = \sup\{f(t) : 0 \leq t \leq x\} \sim f(x)$*;*

(ii) $\underline{f}(x) = \inf\{f(t) : t \geq x\} \sim f(x)$*.*

If $\rho < 0$ then $\sup\{f(t) : t \geq x\} \sim f(x)$ *and* $\inf\{f(t) : a \leq t \leq x\} \sim f(x)$*.*

Theorem 11 (Potter bounds)**.** *(i) Let ℓ be a slowly varying function. Then for each $A > 1$, $\delta > 0$ there exists x_0 such that for each $x, y \geq x_0$*

$$\frac{\ell(x)}{\ell(y)} \leq A \max\left\{ \left(\frac{x}{y}\right)^\delta, \left(\frac{y}{x}\right)^\delta \right\}.$$

(ii) If ℓ is bounded away from 0 and ∞ on every compact subset of $[0,\infty)$ then for each $\delta > 0$ there exists and $A = A(\delta)$ such that for each x, y

$$\frac{\ell(x)}{\ell(y)} \leq A \max\left\{ \left(\frac{x}{y}\right)^{\delta}, \left(\frac{y}{x}\right)^{\delta} \right\}.$$

(iii) If $f \in \mathcal{RV}_{\rho}$ then for each $A > 1$, $\delta > 0$ there exist $x_0 > 0$ such that for $x, y \geq x_0$

$$\frac{f(x)}{f(y)} \leq A \max\left\{ \left(\frac{x}{y}\right)^{\rho+\delta}, \left(\frac{x}{y}\right)^{\rho-\delta} \right\}.$$

Proof. (i) follows from the representation theorem. (iii) is immediate from (i). (ii) follows from the local boundedness and strict positivity. □

Proposition 3. *(i) If $f \in \mathcal{RV}_{\rho}$ then $f^{\alpha} \in \mathcal{RV}_{\rho\alpha}$.*
(ii) If $f_i \in \mathcal{RV}_{\rho_i}$, $i = 1, 2$, and $f_2(x) \to \infty$, then $f_1(f_2(x)) \in \mathcal{RV}_{\rho_1\rho_2}$.
(iii) If $f_i \in \mathcal{RV}_{\rho_i}$, $i = 1, 2$, then $f_1 + f_2 \in \mathcal{RV}_{\max\{\rho_1, \rho_2\}}$.

5.1 Exercises

9. Prove Proposition 3.

6 Karamata's theorem

Proposition 4. *Let $\ell \in \mathcal{SV}$ be locally bounded on $[a, \infty)$, $\alpha > -1$. Then*

$$\int_a^x t^{\alpha} \ell(t) \mathrm{d}t \sim x^{\alpha+1} \ell(x) \frac{1}{\alpha+1}.$$

Proof. We have

$$\begin{aligned} \frac{\int_{a'}^x t^{\alpha} \ell(t) \mathrm{d}t}{x^{\alpha+1} \ell(x)} &= \int_{a'/x}^1 u^{\alpha} \frac{\ell(ux)}{\ell(x)} \mathrm{d}u \\ &= \int_0^1 u^{\alpha} \frac{\ell(ux)}{\ell(x)} I_{[a'/x, 1]}(u) \mathrm{d}u. \end{aligned}$$

The integrand converges pointwise to u^{α}. Choose a' so that the Potter bound can be applied to the ratio with $A = 2$ and $\delta < \alpha + 1$. The statement follows from Lebesgue's dominated convergence theorem. □

We need $\alpha > -1$ for the integrability of the integrand. However, the result hold true in the following sense.

Proposition 5. *Let $\ell \in \mathcal{SV}$ be locally bounded on $[a, \infty)$. Then*

$$\widetilde{\ell}(x) = \int_a^x t^{-1}\ell(t)\mathrm{d}t$$

is slowly varying, and $\widetilde{\ell}(x)/\ell(x) \to \infty$.

Proof. Let $c \in (0, 1)$. For $x > a/c$, by the uniform convergence theorem

$$\begin{aligned}\widetilde{\ell}(x) &= \int_a^x \frac{\ell(t)}{t}\mathrm{d}t \geq \int_{x/c}^x \frac{\ell(t)}{t}\mathrm{d}t \\ &= \int_{1/c}^1 \frac{\ell(xu)}{u}\mathrm{d}u \sim \ell(x)\int_{1/c}^1 \frac{1}{u}\mathrm{d}u \\ &= \ell(x)\log c^{-1}.\end{aligned}$$

Thus

$$\liminf_{x\to\infty} \frac{\widetilde{\ell}(x)}{\ell(x)} \geq \log c^{-1} \to \infty \quad \text{as } c \to 0.$$

To show that $\widetilde{\ell}$ is slowly varying let

$$\varepsilon(x) = \frac{\ell(x)}{\widetilde{\ell}(x)}.$$

We have already shown that $\varepsilon(x) \to 0$ as $x \to \infty$. By the definition of $\widetilde{\ell}$, Lebesgue almost everywhere

$$\widetilde{\ell}'(x) = \frac{\ell(x)}{x} = \frac{\varepsilon(x)\widetilde{\ell}(x)}{x}.$$

Since $\widetilde{\ell}$ is absolutely continuous, so is $\log\widetilde{\ell}$, and

$$\frac{\mathrm{d}}{\mathrm{d}x}\log\widetilde{\ }(x) = \frac{\varepsilon(x)}{x} \quad \text{a.e.}$$

Integrating out, the representation theorem implies the statement. $\square$

The following versions can be proved similarly.

