Almost sure properties of Lévy driven supOU processes

Péter Kevei

University of Szeged

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Joint work with Danijel Grahovac (Osijek).

Outline

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Classical Ornstein-Uhlenbeck

Langevin equation:

$$\mathrm{d}X_t = -\mu X_t \mathrm{d}t + \mathrm{d}W_t$$

$$X_t = e^{-\mu t} x_0 + \int_0^t e^{-\mu(t-s)} \mathrm{d}W_s$$

stationary version:

$$X_t = \int_{-\infty}^t e^{-\mu(t-s)} \mathrm{d}W_s$$

stationary Gaussian process

Lévy driven OU

$$\mathrm{d}X_t = -\mu X_t \mathrm{d}t + \mathrm{d}L_t$$

$$X_t = \int_{-\infty}^{t} e^{-\mu(t-s)} \mathrm{d}L_s$$

flexible tail behavior, self-decomposable distribution
generalized OU process: Maller, Lindner, Behme

Lévy driven OU

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flexible tail behavior, self-decomposable distribution
generalized OU process: Maller, Lindner, Behme
Covariance function:

$$\rho(t) = e^{-\mu t} c.$$

flexible covariance structure?

superposition of OU processes

Barndorff-Nielsen (2000) (turbulence, financial data):

$$X(t) = \int\int_{(0,\infty)\times(-\infty,t]} e^{-x(t-s)} \Lambda(\mathrm{d}x, \mathrm{d}s),$$

where Λ is a Lévy basis.

superposition of OU processes

Barndorff-Nielsen (2000) (turbulence, financial data):

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where Λ is a Lévy basis.
Barndorff-Nielsen, Espen Benth, Veraart (ambit);
Barndorff-Nielsen, Stelzer (multivariate), Fasen, Klüppelberg (extremes), Grahovac, Leonenko, Taqqu (limit theorems), ...

Lévy basis

Lévy basis (infinitely divisible independently scattered random measure)

$$\Lambda(\mathrm{d}x, \mathrm{d}s) = \int_{(0,1]} z(\mu - \nu)(\mathrm{d}x, \mathrm{d}s, \mathrm{d}z) + \int_{(1,\infty)} z\mu(\mathrm{d}x, \mathrm{d}s, \mathrm{d}z),$$

μ is a Poisson random measure with intensity

$$\nu(\mathrm{d}x, \mathrm{d}s, \mathrm{d}z) = \pi(\mathrm{d}x)\mathrm{d}s\lambda(\mathrm{d}z),$$

λ is a Lévy measure, π is a measure such that $\int x^{-1}\pi(\mathrm{d}x) < \infty$.
Integration theory: Rajput & Rosinski (1989)

supOU

$$X(t) = \int \int e^{-x(t-s)} \mathbb{I}(s \leq t) \Lambda(\mathrm{d}x, \mathrm{d}s),$$

- existence: $\int x^{-1} \pi(\mathrm{d}x) < \infty$, $\int_{(1,\infty)} \log z \lambda(\mathrm{d}z) < \infty$;
- strictly stationary
- covariance function:

$$\rho(t) = \int x^{-1} e^{-xt} \pi(\mathrm{d}x)$$

- tail - λ, dependence - π

supOU - short-range vs long-range dependence

$$\rho(t) = \int x^{-1} e^{-xt} \pi(\mathrm{d}x)$$

If $\pi(\mathrm{d}x) = x^{\alpha}\mathrm{d}x$, then $\rho(t) \sim t^{-\alpha}$.

short-range dependence:

$$\int_0^{\infty} \rho(t)\mathrm{d}t < \infty \Longleftrightarrow \int x^{-2}\pi(\mathrm{d}x) < \infty.$$

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Marginals - Lévy driven OU

$$X(t) = \iint_{(0,\infty)\times(-\infty,t)} e^{-x(t-s)} \Lambda(\mathrm{d}x, \mathrm{d}s)$$

$$\mathbf{E} e^{\mathrm{i}\theta X(t)} = \exp\left\{ \mathrm{i}\theta A + \int_{(0,\infty)} \left(e^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta y \mathbb{I}(y \leq 1) \right) \eta(\mathrm{d}y) \right\},$$

where

$$\eta(B) = m_{-1}(\pi) \,\mathrm{Leb} \times \lambda \left(\{(u, z) : e^{-u} z \in B\} \right)$$

Note: doesn't depend on π

tail

$$\overline{\eta}(r) = \eta((r,\infty)) = \int\int \mathbb{I}(e^{-u}z > r)\mathrm{d}u\lambda(\mathrm{d}z)$$
$$= \int_{(r,\infty)} \log\frac{z}{r}\,\lambda(\mathrm{d}z) = \int_r^\infty \frac{\overline{\lambda}(z)}{z}\mathrm{d}z.$$

$\mathbf{E}X(t)^\beta < \infty$ iff $\int_{(1,\infty)} z^\beta \lambda(\mathrm{d}z) < \infty$.

