Nearly degenerate branching processes

Péter Kevei, Kata Kubatovics

University of Szeged

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Outline

Introduction
- Varying environment
- Nearly critical processes

Conditioning – Yaglom-type results
- Results
- Functional limit theorem

Immigration
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Varying environment

$X_0 = 1$, and

$$X_n = \sum_{j=1}^{X_{n-1}} \xi_{n,j},$$

where $\{\xi_{n,j}\}_{n,j\in\mathbb{N}}$ are independent random variables, such that for each n, $\{\xi_n, \xi_{n,j}\}_{j\in\mathbb{N}}$ are identically distributed.

- 1970's: Church, Fearn, Jagers, Agresti
- 2017 Kersting, 2020 Kersting & Vatutin monograph (BPV/RE)
- 2020s: Bhattacharya & Perlman, Dolgopyat et al., Cardona-Tobón & Palau, González & Minuesa & del Puerto, ...

Varying environment – immigration

Inhomogeneous Galton–Watson process with immigration:
$Y_0 = 0$,

$$Y_n = \sum_{j=1}^{Y_{n-1}} \xi_{n,j} + \varepsilon_n$$

where $\{\xi_{n,j}, \varepsilon_n : n, j \in \mathbb{N}\}$ are independent nonnegative integer valued random variables, $\{\xi_{n,j} : j \in \mathbb{N}\}$ are iid.

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Nearly critical process

$\bar{f}_n = f_n'(1) = \mathbf{E}\xi_n$.

(C1) $\bar{f}_n < 1$, $\lim_{n\to\infty} \bar{f}_n = 1$, $\sum_{n=1}^{\infty}(1 - \bar{f}_n) = \infty$,
(more generally $\lim_{n\to\infty} \bar{f}_n = 1$, $\sum_{n=1}^{\infty}(1 - \bar{f}_n)_+ = \infty$,
$\sum_{n=1}^{\infty}(\bar{f}_n - 1)_+ < \infty$),

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(C3) $\lim_{n\to\infty} \frac{f_n'''(1)}{1-\bar{f}_n} = 0$, if $\nu > 0$.

subcritical in Kersting's (2017) characterization of BPVE

C1

$\overline{f}_n < 1$, $\lim_{n\to\infty} \overline{f}_n = 1$, $\sum_{n=1}^{\infty}(1 - \overline{f}_n) = \infty$

$$X_n = \sum_{j=1}^{X_{n-1}} \xi_{n,j},$$

$$\mathbf{E}X_n = \mathbf{E}\xi_1 \mathbf{E}\xi_2 \dots \mathbf{E}\xi_n = \prod_{i=1}^{n} \overline{f}_i \to 0,$$

so (X_n) dies out a.s.

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- conditioning on $X_n > 0$, Yaglom-type limit results
- adding immigration

INAR(1)

If the offspring distribution is Bernoulli(ρ_n): integer-valued autoregressive (INAR(1)) time series:

$$X_n = \rho_n \circ X_{n-1} + \varepsilon_n,$$

where $\rho \circ X$ is a Bernoulli thinning of X, $\circ$ is the *Steutel and van Harn operator*.

- introduced by Laci Györfi, Márton Ispány, Gyula Pap and Katalin Varga (2007)
- K (2011), weakening the Bernoulli offspring assumption
- Györfi, Ispány, K, Pap (2014): multitype setup

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Yaglom's theorem in the classical setup

Theorem (Yaglom)

If $m < 1$ *then* $\mathcal{L}(X_n | X_n > 0)$ *converges in distribution.*

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If $m = 1$ *then* $\mathcal{L}(X_n/n | X_n > 0)$ *converges to the exponential distribution.*

Yaglom-type results

Theorem (K & Kubatovics (2024))

(C1) $\overline{f}_n \to 1$, $\overline{f}_n < 1$, $\sum_n (1 - \overline{f}_n) = \infty$

(C2) $\lim_{n\to\infty} \frac{f_n''(1)}{1-\overline{f}_n} = \nu \in [0, \infty)$,

(C3) $\lim_{n\to\infty} \frac{f_n'''(1)}{1-\overline{f}_n} = 0$, *if* $\nu > 0$.

Then

$$\mathcal{L}(X_n | X_n > 0) \xrightarrow{\mathcal{D}} \text{Geom}\left(\frac{2}{2+\nu}\right) \quad \textit{as } n \to \infty,$$

Consequence: $\mathbf{P}(X_n > 0) \sim \frac{2}{2+\nu} \overline{f}_{0,n}$.