Proposition 6. *If* $\int_x^\infty \ell(t)/t\,\mathrm{d}t < \infty$ *then*

$$\widetilde{\ell}(x) = \int_x^\infty \frac{\ell(t)}{t}\mathrm{d}t$$

is slowly varying and $\widetilde{\ell}(x)/\ell(x) \to \infty$.

Proposition 7. *Let* $\ell \in \mathcal{SV}$, $\alpha < -1$. *Then*

$$\int_x^\infty t^\alpha \ell(t)\mathrm{d}t < \infty$$

and

$$\frac{x^{\alpha+1}\ell(x)}{\int_x^\infty t^\alpha \ell(t)\mathrm{d}t} \to -\alpha - 1.$$

Summarizing, we proved the following.

Theorem 12 (Karamata's theorem, direct part)**.** *Let* $f \in \mathcal{RV}_\rho$ *be locally bounded on* $[a, \infty)$. *Then*

(i) for $\sigma \geq -(\rho + 1)$

$$\frac{x^{\sigma+1}f(x)}{\int_a^x t^\sigma f(t)\mathrm{d}t} \to \sigma + \rho + 1;$$

(ii) for $\sigma < -(\rho + 1)$

$$\frac{x^{\sigma+1}f(x)}{\int_x^\infty t^\sigma f(t)\mathrm{d}t} \to -(\sigma + \rho + 1).$$

(The latter also holds for $\sigma = -(\rho + 1)$ *if the integral is finite.)*

It turns out that this behavior also characterizes regular variation.

Theorem 13 (Karamata's theorem, converse part)**.** *Let* f *be a positive, measurable, locally integrable function on* $[a, \infty)$.

(i) If for some $\sigma > -(\rho + 1)$

$$\frac{x^{\sigma+1}f(x)}{\int_a^x t^\sigma f(t)\mathrm{d}t} \to \sigma + \rho + 1,$$

then $f \in \mathcal{RV}_\rho$*;*

(ii) If for $\sigma < -(\rho+1)$

$$\frac{x^{\sigma+1}f(x)}{\int_x^\infty t^\sigma f(t)\mathrm{d}t} \to -(\sigma+\rho+1),$$

then $f \in \mathcal{RV}_\rho$.

Proof. We only prove (i), the other is similar. Put

$$g(x) = \frac{x^{\sigma+1}f(x)}{\int_a^x t^\sigma f(t)\mathrm{d}t}.$$

Then $g(x) \to \sigma+\rho+1$, and for some $b > a$ fix

$$\int_b^x \frac{g(t)}{t}\mathrm{d}t = \log\left(\int_a^x t^\sigma f(t)\mathrm{d}t/C\right),$$

with $C = \int_a^b t^\sigma f(t)\mathrm{d}t$. This follows by differentiating both sides. Then

$$f(x) = Cb^{-(\rho+\sigma+1)}g(x)x^\rho \exp\left\{\int_b^\sigma \varepsilon(t)/t\mathrm{d}t\right\},$$

and the result follows from the representation theorem. $\square$

6.1 Exercises

10. Let ℓ be a slowly varying function which is locally bounded on $[0,\infty)$. Assume further that $\int_1^\infty \ell(t)/t\,\mathrm{d}t < \infty$. Show that $\widetilde{\ell}(x) = \int_x^\infty \ell(t)/t\,\mathrm{d}t$ is slowly varying and $\widetilde{\ell}(x)/\ell(x) \to \infty$ as $x \to \infty$.

11. Let $\ell_0(x) \equiv 1$, and let $\ell_{i+1}(x) = \int_1^x \ell_i(t)/t\,\mathrm{d}t$, $i = 0, 1, 2, \ldots$. Find ℓ_i.

12. Let ℓ be slowly varying, locally boundend, and $\alpha < -1$. Show that $\int_x^\infty t^\alpha \ell(t)\mathrm{d}t < \infty$, and

$$\lim_{x\to\infty} \frac{x^{\alpha+1}\ell(x)}{\int_x^\infty t^\alpha \ell(t)\mathrm{d}t} = -\alpha - 1.$$

7 Monotone density theorem

Karamata's theorems show how to integrate regularly varying function. Next we turn to the question of differentiating absolutely continuous regularly varying functions. Assume that

$$U(x) = \int_0^x u(t)\mathrm{d}t,$$

for some nonnegative measurable u. Assume that U is regularly varying. Under some additional assumption it follows that u is regularly varying too. A function is *ultimately monotone* if it is monotone (increasing or decreasing) for x large enough.