Lemma (Fasen, Klüppelberg (2007))

$\overline{\lambda} \in \mathcal{RV}_{-\gamma}$ iff $\overline{\eta} \in \mathcal{RV}_{-\gamma}$, and

$$\overline{\eta}(r) \sim \frac{1}{\gamma}\overline{\lambda}(r), \quad r \to \infty.$$

Outline

Proposition (Grahovac & K 2025)

Assume that $\int_{(0,1]} z\lambda(\mathrm{d}z) < \infty$. *If* $m_0(\pi) = \pi((0,\infty)) = \infty$, *then*

$$\sup_{t\in[0,1]} X(t) \geq \sup\{z : \overline{\lambda}(z) > 0\}.$$

In particular, if the jump sizes are unbounded, then

$$\sup_{t\in[0,1]} X(t) = \infty \quad \textit{a.s.}$$

Maejima (1983): linear fractional stable motion
Chong, Dalang, Humeau (2019): SHE with Lévy noise

Proof

$$X(t) = \iint e^{-x(t-s)} \Lambda(\mathrm{d}x, \mathrm{d}s) = \sum_{k:\tau_k \le t} e^{-\xi_k(t-\tau_k)} \zeta_k$$

thus

$$\sup_{t\in[0,1]} X(t) \ge \sup\{\zeta_k : \tau_k \in [0,1]\}.$$

Since

$$\pi \times \mathrm{Leb}\left(\{(x,s) : 0 \le s \le 1\}\right) = \pi((0,\infty)) = \infty,$$

there are infinitely many k's such that $\tau_k < 1$. The corresponding ζ_k's are independent, thus the result follows.

Càdlàg paths

Work in progress: conditions for càdlàg path for process of the form

$$X(t) = \int f(x, t-s)\Lambda(\mathrm{d}x, \mathrm{d}s).$$

Theorem (Grahovac & K & Sauri 2025+)

Let $X(t)$ be a supOU process and π a probability measure such that $\int_{[1,\infty)} x^{\varepsilon}\pi(\mathrm{d}x) < \infty$ for some $\varepsilon > 0$. Then X has a càdlàg modification.

For symmetric α-stable jump measure λ with $\alpha \in (1, 2)$, the existence of a càdlàg modification was obtained by Basse-O'Connor 2020.

Outline

Growth rate

Theorem

Assume that $\int_{(0,1]} z\lambda(\mathrm{d}z) < \infty$ *and* $m_0(\pi) < \infty$*.*

$$\int_1^\infty \overline{\lambda}(Kf(t))\mathrm{d}t = \infty \;\Rightarrow\; \limsup_{t\to\infty} \frac{X(t)}{f(t)} \geq K \quad a.s.$$

$$\int_1^\infty \left(\overline{\lambda}(Lf(t)) + \overline{\eta}(Lf(t))\right) \mathrm{d}t < \infty \;\Rightarrow\; \limsup_{t\to\infty} \frac{X(t)}{f(t)} \leq 2L \quad a.s.$$

Heavy tails

Corollary

Assume that $\int_{(0,1]} z\lambda(\mathrm{d}z) < \infty$*,* $m_0(\pi) < \infty$*,* $\overline{\lambda} \in \mathcal{RV}$*. Then*

$$\limsup_{t\to\infty} \frac{X(t)}{f(t)} = 0 \ \ \textit{or} \ \ \ \infty, \ \ \textit{as} \ \ \int_1^\infty \overline{\lambda}(f(t))\mathrm{d}t < \infty \ \ \textit{or} \ \ = \infty.$$

In particular, if $\overline{\lambda} \in \mathcal{RV}_{-\gamma}$, then almost surely $\limsup_{t\to\infty} \frac{X(t)}{t^a} = 0$ for $a > 1/\gamma$ and $\limsup_{t\to\infty} \frac{X(t)}{t^a} = \infty$ for $a < 1/\gamma$.