Proof – Notation

$f_n(s) = \mathbf{E}s^{\xi_n}$ g.f. in generation n.
For the composite g.f. $f_{n,n}(s) = s$, and for $j < n$

$$f_{j,n}(s) = f_{j+1} \circ \ldots \circ f_n(s),$$

and for the corresponding means $\overline{f}_{n,n} = 1$,

$$\overline{f}_{j,n} = \overline{f}_{j+1} \ldots \overline{f}_n, \quad j < n.$$

Then $\mathbf{E}s^{X_n} = f_{0,n}(s)$ and $\mathbf{E}X_n = \overline{f}_{0,n}$.

Proof – Shape function

For a g.f. f, with mean $\overline{f}$, define the *shape function* (Kersting 2017)

$$\varphi(s) = \frac{1}{1 - f(s)} - \frac{1}{\overline{f}(1-s)}, \; 0 \leq s < 1, \quad \varphi(1) = \frac{f''(1)}{2f'(1)^2}.$$

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$$\frac{1}{1 - f_{0,n}(s)} = \frac{1}{\overline{f}_1(1 - f_{1,n}(s))} + \varphi_1(f_{1,n}(s)),$$

therefore

$$\frac{1}{1 - f_{0,n}(s)} = \frac{1}{\overline{f}_{0,n}(1-s)} + \varphi_{0,n}(s),$$

where

$$\varphi_{0,n}(s) = \sum_{k=1}^{n} \frac{\varphi_k(f_{k,n}(s))}{\overline{f}_{0,k-1}}.$$

Proof – Example

Linear fractional g.f.:

$$f(s) = 1 - a\frac{1-s}{1-qs}, \quad f[k] = a(1-q)q^{k-1}, k > 0.$$

Then $\bar{f} = \frac{a}{1-q}$,

$$\frac{1}{1-f(s)} = \frac{1}{\bar{f} \cdot (1-s)} + \frac{q}{a}.$$

That is $\varphi(s) = \frac{q}{a}$.

Proof

Lemma (Kersting)

Assume $0 < \overline{f} < \infty$, $f''(1) < \infty$ *and let* $\varphi(s)$ *be the shape function of* f. *Then, for* $0 \leq s \leq 1$,

$$\frac{1}{2}\varphi(0) \leq \varphi(s) \leq 2\varphi(1).$$

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Setup

Work in progress.
Simplify:

(C1) $\overline{f}_n = 1 - \frac{1}{n}$, $n \geq 2$, $\overline{f}_1 = 1$,

(C2) $\lim_{n\to\infty} nf_n''(1) = \nu \in [0, \infty)$,

(C3) $\lim_{n\to\infty} nf_n'''(1) = 0$, if $\nu > 0$.

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$$\bar{f}_{0,n} = \prod_{j=1}^{n} \bar{f}_j = \frac{1}{2} \cdot \frac{2}{3} \cdot \ldots \cdot \frac{n-1}{n} = \frac{1}{n}.$$

Consider X_{nt}, $t > 0$, given $X_n > 0$.

Theorem (K - Kubatovics, 2024+)

Let $0 < \varepsilon \leq 1$,

$$\mathcal{L}((X_{nt})_{t \geq \varepsilon} | X_n > 0) \xrightarrow{\mathcal{D}} \mathcal{L}((Z(\log t))_{t \geq \varepsilon} | Z(0) > 0),$$

where $(Z(s))_{s \geq \log \varepsilon}$ *is a simple birth and death process with* $Z(\log \varepsilon) \sim Geom(\frac{2}{2+\nu})$, *birth rate* $\lambda = \frac{\nu}{2}$ *and death rate* $\mu = 1 + \frac{\nu}{2}$.

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$\mathrm{Geom}(\frac{2}{2+\nu})$ is the extremal quasi-stationary distribution of the birth and death process, see Collet, Martínez, San Martín, Quasi-stationary distributions (2013).

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Varying environment- immigration

$Y_0 = 0,$

$$Y_n = \sum_{j=1}^{Y_{n-1}} \xi_{n,j} + \varepsilon_n$$

$\{\xi_{n,j}, \varepsilon_n : n, j \in \mathbb{N}\}$ independent nonnegative, $\{\xi_{n,j} : j \in \mathbb{N}\}$ iid.

