Branching processes with immigration in a random environment

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Outline

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GWI subcritical

Let $X_0 = 0$,

$$X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1} =: \theta_{n+1} \circ X_n + B_{n+1}, \quad n \geq 0,$$

offsprings $\{A_i^{(n)} : \ i = 1, 2, \ldots, n = 1, 2, \ldots\}$ iid, immigrants $\{B_n : \ n = 1, 2, \ldots\}$ iid.
Subcritical: $\mathbf{E}A < 1$.

Stationary distribution – existence

Theorem (Quine (1970), Foster & Williamson (1971))

Unique stationary distribution exists iff

$$\int_0^1 \frac{1 - \mathbf{E} s^B}{\mathbf{E} s^A - s} \mathrm{d}s < \infty.$$

$$X_\infty = B_1 + \theta_1 \circ B_2 + \theta_1 \circ \theta_2 \circ B_3 + \ldots = \sum_{i=0}^{\infty} \Pi_i \circ B_{i+1}.$$

If $m = \mathbf{E} A < 1$, then $\mathbf{E} \log B < \infty$ is necessary and sufficient.
If $m = 1$, then the condition holds if $\mathbf{P}(A > n) \sim \ell_A(n) n^{-1-\alpha}$, $\mathbf{E} B < \infty$, $\alpha \in (0, 1)$, or $\mathbf{P}(B > n) \sim \ell_B(n) n^{-\beta}$, $\beta > \alpha$.

Stationary distribution – tail

Theorem (Basrak & Kulik & Palmowski (2013))

(i) *If* $m = \mathbf{E}A < 1$, $\mathbf{E}A^2 < \infty$, *and* $\mathbf{P}(B > x)$ *is regularly varying with index* $-\alpha \in (-2, 0)$, *then*

$$\mathbf{P}(X_\infty > x) \sim c\,\mathbf{P}(B > x), \qquad c > 0.$$

(ii) *If* $m = \mathbf{E}A < 1$, $\mathbf{P}(A > x)$ *is regularly varying with index* $-\alpha \in (-2, -1)$, *and* $\mathbf{P}(B > x) \sim c'\mathbf{P}(A > x)$, $c' \geq 0$ *then*

$$\mathbf{P}(X_\infty > x) \sim c\,\mathbf{P}(A > x), \qquad c > 0.$$

More general tail behavior: Foss & Miyazawa (2020)
Second order GWI: Barczy & Bősze & Pap (2020)

Regular and slow variation

ℓ is slowly varying if for any $\lambda > 0$

$$\lim_{x\to\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1.$$

Examples: $\lim_{x\to\infty} \ell(x) \in (0,\infty)$, $\ell(x) = \log x$, $\ell(x) = (\log x)^\beta$.
f is regularly varying with index α if

$$f(x) = x^\alpha \ell(x).$$

Critical case, $m = 1$

Theorem (Guo & Hong (2024))

Assume $\mathbf{P}(A > n) \sim \ell_A(n) n^{-1-\alpha}$, $\alpha \in (0,1)$, $\mathbf{P}(B > n) \sim \ell_B(n) n^{-\beta}$, $\beta > \alpha$, *additional assumption on* ℓ_A, ℓ_B. *Then*

$$\mathbf{P}(X_\infty > x) \sim \ell(x) x^{-(\beta-\alpha)}.$$

Critical case, $m = 1$

Assume $\mathbf{P}(A > n) \sim \ell_A(n) n^{-1-\alpha}$, $\alpha \in (0,1)$, $\mathbf{E}B < \infty$.
K & Kubatovics (2025+, work in progress):

$$\mathbf{P}(X_\infty > x) \sim \ell(x) x^{-(1-\alpha)}.$$

Stationary chain $(X_n)_{n \geq 0}$ is regularly varying, tail process:

$$\mathcal{L}\left((X_n/x) | X_0 > x \right) \longrightarrow Y(1, 1, \ldots),$$

$\mathbf{P}(Y > y) = y^{-(1-\alpha)}$. The anticlustering condition does not hold.
Explicit calculations are possible: Alsmeyer & Hoang (2025):
Power fractional distributions, Lindo & Sagitov (2016):
θ-branching

Outline

GWRE with immigration (GWIRE)

- Δ probability measures on $\mathbb{N} = \{0, 1, \ldots\}$
- $\xi, \xi_1, \ldots$ iid on Δ^2 (environment), $\xi = (\nu_\xi, \nu_\xi^\circ)$

GWRE with immigration (GWIRE)

- Δ probability measures on $\mathbb{N} = \{0, 1, \ldots\}$
- $\xi, \xi_1, \ldots$ iid on Δ^2 (environment), $\xi = (\nu_\xi, \nu_\xi^\circ)$
- $X_0 = 0$,

$$X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1} =: \theta_{n+1} \circ X_n + B_{n+1}, \quad n \geq 0,$$

conditioned on $\mathcal{E}$, $\{A_i^{(n)}, B_n : i = 1, 2, \ldots, n = 1, 2, \ldots\}$ are independent and for n fix $(A_i^{(n)})_{i=1,2,\ldots}$ are iid with distribution ν_{ξ_n}, and B_n has distribution $\nu_{\xi_n}^\circ$.