Theorem 14. *Let $U(x) = \int_0^x u(t)\mathrm{d}t \sim cx^\rho \ell(x)$ as $x \to \infty$ for $c \geq 0$, $\rho \geq 0$, ℓ slowly varying, and assume that u is ultimately monotone. Then*

$$u(x) \sim c\rho x^{\rho-1}\ell(x).$$

Proof. Assume that u is eventually nondecreasing. Then for $a < b$

$$U(bx) - U(ax) = \int_{ax}^{bx} u(t)\mathrm{d}t \leq (b-a)xu(bx).$$

Dividing both sides by $x^\rho \ell(x)$ we obtain

$$\limsup \frac{u(ax)}{x^{\rho-1}\ell(x)} \leq c\frac{b^\rho - a^\rho}{b-a}.$$

Choosing $a = 1$ and letting $b \downarrow 1$ we obtain

$$\limsup \frac{u(x)}{x^{\rho-1}\ell(x)} \leq c\rho.$$

The $\liminf$ result can be shown similarly, and the statement follows. □

Versions of this theorem remain true.

Theorem 15. *Let $U(x) = \int_0^x u(t)\mathrm{d}t \sim cx^\rho \ell(x)$ as $x \downarrow 0$ for $c \geq 0$, $\rho \geq 0$, ℓ slowly varying at 0, and assume that u is ultimately monotone. Then as $x \downarrow 0$*

$$u(x) \sim c\rho x^{\rho-1}\ell(x).$$

8 Inversion

Let f be poisitive locally bounded function on $[a,\infty)$ tending to ∞. Put

$$f^{\leftarrow}(x) = \inf\{y \geq a \,:\, f(y) > x\}.$$

Clearly $f^{\leftarrow}$ is monotone increasing.

Theorem 16. *For $f \in \mathcal{RV}_\alpha$, $\alpha > 0$, there exists $g \in \mathcal{RV}_{1/\alpha}$ such that*

$$f(g(x)) \sim g(f(x)) \sim x \qquad \text{as } x \to \infty.$$

Furthermore, g is uniquely determined up to asymptotic equivalence, and a version of g is $f^{\leftarrow}$.

Proof. We prove that $f(f^{\leftarrow}(x)) \sim x$. Let $A > 1$, $\lambda > 1$, $\delta > 0$. By Potter's bound there is an x_0 such that for $u \geq x_0$

$$\frac{1}{A\lambda^{\alpha+\delta}} \leq \frac{f(u)}{f(v)} \leq A\lambda^{\alpha+\delta} \qquad \text{for } v \in [u/\lambda, u\lambda].$$

Choose x so large that $f^{\leftarrow}(x) \geq x_0$. There exists $y \in [f^{\leftarrow}(x), \lambda f^{\leftarrow}(x)]$ such that $f(y) > x$, and there exists $y' \in [\lambda^{-1} f^{\leftarrow}(x), f^{\leftarrow}(x)]$ such that $f(y') \leq x$. Choosing $u = f^{\leftarrow}(x)$ we obtain

$$\frac{1}{A\lambda^{\alpha+\delta}} \leq \liminf_{x\to\infty} \frac{f(f^{\leftarrow}(x))}{x} \leq \limsup_{x\to\infty} \frac{f(f^{\leftarrow}(x))}{x} \leq A\lambda^{\alpha+\delta}.$$

Letting $A \downarrow 1$, $\lambda \downarrow 1$, the statement follows.

Next we show that $f^{\leftarrow}$ is regularly varying with index $1/\alpha$. Fix $\lambda > 1$. We have

$$\frac{f(\lambda^{1/\alpha} f^{\leftarrow}(x))}{f(f^{\leftarrow}(\lambda x))} = \frac{\lambda x}{f(f^{\leftarrow}(\lambda x))} \frac{f(f^{\leftarrow}(x))}{x} \frac{f(\lambda^{1/\alpha} f^{\leftarrow}(x))}{\lambda f(f^{\leftarrow}(x))},$$

where each factor in the product tends to 1. The first two by the fact that $f(f^{\leftarrow}(x)) \sim x$, the third by the regular variation of f. Therefore

$$\frac{f(\lambda^{1/\alpha} f^{\leftarrow}(x))}{f(f^{\leftarrow}(\lambda x))} \to 1.$$

The regular variation of f implies that

$$f^{\leftarrow}(\lambda x) \sim \lambda^{1/\alpha} f^{\leftarrow}(x),$$

i.e. $f^{\leftarrow}$ is regularly varying with index $1/\alpha$.

Next we show that $f^{\leftarrow}(f(x)) \sim x$. Since $f(f^{\leftarrow}(x)) \sim x$ we have

$$f(f^{\leftarrow}(f(x))) \sim f(x),$$

which, by the regular variation of f implies $f^{\leftarrow}(f(x)) \sim x$.

Finally, $g(f(x)) \sim x$ implies $g(f(f^{\leftarrow}(x))) \sim f^{\leftarrow}(x)$, thus $g(x) \sim f^{\leftarrow}(x)$ as claimed. $\square$

As a simple consequence we obtain the following.

Theorem 17 (de Bruijn conjugate)**.** *For any $\ell \in \mathcal{SV}$ there exists $\ell^{\sharp} \in \mathcal{SV}$ unique up to asymptotic equivalence such that*

$$\ell(x)\ell^{\sharp}(x\ell(x)) \to 1 \quad \textit{and} \quad \ell^{\sharp}(x)\ell(x\ell^{\sharp}(x)) \to 1.$$

Moreover, $(\ell^{\sharp})^{\sharp} \sim \ell$.

8.1 Exercises

13. Find an asymptotic inverse of the following functions and prove that it is indeed an asymptotic inverse.

(a) $f_1(x) = x \log x$;

(b) $f_2(x) = x^2 \log\log x$;

(c) $f_3(x) = x^2 (\log x)^3$.