Exponential tails

Corollary

Assume that $\int_{(0,1]} z\lambda(\mathrm{d}z) < \infty$, $m_0(\pi) < \infty$, *and that for some* $0 < c < c' < \infty$

$$\int_{(1,\infty)} e^{cz}\lambda(\mathrm{d}z) < \infty, \quad \int_{(1,\infty)} e^{c'z}\lambda(\mathrm{d}z) = \infty.$$

Then

$$\frac{1}{c'} \leq \limsup_{t\to\infty} \frac{X(t)}{\log t} \leq \frac{2}{c} \quad a.s.$$

Bounded support

Theorem (Grahovac & K 2025)
Assume $\int_{(0,1]} z\lambda(\mathrm{d}z) < \infty$, $\inf\{y : \overline{\lambda}(y) = 0\} < \infty$, *and* $m_0(\pi) < \infty$. *Then*

$$0 < \limsup_{t\to\infty} \frac{\log\log t}{\log t} X(t) < \infty.$$

Outline

Integrated supOU

Consider the integrated process

$$
\begin{aligned}
X^*(t) &= \int_0^t X(u)\mathrm{d}u \\
&= \iint_{(0,\infty)\times(-\infty,0]} x^{-1}(1-e^{-xt})e^{xs}\Lambda(\mathrm{d}x,\mathrm{d}s) \\
&\quad + \iint_{(0,\infty)^2} x^{-1}(1-e^{-x(t-s)})\mathbb{I}(t>s)\Lambda(\mathrm{d}x,\mathrm{d}s) \\
&=: X^*_-(t) + X^*_+(t) = \iint f_t(x,s)\Lambda(\mathrm{d}x,\mathrm{d}s)
\end{aligned}
$$

Lemma (Barndorff-Nielsen, 2001)

$X^*(t)$ is infinitely divisible

$$\mathbf{E}e^{\mathrm{i}\theta X^*(t)} = \exp\left\{\mathrm{i}A\theta + \int_{(0,\infty)} (e^{\mathrm{i}\theta y} - 1 - \mathrm{i}\theta y \mathbb{I}(y \le 1))\eta_t^*(\mathrm{d}y)\right\},$$

where

$$\eta_t^*(B) = \nu\left(\{(x,s,z) : f_t(x,s)z \in B\}\right).$$

Furthermore,

$$\overline{\eta}_t^*(r) = \int_{(0,\infty)} x^{-1}\pi(\mathrm{d}x) \int_{\frac{xr}{1-e^{-xt}}}^{\infty} \overline{\lambda}(z) \frac{1}{z - xr} \mathrm{d}z.$$

Outline

$\pi(\mathrm{d}x) = x^{\alpha}\mathrm{d}x$.
λ the jump measure
Limit theorems for X^*: Grahovac, Leonenko, Taqqu (2019, 2020)

Variance

$$\mathbf{Var}(X^*(t)) = m_2(\lambda) \int_{(0,\infty)} \frac{e^{-xt} - 1 + xt}{x^3} \pi(\mathrm{d}x),$$

thus if $m_{-2}(\pi) = \int_0^\infty x^{-2} \pi(\mathrm{d}x) < \infty$, then as $t \to \infty$

$$\mathbf{Var}(X^*(t)) \sim t m_2(\lambda) m_{-2}(\pi).$$

Finite variance - $\int z^2 \lambda(\mathrm{d}z) < \infty$, GLT19

$\pi(\mathrm{d}x) = x^\alpha \mathrm{d}x$, $\lambda((z,1)) \sim z^{-\beta}$

- If $\alpha \geq 1$
$$T^{-1/2} X^*(T\cdot) \stackrel{\text{fdd}}{\to} \text{SBM},$$
- If $\alpha \in (0,1)$ and Λ has Gaussian part
$$T^{-(1-\alpha/2)} X^*(T\cdot) \stackrel{\text{fdd}}{\to} \text{fractional BM } H = 1 - \alpha/2,$$

Finite variance - $\int z^2 \lambda(\mathrm{d}z) < \infty$, GLT19

$\pi(\mathrm{d}x) = x^{\alpha}\mathrm{d}x$, $\lambda((z,1)) \sim z^{-\beta}$

- If $\alpha \in (0,1)$, no Gaussian part, $\beta_{BG} < 1 + \alpha$

$$T^{-1/(1+\alpha)} X^*(T\cdot) \overset{\text{fdd}}{\to} L_{1+\alpha}(\cdot) \quad (1+\alpha)\text{-stable Lévy}$$