Subcritical / critical / supercritical: $\mathbf{E} \log m(\xi) < / = / > 0$.
Kersting, Vatutin: Discrete Time Branching Processes in Random Environment, 2017, Wiley.

Stationary distribution – existence

Theorem (Key (1987))

If $\mathbf{E} \log m(\xi) < 0$ *(offspring) and* $\mathbf{E} \log^+ m^\circ(\xi) < \infty$ *(immigration) then there exists a unique stationary distribution*

$$X_\infty = B_1 + \theta_1 \circ B_2 + \theta_1 \circ \theta_2 \circ B_3 + \ldots = \sum_{i=0}^{\infty} \Pi_i \circ B_{i+1}.$$

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Kesten–Grincevičius–Goldie setup

$$X_\infty = B_1 + \theta_1 \circ B_2 + \theta_1 \circ \theta_2 \circ B_3 + \ldots = \sum_{i=0}^{\infty} \Pi_i \circ B_{i+1}.$$

Theorem (Basrak & K 2022)

Assume: $\mathbf{E} m(\xi)^\kappa = 1$, $\mathbf{E} A^\kappa < \infty$, $\mathbf{E} B^\kappa < \infty$, $\mathbf{E} m(\xi)^\kappa \log m(\xi) < \infty$, $\log m(\xi)$ *is non-arithmetic. Then*

$$\mathbf{P}(X_\infty > x) \sim C x^{-\kappa} \quad x \to \infty,$$

with $C > 0$*.*

Theorem (Basrak & K 2022)

($\mathbf{E}m(\xi)^\kappa = 1$, $\mathbf{E}A^\kappa < \infty$, $\mathbf{E}B^\kappa < \infty$, $\overline{F}_\kappa(x) = \ell(x)x^{-\alpha}$) or ($\mathbf{E}m(\xi)^\kappa < 1$ and F_κ is locally subexponential, $\mathbf{E}B^\kappa < \infty$)

$$\mathbf{P}(X_\infty > x) \sim Cx^{-\kappa}L(x) \quad x \to \infty,$$

where L is slowly varying, $C \geq 0$, and if $\kappa \geq 1$ $C > 0$.

Related papers

- Afanasyev (2001): $\mathbf{P}(\sup_n X_n > x) \sim cx^{-\kappa}$, $c > 0$.
- Large deviation results: Buraczewski & Dyszewski (2022), Guo & Hong & Sun (2025)
- Arithmetic case: Jelenković and Olvera-Cravioto (2012), K (2017): implicit renewal theory in the arithmetic case

Grincevičius – Grey setup

Theorem (K 2024)

Assume: $\mathbf{E}(m(\xi)^{1\vee\kappa}) < 1$, $\mathbf{E}(A^{(1\vee\kappa)+\delta}) < \infty$.
Let ℓ *be a slowly varying function. Then*

$$\mathbf{P}(B > x) \sim \frac{\ell(x)}{x^\kappa}, \quad \text{as} \;\; x \to \infty,$$

if and only if

$$\mathbf{P}(X_\infty > x) \sim \frac{\ell(x)}{x^\kappa} \frac{1}{1 - \mathbf{E}(m(\xi)^\kappa)}, \quad \text{as} \;\; x \to \infty.$$

Outline

Setup

- $X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1} =: \theta_{n+1} \circ X_n + B_{n+1}$, $n \in \mathbb{Z}$, strictly stationary
- $\mathbf{P}(X_0 > x) \sim c\ell(x)x^{-\kappa}$
- a_n is defined by $n\mathbf{P}(X_0 > a_n) \sim 1$.

Asymptotic properties of Markov chain

- tail process (Basrak & Segers (2009))
- point process convergence (ergodicity, anticlustering)
- convergence of partial sums (vanishing small values)

CLT

Theorem (Basrak & K 2022)

Let $b_n = 0$, $\kappa < 1$, $b_n = n\mathbf{E}(X_\infty / a_n I(X_\infty \leq a_n))$, $\kappa \in [1, 2)$. *Then*

$$V_n = \sum_{k=1}^{n} \frac{X_k}{a_n} - b_n \xrightarrow{\mathcal{D}} V, \qquad n \to \infty,$$

with V κ*-stable. If* $\kappa > 2$,

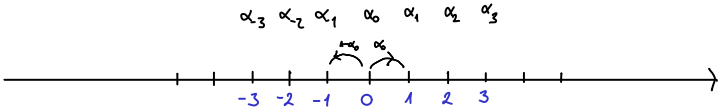
$$\frac{1}{\sqrt{n}\sigma} \sum_{j=1}^{n} (X_i - \mathbf{E}X_\infty) \xrightarrow{\mathcal{D}} Z \sim N(0, 1).$$

Random walk in random environment

Kozlov and Solomon:

- $\{\alpha_i\}_{i\in\mathbb{Z}}$ iid in $[0,1]$
- $\mathcal{A} = \sigma(\alpha_i : i \in \mathbb{Z})$, σ-algebra generated by the environment,
- $X_0 = 0$,

$$\mathbf{P}(W_{n+1} = W_n + 1 | \mathcal{A}, W_0, \ldots, W_n) = \alpha_i \quad \text{on } \{W_n = i\}$$
$$\mathbf{P}(W_{n+1} = W_n - 1 | \mathcal{A}, W_0, \ldots, W_n) = 1 - \alpha_i \quad \text{on } \{W_n = i\}$$



KKS result

Let T_n be the first hitting time of n. $T_n \approx 2 \sum_{k=1}^{n} X_k - n$.