Bernoulli immigration

Theorem (Györfi, Ispány, Pap, Varga (2007))

Let $(Y_n)_{n\in\mathbb{N}}$ be an inhomogeneous INAR(1) process, with $\varepsilon_n \sim \mathrm{Bernoulli}(m_{n,1})$. Assume that

(i) $\overline{f}_n \to 1$, $\overline{f}_n < 1$, $\sum_n (1-\overline{f}_n) = \infty$,

(ii) $\lim_{n\to\infty} \frac{m_{n,1}}{1-\overline{f}_n} = \lambda$.

Then

$$Y_n \xrightarrow{\mathcal{D}} \mathrm{Poisson}(\lambda).$$

Theorem (K 2011)

Let (Y_n) be a Galton–Watson process with immigration, with general offspring and immigration distribution, such that the followings hold:

(i) $\overline{f}_n < 1, \overline{f}_n \to 1, \sum_{n=1}^{\infty}(1 - \overline{f}_n) = \infty$,

(ii) $\frac{f_n''(1)}{1-\overline{f}_n} \to \nu \in (0, \infty)$,

(iii) $\frac{f_n^{(s)}(1)}{1-\overline{f}_n} \to 0$, *for all* $s \geq 3$,

(iv) $\frac{m_{n,1}}{1-\overline{f}_n} \to \lambda$ *and* $\frac{m_{n,2}}{1-\overline{f}_n} \to 0$.

Then

$$Y_n \xrightarrow{\mathcal{D}} \mathrm{NB}(2\lambda/\nu, \nu/(2+\nu)).$$

Assumptions

$\bar{f}_n = f_n'(1) = \mathbf{E}\xi_n$.

(C1) $\bar{f}_n < 1$, $\lim_{n\to\infty} \bar{f}_n = 1$, $\sum_{n=1}^{\infty}(1-\bar{f}_n) = \infty$,

(C2) $\lim_{n\to\infty} \frac{f_n''(1)}{1-\bar{f}_n} = \nu \in [0,\infty)$,

(C3) $\lim_{n\to\infty} \frac{f_n'''(1)}{1-\bar{f}_n} = 0$, if $\nu > 0$.

Theorem (K - Kubatovics (2024))

Assume (C1)–(C3) and

(C4) $\lim_{n\to\infty} \frac{m_{n,k}}{k!(1-\overline{f}_n)} = \lambda_k, \quad k = 1, 2, \ldots, K$ *and* $\lambda_K = 0$*, or*

(C4') $\lim_{n\to\infty} \frac{m_{n,k}}{k!(1-\overline{f}_n)} = \lambda_k$, $k = 1, 2, \ldots$, *such that* $\limsup_{n\to\infty} \lambda_n^{1/n} \leq 1$.

Then

$$Y_n \xrightarrow{\mathcal{D}} Y \quad \text{as } n \to \infty,$$

where Y is compound-Poisson with g.f.

$$\exp\left\{ -\sum_{k=1}^{K-1} \frac{2^k \lambda_k}{\nu^k} \left(\log\left(1 + \frac{\nu}{2}(1-s)\right) + \sum_{i=1}^{k-1} (-1)^i \frac{\nu^i}{i2^i} (1-s)^i \right) \right\}.$$

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Consider Y_{nt}, $t > 0$.

Theorem

Assume (C1)–(C3) and (C4) or (C4'). For any $0 < \varepsilon \leq 1$,

$$\mathcal{L}((Y_{nt})_{t \geq \varepsilon}) \stackrel{\mathcal{D}}{\Longrightarrow} (W(\log t))_{t \geq \varepsilon},$$

where $(W(s))_{s \geq \log \varepsilon}$ *is a stationary continuous time branching process with immigration.*

Limit

$(W(t))_{t \geq \log \varepsilon}$ continuous time branching process with immigration, with α, β, $f(s) = \sum_{k=0}^{\infty} \mathbf{P}(\xi = k) s^k$, and $h(s) = \sum_{k=0}^{\infty} \mathbf{P}(\varepsilon = k) s^k$.
Then $G(s,t) = \mathbf{E}(s^{W(t)})$ satisfies the Kolmogorov forward equation (Li, Chen, Pakes, JOTP 2012),

$$\frac{\partial}{\partial t} G(s,t) = a(s) \frac{\partial}{\partial s} G(s,t) + b(s) G(s,t)$$

where $a(s) = \alpha \left(f(s) - s \right)$, $b(s) = \beta \left(h(s) - 1 \right)$.

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