- If $\alpha \in (0,1)$, no Gaussian part, $1 + \alpha < \beta$
  ($\lambda((x,\infty)) \sim x^{-\beta}$, $x \to 0$)

$$T^{-(1-\alpha/\beta)} X^*(T\cdot) \overset{\text{fdd}}{\to} Z(\cdot) \quad \beta\text{-stable, not independent increments}$$

limit process Z appears in Puplinskaite, Surgailis (2010) - aggregation of AR(1), Barczy, Nedényi, Pap (2017-21): aggregation of Galton–Watson processes

Infinite variance, GLT20

$\pi(\mathrm{d}x) = x^\alpha \mathrm{d}x$, $\lambda((z,1)) \sim z^{-\beta}$, $\overline{\lambda}(z) \sim z^{-\gamma}$, no Gaussian component.

- If $\gamma < 1 + \alpha$

$$T^{-1/\gamma} X^*(T\cdot) \overset{\text{fdd}}{\to} L_\gamma(\cdot), \quad \gamma\text{-stable Lévy}$$

- If $\gamma > 1 + \alpha$, $\beta < 1 + \alpha$

$$T^{-1/(1+\alpha)} X^*(T\cdot) \overset{\text{fdd}}{\to} L_{1+\alpha}(\cdot) \quad (1+\alpha)\text{-stable Lévy}$$

- If $\gamma > 1 + \alpha$, $1 + \alpha < \beta$ ($\lambda((x,\infty)) \sim x^{-\beta}$, $x \to 0$)

$$T^{-(1-\alpha/\beta)} X^*(T\cdot) \overset{\text{fdd}}{\to} Z(\cdot) \quad \beta\text{-stable, not independent increments}$$

Infinite variance, GLT20

$\pi(\mathrm{d}x) = x^{\alpha}\mathrm{d}x$, $\lambda((z,1)) \sim z^{-\beta}$, $\overline{\lambda}(z) \sim z^{-\gamma}$, Gaussian component.

- If $\alpha > 1$, or $\alpha \in (0,1)$ and $\gamma < 2/(2-\alpha)$,

$$T^{-1/\gamma} X^*(T\cdot) \stackrel{\text{fdd}}{\to} L_\gamma(\cdot), \quad \gamma\text{-stable Lévy}$$

- If $\alpha \in (0,1)$ and $\gamma > 2/(2-\alpha)$

$$T^{-(1-\alpha/2)} X^*(T\cdot) \stackrel{\text{fdd}}{\to} \text{fBm}, H = 1 - \alpha/2$$

Outline

Gaussian case

Theorem (Grahovac & K 2025)

$\mu \equiv 0$, If $\int x^{-2}\pi(\mathrm{d}x) < \infty$, then

$$\limsup_{t\to\infty} \frac{|X^*(t)|}{\sqrt{2bm_{-2}(\pi)t\log\log t}} = 1 \quad a.s.$$

If π has a density p, $p(x) \sim \alpha\ell(x^{-1})x^{\alpha}$, $\alpha \in (0,1)$, then

$$\limsup_{t\to\infty} \frac{|X^*(t)|}{\widetilde{\sigma}\ell(t)^{\frac{1}{2}}t^{1-\frac{\alpha}{2}}\sqrt{2\log\log t}} = 1 \quad a.s.$$

where $\widetilde{\sigma}^2 = b\frac{\Gamma(1+\alpha)}{(2-\alpha)(1-\alpha)}$.

Consequence of LIL for Gaussian processes (Orey, 1972)

Finite variation

Theorem (Grahovac & K 2025)

Assume $\int_{|z|\leq 1} |z|\lambda(\mathrm{d}z) < \infty$. *If for some* $\gamma \in (0,2)$

$$\iint_{(0,\infty)\times\mathbb{R}} \frac{|z|^\gamma}{x^\gamma} \mathbb{I}(|z| > x)\pi(\mathrm{d}x)\lambda(\mathrm{d}z) < \infty, \qquad (*)$$

then

$$\lim_{t\to\infty} \frac{X^*(t) - \mathbb{I}(\gamma \geq 1)\mathbf{E}X^*(t)}{t^{1/\gamma}} = 0 \quad a.s.$$