14. Let $f \in \mathcal{RV}_\alpha$, and g is a positive measurable function such that

$$\lim_{x\to\infty} \frac{f(g(x)\lambda^{1/\alpha})}{f(g(\lambda x))} = 1.$$

Show that $g \in \mathcal{RV}_{1/\alpha}$.

9 Laplace–Stieltjes transforms

In the following U is a nondecreasing right-continuous function on $\mathbb{R}$ such that $U(x) = 0$ for $x < 0$. Its Laplace–Stieltjes transform is

$$\widehat{U}(s) = \int_{[0,\infty)} e^{-sx} \mathrm{d}U(x).$$

Theorem 18. *Let U be as above, $c \geq 0$, $\rho \geq 0$, $\ell \in \mathcal{SV}$. The following are equivalent:*

(i) $U(x) \sim c x^\rho \ell(x) \frac{1}{\Gamma(1+\rho)}$ *as* $x \to \infty$*;*

(ii) $\widehat{U}(s) \sim c s^{-\rho} \ell(1/s)$ *as* $s \downarrow 0$.

The following version can be proved in the same way.

Theorem 19. *Let U be as above, $c \geq 0$, $\rho \geq 0$, $\ell \in \mathcal{SV}$. The following are equivalent:*

(i) $U(x) \sim c x^\rho \ell(x) \frac{1}{\Gamma(1+\rho)}$ *as* $x \downarrow 0$*;*

(ii) $\widehat{U}(s) \sim c s^{-\rho} \ell(1/s)$ *as* $s \to \infty$.

9.1 Exercices

15. Show that $\sum_{n=1}^\infty e^{-2^n} 2^{\rho n} < \infty$ for any ρ.

10 Tails of nonnegative random variables

In the following let X be a nonnegative random variable, and $F(x) = \mathbf{P}(X \leq x)$ its distribution function. The tail of the distribution function is $\overline{F}(x) = 1 - F(x)$. The Laplace transform of F, or X is

$$\widehat{F}(s) = \mathbf{E} e^{-sX} = \int_{[0,\infty)} e^{-sx} \mathrm{d}F(x), \quad s \geq 0.$$

Further, let μ_n denote the moments of F, i.e.

$$\mu_n = \mathbf{E} X^n = \int_{[0,\infty)} x^n \mathrm{d}F(x).$$

We are interested in the relation of $\overline{F}$ at infinity and $\widehat{F}$ at zero. By the Taylor formula, whenever $\mathbf{E} X^n = \mu_n < \infty$

$$\widehat{F}(s) = \sum_{k=0}^n \mu_k \frac{(-s)^k}{k!} + o(s^n) \quad \text{as} \quad s \downarrow 0.$$

Introduce the notation for $n \geq 0$

$$\begin{aligned} f_n(s) &= (-1)^{n+1} \left(\widehat{F}(s) - \sum_{k=0}^{n} \mu_k \frac{(-s)^k}{k!} \right) \\ g_n(s) &= \frac{\mathrm{d}^n}{\mathrm{d}s^n} f_n(s) = \mu_n + (-1)^{n+1} \widehat{F}^{(n)}(s). \end{aligned} \tag{5}$$

In particular, $f_0(s) = g_0(s) = 1 - \widehat{F}(s)$.

The following theorem is due to Bingham and Doney (1974), see Theorem 8.1.6 in [1].

Theorem 20. *Let $\ell \in \mathcal{SV}$, $\mu_n < \infty$, $\alpha = n + \beta$ for $\beta \in [0, 1]$. The following are equivalent:*

(i) $f_n(s) \sim s^\alpha \ell(1/s)$ *as* $s \downarrow 0$*;*

(ii) $g_n(s) \sim \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)} s^\beta \ell(1/s)$ *as* $s \downarrow 0$*;*

(iii) as $x \to \infty$

$$\begin{aligned} \int_{(x,\infty)} t^n \mathrm{d}F(t) &\sim n! \ell(x) \quad \textit{if} \;\; \beta = 0 \\ \overline{F}(x) &\sim \frac{(-1)^n}{\Gamma(1-\alpha)} x^{-\alpha} \ell(x) \quad \textit{if} \;\; \beta \in (0,1) \\ \int_{[0,x]} t^{n+1} \mathrm{d}F(t) &\sim (n+1)! \ell(x) \quad \textit{if} \;\; \beta = 1. \end{aligned}$$

For $\beta > 0$ *these are further equivalent to*

(iv) $(-1)^{n+1} \widehat{F}^{(n+1)}(s) \sim \frac{\Gamma(\alpha+1)}{\Gamma(\beta)} s^{\beta-1} \ell(1/s)$ *as* $s \downarrow 0$*.*

Proof. The equivalence of (i) and (ii) follows from the monotone density theorem. By the same reason, for $\beta > 0$ these are equivalent to (iv).

For $\beta = 1$ the function $(-1)^{n+1} \widehat{F}^{(n+1)}(s)$ is the Laplace–Stieltjes transform of $\int_{[0,x]} t^{n+1} \mathrm{d}F(t)$, thus the equivalence of (iii) and (iv) are follows from the Tauberian theorem for the Laplace transform. Thus in the following we may assume that $\beta < 1$.