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Theorem (Kesten & Kozlov & Spitzer 1975)

For $\kappa \in (0,2)$,

$$n^{-1/\kappa}(T_n - A_n) \xrightarrow{\mathcal{D}} \kappa - \text{stable r.v.}$$

where $A_n \equiv 0$ if $\kappa < 1$, $A_n = nc_1$ if $\kappa > 1$.
For $\kappa > 2$

$$n^{-1/2}(T_n - nc) \xrightarrow{\mathcal{D}} N(0,1).$$

Moreover, $n^{-\kappa}(W_n - B_n)$ also converges.

Outline

$(A_n, B_n)_n$ iid random vectors, and X_0 a random variable independent of them. The stochastic recurrence equation is

$$X_{n+1} = A_{n+1} X_n + B_{n+1}.$$

The stationary solution should be

$$X_\infty = B_1 + A_1 B_2 + \ldots + A_1 A_2 \ldots A_n B_{n+1} + \ldots =: \sum_{n=0}^{\infty} \Pi_n B_{n+1}.$$

Satisfies the fixed point equation

$$X \stackrel{\mathcal{D}}{=} AX + B,$$

where (A, B) and X on the RHS are independent.

Tail of the stationary distribution

Theorem (Grincevičius – Kesten – Goldie)

If $\mathbf{E}A^\kappa = 1, \mathbf{E}A^\kappa \log_+ A < \infty$, $\log A$ *is nonarithmetic,* $\mathbf{E}B^\kappa < \infty$ *then for the solution to the equation* $X \stackrel{\mathcal{D}}{=} AX + B$ *we have*

$$\mathbf{P}(X > x) \sim cx^{-\kappa},$$

with $c > 0$*.*

Tail of the stationary distribution

Theorem (Grincevičius – Grey)

If $A \geq 0$, $\mathbf{E}A^{\kappa} < 1$, $\mathbf{E}A^{\kappa+\epsilon} < \infty$ then the tail of X is regularly varying with parameter $-\kappa$ if and only if the tail of B is.

Damek & Kołodziejek 2020: Between Kesten and Grincevičius – Grey

Outline

Goldie's setup - stochastic fixed point equations

$$X_{n+1} = \sum_{i=1}^{X_n} A_i^{(n+1)} + B_{n+1} =: \theta_{n+1} \circ X_n + B_{n+1}, \quad n \geq 0,$$

X stationary law:

$$X \stackrel{\mathcal{D}}{=} \sum_{i=1}^{X} A_i + B = \theta \circ X + B$$

(θ, B) and X are independent.

Examples

- Perpetuity: $X \stackrel{\mathcal{D}}{=} AX + B$, (A, B) and X are independent.

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Stationary distributions of a Markov chain.

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- $X \stackrel{\mathcal{D}}{=} \sum_{i=1}^{X} A_i + B$

Stationary distributions of a Markov chain.
Buraczewski, Damek, Mikosch: Stochastic models with power law tails. The equation $X = AX + B$. (2016)
Iksanov: Renewal theory for perturbed random walks and similar processes. (2016)

General: $X \stackrel{\mathcal{D}}{=} \Psi(X)$, where $\Psi : \mathbb{R} \times \Omega \to \mathbb{R}$ random operator, independent of X.

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Theorem (Goldie (1991), Grincevicius (1975))

X is the solution to $X \stackrel{\mathcal{D}}{=} \Psi(X)$, assume $\mathbf{E}|(\Psi(X))^\kappa - (AX)^\kappa| < \infty$. Then $\mathbf{P}(X > x) \sim cx^{-\kappa}$, where $c = \mathbf{E}(\Psi(X)^\kappa - (AX)^\kappa)/\mathbf{E}(A^\kappa \log A) \geq 0$.

$$\mathbf{P}(X > x) \sim cx^{-\kappa}$$

- Problem: $c = 0$ is possible!
- If $\mathbf{E}X^{\kappa} < \infty$, then $c = 0$.
- Idea: $\Psi(x) \sim Ax$, $x \to \infty$. $(x \to \pm\infty)$
- Alsmeyer, Brofferio, Buraczewski: Asymptotically linear iterated function systems on the real line (2023)
- K (2016): additional slowly varying factor, or $\mathbf{E}A^{\kappa} < 1$ is possible