If $(*)$ *holds for* $\gamma = 2$, *then*

$$\limsup_{t\to\infty} \frac{|X^*(t) - \mathbf{E}X^*(t)|}{\sqrt{2t\log\log t}} = \sqrt{\mathbf{Var}X^*(1)} \quad a.s.$$

Marcinkiewicz–Zygmund SLLN

$X, X_1, X_2, \ldots$ iid, $\mathbf{E}|X|^\gamma < \infty$, $\gamma \in (0,2)$ then

$$\frac{1}{n^{1/\gamma}}(S_n - n\mathbf{E}(X)\mathbb{I}(\gamma \geq 1)) \to 0$$

Main idea

(ξ_k, τ_k, ζ_k) the points of the Poisson random measure μ.

$$X_+^*(t) = \sum_{k:\tau_k \in [0,t)} \frac{\zeta_k}{\xi_k} \left(1 - e^{-\xi_k(t-\tau_k)}\right)$$

$$= \sum_{k:\tau_k \in [0,t)} \frac{\zeta_k}{\xi_k} - \sum_{k:\tau_k \in [0,t)} \frac{\zeta_k}{\xi_k} e^{-\xi_k(t-\tau_k)} =: I_1(t) - I_2(t)$$

$$X_-^*(t) = \sum_{k:\tau_k < 0} \frac{\zeta_k}{\xi_k} (1 - e^{-\xi_k t}) e^{\tau_k \xi_k}.$$

Main term - subordinator

Lemma

Assume

$$\iint_{(0,\infty)^2} \left(\frac{z}{x}\right)^{\gamma} \mathbb{I}(z > x)\pi(\mathrm{d}x)\lambda(\mathrm{d}z) < \infty.$$

Then for $I_1(t) = \sum_{\tau_i \in (0,t]} \zeta_i/\xi_i$

$$\lim_{t\to\infty} t^{-1/\gamma} \left(I_1(t) - \mathbb{I}(\gamma \geq 1) t m_1(\lambda) m_{-1}(\pi) \right) = 0 \quad a.s.$$

If $\gamma = 2$ *then the LIL holds, i.e.*

$$\limsup_{t\to\infty} \frac{|I_1(t) - t m_1(\lambda) m_{-1}(\pi)|}{\sqrt{2t \log\log t}} = \sqrt{\mathbf{Var}(I_1(1))}.$$

Outline

$$\beta_0 = \inf\left\{\beta \geq 0 : \int_{|z|\leq 1} |z|^\beta \lambda(\mathrm{d}z) < \infty\right\},$$

$$\eta_\infty = \sup\left\{\eta \geq 0 : \int_{|z|>1} |z|^\eta \lambda(\mathrm{d}z) < \infty\right\},$$

$$\alpha_0 = \sup\left\{\alpha \geq 0 : \int_{(0,1]} x^{-\alpha-1} \pi(\mathrm{d}x) < \infty\right\}.$$

Theorem (Grahovac & K 2025)

Assume that $\int_{|z|\leq 1} |z|\lambda(\mathrm{d}z) = \infty$*. If* $\beta \leq 1+\alpha$*, assume* $\gamma \in (0,2]$

$$\iint_{(0,\infty)\times\mathbb{R}} \frac{|z|^\gamma}{x^\gamma}\mathbb{I}(|z|>x)\pi(\mathrm{d}x)\lambda(\mathrm{d}z) < \infty,$$

Then

$$\limsup_{t\to\infty} \frac{|X^*(t) - \mathbb{I}(\gamma \geq 1)\mathbf{E}X^*(t)|}{t^{1/\gamma}\log t} \leq 1 \quad a.s.$$

Theorem (continued)

If $\beta \geq 1 + \alpha$, assume that

$$\iint_{(0,\infty)\times\mathbb{R}} \frac{|z|^\gamma}{x^\gamma} \mathbb{I}(|z| > x, |z| > 1)\pi(\mathrm{d}x)\lambda(\mathrm{d}z) < \infty$$

holds with $\gamma < 1/(1 - \alpha/\beta)$. Then

$$\lim_{t\to\infty} \frac{X^*(t) - \mathbb{I}(\gamma \geq 1)\mathbf{E}X^*(t)}{t^{1/\gamma}} = 0 \quad a.s.$$

Extension

Mixed moving average process:

$$X(t) = \int_{V \times \mathbb{R}} f(x, t-s) \Lambda(\mathrm{d}x, \mathrm{d}s)$$

Work in progress: Grahovac, K, Mihalčić (2025+)