Next we show the equivalence of (ii) and (iii). Put

$$U(x) = \int_0^x \int_{(t,\infty)} y^n \mathrm{d}F(y) \, \mathrm{d}t.$$

Then integrations by parts shows

$$\widehat{U}(s) = \int_{[0,\infty)} e^{-sx} \mathrm{d}U(x) = s^{-1}\left[\mu_n + (-1)^{n+1}\widehat{F}^{(n)}(s)\right] = \frac{g_n(s)}{s}.$$

Thus by the Tauberian theorem

$$\text{(ii)} \quad \Longleftrightarrow \quad U(x) \sim \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(2-\beta)} x^{1-\beta}\ell(x). \tag{6}$$

By the monotone density theorem again, the right-hand side of (6) is further equivalent to

$$\begin{aligned} T_n(x) := \int_{(x,\infty)} y^n \mathrm{d}F(y) &\sim \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(2-\beta)}(1-\beta)x^{-\beta}\ell(x) \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(1-\beta)} x^{-\beta}\ell(x). \end{aligned} \tag{7}$$

Thus the statement is proved for $\beta = 0$. Assume now $\beta \in (0,1)$. Then integration by parts gives

$$T_n(x) = x^n \overline{F}(x) + n \int_x^\infty y^{n-1}\overline{F}(y)\mathrm{d}y.$$

If (iii) holds then by Karamata's theorem (7), and thus (ii) follows. For the converse, assume that (ii), thus (7) holds. Then, after some integration by parts formulas, we obtain

$$\frac{x^n \overline{F}(x)}{T_n(x)} = 1 - \frac{n x^n}{T_n(x)} \int_x^\infty y^{-n-1} T_n(y)\mathrm{d}y.$$

Thus the theorem follows again by an application of Karamata's theorem. □

The most important special case is when $n = 0$.

Corollary 4. *Let $\ell \in \mathcal{SV}$, $\alpha \in [0,1]$. Then the following are equivalent:*

(i) $1 - \widehat{F}(s) \sim s^\alpha \ell(1/s)$ *as* $s \downarrow 0$*;*

(ii) as $x \to \infty$

$$\begin{aligned} \overline{F}(x) &\sim \frac{1}{\Gamma(1-\alpha)} x^{-\alpha}\ell(x) \quad \text{if } \alpha \in [0,1) \\ \int_{[0,x]} t\mathrm{d}F(t) &\sim \ell(x) \quad \text{if } \alpha = 1 \\ \int_0^x \overline{F}(t)\mathrm{d}t &\sim \ell(x) \quad \text{if } \alpha = 1. \end{aligned}$$

The importance of the tail behavior of random variables is explained by the following classical result.

Theorem 21 (Doeblin, Gnedenko)**.** *Let $X, X_1, X_2, \dots$ be iid random variables with distribution function F, and let $S_n = X_1 + \ldots + X_n$ denote their partial sum. Then there exist centering and norming sequences a_n and c_n such that $(S_n - c_n)/a_n$ converges in distribution to a nondegenerate random variable Z if and only if one of the following two conditions holds:*

(i) Z a normal, and the truncated second moment

$$V(x) = \int_{[-x,x]} y^2 \mathrm{d}F(y)$$

is slowly varying;

(ii) for some $\alpha \in (0,2)$ and a slowly varying function ℓ

$$F(-x) + 1 - F(x) = \frac{\ell(x)}{x^\alpha}$$

and $\lim_{x\to\infty} F(-x)/(1-F(x))$ exists (0 or ∞ allowed).

Example 2. Let X be a nonnegative random variable with distribution function $F(x) = 1 - x^{-\alpha}$, $x \geq 1$. This is the Pareto distribution with parameter $\alpha > 0$. By Theorem 20

$$1 - \mathbf{E}e^{-sX} \sim \Gamma(1-\alpha)s^\alpha \quad \text{as} \;\; s \downarrow 0.$$

Therefore, for the partial sum $S_n = X_1 + \ldots + X_n$ with the sequence $a_n = n^{1/\alpha}$

$$\mathbf{E}e^{s\frac{S_n}{a_n}} = \exp\left\{ n \log \mathbf{E}e^{-sX/a_n} \right\} \sim e^{-\Gamma(1-\alpha)s^\alpha},$$

which implies that $S_n/n^{1/\alpha}$ converges in distribution.

10.1 Exercises

16. Determine the Laplace transform of the following distributions.

(a) $X \sim \text{Bernoulli}(p)$;

(b) $X \sim \text{Binomial}(n,p)$;

(c) $X \sim \text{Poisson}(\lambda)$;

(d) $X \sim \text{Uniform}(a, b)$;

(e) $X \sim \text{Exp}(\lambda)$.

17. Let $X \geq 0$, $\alpha > 0$. Show that $\mathbf{E}X^\alpha < \infty$ implies $\lim_{x\to\infty} x^\alpha[1 - F(x)] = 0$. Give a counterexample to show that the converse is not true. (It is almost true, see the next exercise.)

18. Let $X \geq 0$, $\alpha > 0$. Show that $\lim_{x\to\infty} x^\alpha[1-F(x)] = 0$ implies $\mathbf{E}X^\beta < \infty$ for any $\beta < \alpha$.

19. Let X be a nonnegative random variable, F its distribution function, and $\widehat{F}(s) = \int_{[0,\infty)} e^{-sx} \mathrm{d}F(x)$ its Laplace transform. Assume that $\mu_n = \mathbf{E}X^n < \infty$. Define

$$
\begin{aligned}
f_n(s) &= (-1)^{n+1} \left(\widehat{F}(s) - \sum_{k=0}^{n} \mu_k (-s)^k / k! \right) \\
g_n(s) &= \frac{\mathrm{d}^n}{\mathrm{d}s^n} f_n(s).
\end{aligned}
$$

Let ℓ be a slowly varying function, $\alpha = n + \beta$ with $\beta \in [0, 1]$. Show that $f_n(s) \sim s^\alpha \ell(1/s)$ if and only if $g_n(s) \sim \Gamma(\alpha + 1)/\Gamma(\beta + 1)\, s^\beta \ell(1/s)$.

20. Show that the Laplace transform of the standard normal distribution is $e^{s^2/2}$.

11 Sum and maxima of iid random variables

In the following $X, X_1, X_2, \ldots$ are nonnegative iid random variables with distribution function $\mathbf{P}(X \leq x) = F(x)$. Let $M_n = \max\{X_1, \ldots, X_n\}$ and $S_n = X_1 + \ldots + X_n$ denote the partial maxima and partial sum. We are interested in the behavior of the ration M_n/S_n.

Darling [3] proved that if $\overline{F}(x) = 1 - F(x)$ is slowly varying then the maximum term dominates the whole sum.

Theorem 22. *If $\overline{F}$ is slowly varying then $S_n/M_n \to 1$ in probability (and in L^1).*

Before the proof we need the condtional distribution of S_n given M_n.

Lemma 4. *Assume that F is continuous with density function f. Then*

$$\mathcal{L}(S_n|M_n = m) = \mathcal{L}(S_{n-1}^{(m)} + m),$$

where $S_k^{(m)} = Y_1^{(m)} + \ldots + Y_k^{(m)}$, with $Y^{(m)}, Y_1^{(m)}, \ldots$ being iid random variables with distribution function $\mathbf{P}(Y^{(m)} \leq y) = \mathbf{P}(X \leq y|X \leq x)$.

Proof. It is a long but straigthforward calculation. □

Next we prove the theorem.

Proof of Theorem 22. Assume that F is continuous. This can be dropped by adding iid uniform$(0,1)$ random variables.

Note that S_n/M_n for n fix is a bounded nonnegative random variable which is ≥ 1. Therefore its mean can be calculated as the derivative of its Laplace transform at 0. Since $S_n/M_n \geq 1$, it is enough to show that $\mathbf{E}S_n/M_n \to 1$ as $n \to \infty$.

Let $\lambda \geq 0$. Using Lemma 4 (and the notation there) we have

$$\begin{aligned} \phi_n(\lambda) := \mathbf{E}e^{-\lambda \frac{S_n}{M_n}} &= \int_{[0,\infty)} \mathbf{E}^{-\lambda \frac{S_{n-1}^{(x)}+x}{x}} \mathrm{d}\mathbf{P}(M_n \leq x) \\ &= \int_{[0,\infty)} ne^{-\lambda} \left(\int_{[0,x]} e^{-\lambda y/x} \mathrm{d}F(y) \right)^{n-1} \mathrm{d}F(x). \end{aligned} \tag{8}$$

Differentiating and substituting $\lambda = 0$

$$\mathbf{E}\frac{S_n}{M_n} = -\phi'(0) = 1 + \int_{[0,\infty)} n(n-1)F(x)^{n-2} \int_{[0,x]} \frac{y}{x} \mathrm{d}F(y)\mathrm{d}F(x). \tag{9}$$

Integration by parts gives

$$\int_{[0,x]} y\mathrm{d}F(y) = x \int_0^1 \left[\overline{F}(ux) - \overline{F}(x)\right] \mathrm{d}u.$$

Substituting back into (9)

$$\mathbf{E}\frac{S_n}{M_n} = 1 + \int_{[0,\infty)} n(n-1)F(x)^{n-2}\overline{F}(x)A(x)\mathrm{d}F(x), \tag{10}$$

where

$$A(x) = \int_0^1 \left(\frac{\overline{F}(ux)}{\overline{F}(x)} - 1 \right) \mathrm{d}u.$$

The integrand in $A(x)$ converges pointwise to 0 by the slow variation of $\overline{F}$, and Potter's bound provides an integrable majorant ($u^{-1/2}$ say). Therefore, by Lebesgue's dominated convergence theorem $\lim_{x\to\infty} A(x) = 0$. Let $\varepsilon > 0$ be fixed. Then there exists x_0 such that $A(x) \le \varepsilon$ for all $x \ge x_0$. Further, there exists n_0 such that $n(n-1)F(x_0)^{n-2} \sup_{y\in[0,x]} A(y) \le \varepsilon$ for $n \ge n_0$. Thus

$$\int_{[0,x_0]} n(n-1)F(x)^{n-2}\overline{F}(x)A(x)\mathrm{d}F(x) \le \varepsilon \int_{[0,\infty)} \mathrm{d}F(x) = \varepsilon.$$

On the other hand

$$\begin{aligned}
&\int_{(x_0,\infty)} n(n-1)F(x)^{n-2}\overline{F}(x)A(x)\mathrm{d}F(x) \\
&\le \varepsilon \int_{(x_0,\infty)} n(n-1)F(x)^{n-2}\overline{F}(x)\mathrm{d}F(x) \\
&\le \varepsilon \int_0^1 n(n-1)u^{n-2}(1-u)\mathrm{d}u = \varepsilon,
\end{aligned}$$

proving the statement. □

In fact the slow variation of $\overline{F}$ is necessary to the domination of the maxima.

Theorem 23 (Maller & Resnick, 1984)**.** *The following are equivalent:*

(i) $M_n/S_n \xrightarrow{\mathbf{P}} 1$*;*

(ii) $\overline{F}$ *is slowly varying.*

The other extremal situation is when the maxima is asymptotically negligable compared to the sum.

Theorem 24 (O'Brien, 1980)**.** *The following are equivalent:*

(i) $M_n/S_n \xrightarrow{\mathbf{P}} 0$*;*

(ii) $\int_{[0,x]} y\mathrm{d}F(y)$ *is slowly varying.*

Next we turn to the intermediate case.

Theorem 25 (Darling, 1952)**.** *If* $\overline{F}$ *is regularly varying with parameter* $-\alpha \in (-1,0)$ *then*

$$\frac{S_n}{M_n} \xrightarrow{\mathcal{D}} W, \qquad \text{where } \mathbf{E}e^{-\lambda W} = \frac{e^{-\lambda}}{1-\alpha\int_0^1 (e^{-\lambda u}-1)u^{-\alpha-1}\mathrm{d}u}.$$

Proof. Assume that F is continuous. Recall from (8) that

$$\phi_n(\lambda) = \int_{[0,\infty)} ne^{-\lambda} \left(\int_{[0,x]} e^{-\lambda y/x} \mathrm{d}F(y) \right)^{n-1} \mathrm{d}F(x).$$

Integration by parts gives

$$\int_{[0,x]} e^{-\lambda y/x} \mathrm{d}F(y) = 1 - \overline{F}(x) - \overline{F}(x) \int_0^1 \left(\frac{\overline{F}(ux)}{\overline{F}(x)} - 1 \right) \lambda e^{-\lambda u} \mathrm{d}u \qquad (11)$$

As $x \to \infty$, by the regular variation combined with Potter bounds and Lebesgue's dominated convergence we have

$$\int_0^1 \left(\frac{\overline{F}(ux)}{\overline{F}(x)} - 1 \right) \lambda e^{-\lambda u} \mathrm{d}u \to \int_0^1 (u^{-\alpha} - 1) \lambda e^{-\lambda u} \mathrm{d}u.$$

Since the integrand is exponentially small on any finite interval, we obtain for any K large

$$\begin{aligned} \phi_n(\lambda) &\sim e^{-\lambda} \int_K^\infty n \left[1 - \overline{F}(x) \left(1 + \int_0^1 (u^{-\alpha} - 1) \lambda e^{-\lambda u} \mathrm{d}u \right) \right] \mathrm{d}F(x) \\ &\sim e^{-\lambda} \mathbf{E} \left[n(1 - Uc_\lambda)^{n-1} I(U < \delta) \right], \end{aligned}$$

where U ~Uniform$(0,1)$, $\delta = \overline{F}(K)$ and

$$c_\lambda = 1 + \int_0^1 (u^{-\alpha} - 1) \lambda e^{-\lambda u} \mathrm{d}u.$$

Now, simple analysis shows that

$$\lim_{n \to \infty} \mathbf{E} \left[n(1 - Uc_\lambda)^{n-1} I(U < \delta) \right] = c_\lambda^{-1},$$

and the theorem follows.

The continuity assumption can be dropped by adding iid uniform$(0,1)$ random variables. □

The converse result is due to Breiman [2].

Theorem 26 (Breiman, 1965)**.** *If S_n/M_n converges in distribution to a non-degenerate limit then $\overline{F}$ is regularly varying with parameter $-\alpha \in (-1, 0)$.*

Proof. Again, assume that F is continuous.

The distributional convergence of S_n/M_n implies that

$$\lim_{n\to\infty} \phi_n(\lambda) = \phi(\lambda) \tag{12}$$

exists for all $\lambda \geq 0$. Put

$$U(\lambda, x) = \int_{[0,x]} e^{-\lambda y} \mathrm{d}F(y). \tag{13}$$

We have seen in (8) that

$$\phi_n(\lambda) = e^{-\lambda} \int_{[0,\infty)} nU(\lambda/x, x)^{n-1} F(\mathrm{d}x).$$

The monotonicity of U and (12) implies that n can be exchanged to the continuous parameter t, i.e.

$$\lim_{t\to\infty} e^{-\lambda} \int_{[0,\infty)} tU(\lambda/x, x)^t F(\mathrm{d}x) = \phi(\lambda). \tag{14}$$

We have seen in (11) that

$$U(\lambda/x, x) = 1 - \overline{F}(x) \left(1 + \int_0^1 \left(\frac{\overline{F}(ux)}{\overline{F}(x)} - 1 \right) \lambda e^{-\lambda u} \mathrm{d}u \right). \tag{15}$$

Note that $U(\lambda/x, x)$ is increasing in x, and it is strictly increasing for x large. Moreover, $\lim_{x\to 0} U(\lambda/x, x) = 0$, and $\lim_{x\to\infty} U(\lambda/x, x) = 1$. For $\lambda \geq 0$ fixed, put

$$V(x) = -\log U(\lambda/x, x), \tag{16}$$

and let $G(t) = \mu_F(\{y : V(y) \leq t\})$. By the transformation theorem

$$\int_{[0,\infty)} U(\lambda/x, x)^t F(\mathrm{d}x) = \int_{[0,\infty)} e^{-tV(x)} F(\mathrm{d}x) = \int_{[0,\infty)} e^{-ty} G(\mathrm{d}y).$$

Thus, by Karamata's Tauberian theorem (14) is equivalent to

$$G(y) \sim y\phi(\lambda)e^{\lambda} \quad \text{as } y \downarrow 0. \tag{17}$$

By the continuity of F

$$G(V(x)) = \mu_F(\{u : V(u) \leq V(x)\}) = \mu_F(\{u : u \geq x\}) = \overline{F}(x-) = \overline{F}(x),$$

which, combined with (17) and (15)

$$\overline{F}(x) \sim e^{\lambda}\phi(\lambda)\left(\overline{F}(x)e^{-\lambda} + \int_0^1 \overline{F}(ux)e^{-\lambda u}\mathrm{d}u\right).$$

Therefore, we obtain that

$$\lim_{x\to\infty}\int_0^1 \frac{\overline{F}(ux)}{\overline{F}(x)}e^{-\lambda u}\mathrm{d}u \tag{18}$$

exists for all λ. We need the following lemma.

Lemma 5. *Let $J_n(u)$ be a sequence of nonincreasing functions such that for all $\lambda \geq 0$*

$$\lim_{n\to\infty}\int_0^1 e^{-\lambda u}J_n(u)\mathrm{d}u = h(\lambda)$$

for some $h(\lambda)$. Then there exists $J(u)$ nonincreasing such that $J_n(x) \to J(x)$ for all $x \in C_J$, and

$$h(\lambda) = \int_0^1 e^{-\lambda u}J(u)\mathrm{d}u.$$

Proof. The statement follows easily from Helly's selection theorem and the continuity theorem. □

The lemma and (18) implies that the limit $\overline{F}(ux)/\overline{F}(x)$ exists for each u, which implies that $\overline{F}$ is regularly varying. □

12 Breiman's conjecture

Breiman's motivation in his 1965 paper was the following. Lte $S_1, S_2, \ldots$ be a simple symmetric random walk, and let $Y, Y_1, Y_2, \ldots$ be the interarrival times between the consecutive zeros of $S_1, S_2, \ldots$. Independently of S, let $X, X_1, X_2, \ldots$ be iid 0/1 random variables such that $\mathbf{P}\{X = 0\} = \frac{1}{2} = \mathbf{P}\{X = 1\}$. Then

$$T_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$$

is the proportion of the time that the random walk spends in $[0, \infty)$.

In this case the well-known arcsine law holds.

Theorem 27 (Arcsine law)**.** *Let the X's and Y's be as above. Then*

$$\lim_{n\to\infty} \mathbf{P}\{T_n \le x\} = \frac{2}{\pi} \arcsin \sqrt{x}.$$

Moreover, in this case $\overline{G}(y) = \mathbf{P}(Y > y) \sim cy^{-1/2}$, in particular it is regularly varying with parameter $1/2$.

In general, let $Y, Y_1, Y_2, \ldots$ be nonnegative iid random variables with distribution function G, and independently let $X, X_1, X_2, \ldots$ be iid random variables with distribution function F, and assume that $\mathbf{E}|X| < \infty$. What is the necessary and sufficient condition on G such that

$$T_n = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i}$$

has a nondegenerate limit as $n \to \infty$?

Remark 1. If $\mathbf{E}Y < \infty$, then

$$\frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n Y_i} = \frac{\frac{\sum_{i=1}^n X_i Y_i}{n}}{\frac{\sum_{i=1}^n Y_i}{n}} \xrightarrow{\text{a.s.}} \mathbf{E}X,$$

so the limit exists, and it is degenerate. Therefore, the interesting situation is when $\mathbf{E}Y = \infty$.

Breiman proved the following.

Theorem 28 (Breiman, 1965)**.** *If T_n converges in distribution for every F, and the limit is non-degenerate for at least one F, then $Y \in D(\alpha)$, for some $\alpha \in [0, 1)$, i.e. $\overline{G}$ is regularly varying with parameter $-\alpha \in (-1, 0]$.*

The idea of his proof is to prove that the existence of the limit for *all* X implies the existence of the distributional limit of

$$\frac{\max\{Y_1, \ldots, Y_n\}}{Y_1 + \ldots + Y_n}$$

which, by Theorem 26 implies the regular variation. The existence of the limit for *all* X is an essential assumption, though Breiman conjectured it is not necessary. This is the Breiman conjecture, which is still open.

Conjecture 1 (Breiman, 1965)**.** *If T_n has a non-degenerate limit for some F, then $Y \in D(\alpha)$ for some $\alpha \in [0, 1)$.*

References

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