

On the solution to the stochastic heat equation with Lévy noise

Péter Kevei

University of Szeged

Bern Probability Seminar

Outline

Introduction

Setup

Motivation

Fix t, x

Existence

Tail

Growth rate

Continuous sampling

Discrete sampling

Spatial growth rate

Parabolic Anderson model

Existence

Growth

Joint work with Carsten Chong (Columbia University).

Outline

Introduction

Setup

Motivation

Fix t, x

Existence

Tail

Growth rate

Continuous sampling

Discrete sampling

Spatial growth rate

Parabolic Anderson model

Existence

Growth

Heat equation - deterministic case

$$\begin{aligned} \partial_t Y(t, x) &= \Delta Y(t, x) + h(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ Y(0, \cdot) &= f \end{aligned}$$

Δ Laplace operator, h external heating/cooling

Heat equation - deterministic case

$$\begin{aligned}\partial_t Y(t, x) &= \Delta Y(t, x) + h(t, x), & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ Y(0, \cdot) &= f\end{aligned}$$

Δ Laplace operator, h external heating/cooling

Solution:

$$Y(t, x) = \int_{\mathbb{R}^d} g(t, x-y) f(y) dy + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y) h(s, y) dy ds,$$

where

$$g(t, x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}.$$

Stochastic heat equation

$$\begin{aligned} \partial_t Y(t, x) &= \Delta Y(t, x) + \xi(t, x), & (t, x) \in (0, \infty) \times \mathbb{R}, \\ Y(0, \cdot) &= f \end{aligned}$$

ξ is a space-time Lévy noise.

Consider the heat equation ($f \equiv 0$)

$$\partial_t Y(t, x) = \Delta Y(t, x) + \dot{\Lambda}(t, x),$$

where

$$\Lambda(dt, dx) = m dt dx + \int_{(1, \infty)} z \mu(dt, dx, dz) + \int_{(0, 1]} z (\mu - \nu)(dt, dx, dz),$$

μ Poisson random measure on $(0, \infty) \times \mathbb{R}^d \times (0, \infty)$, intensity $\nu(dt, dx, dz) = dt dx \lambda(dz)$. Solution:

$$Y(t, x) = \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y) \Lambda(ds, dy) = \sum_{\tau_i \leq t} g(t - \tau_i, x - \eta_i) \zeta_i.$$

$$g(t, x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}$$

Gaussian noise

ξ space-time white noise - $d = 1$

Khoshnevisan, Kim, Xiao, Conus, Foondun, Joseph, ..., 2010-

Khoshnevisan, *Analysis of Stochastic Partial Differential Equations*, 2014, AMS

Outline

Introduction

Setup

Motivation

Fix t, x

Existence

Tail

Growth rate

Continuous sampling

Discrete sampling

Spatial growth rate

Parabolic Anderson model

Existence

Growth

Moments

$$\begin{aligned} \mathbf{E}[Y(t, x)] &= \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y) dy ds \\ &= \int_0^t 1 ds = t, \quad (t, x) \in [0, \infty) \times \mathbb{R}^d. \end{aligned}$$

Moments

$$\begin{aligned} \mathbf{E}[Y(t, x)] &= \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y) dy ds \\ &= \int_0^t 1 ds = t, \quad (t, x) \in [0, \infty) \times \mathbb{R}^d. \end{aligned}$$

$$\text{Var} Y(t, x) = \begin{cases} \frac{1}{4\sqrt{2\pi}} \sqrt{t}, & \text{for } d = 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Proposition

Let t_n be a sequence increasing to infinity. Then

$$\lim_{n \rightarrow \infty} \frac{Y_0(t_n)}{t_n} = 1 \quad \text{a.s.}$$

holds, if for some $\varepsilon > 0$

$$\begin{cases} \sum_{n=1}^{\infty} t_n^{\varepsilon-9/4} < \infty, & \text{for } d = 1, \\ \sum_{n=1}^{\infty} t_n^{\varepsilon-(1+2/d)} < \infty, & \text{for } d \geq 2. \end{cases}$$

Proposition

Let t_n be a sequence increasing to infinity. Then

$$\lim_{n \rightarrow \infty} \frac{Y_0(t_n)}{t_n} = 1 \quad \text{a.s.}$$

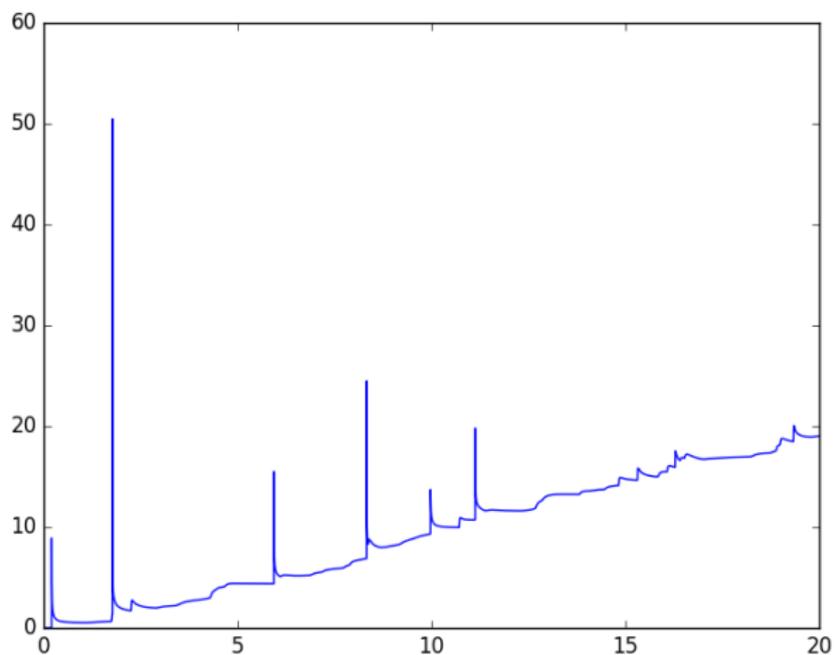
holds, if for some $\varepsilon > 0$

$$\begin{cases} \sum_{n=1}^{\infty} t_n^{\varepsilon-9/4} < \infty, & \text{for } d = 1, \\ \sum_{n=1}^{\infty} t_n^{\varepsilon-(1+2/d)} < \infty, & \text{for } d \geq 2. \end{cases}$$

In any dimension $Y_0(n)/n \rightarrow 1$ a.s.



Motivation





It turns out that

$$\limsup_{t \rightarrow \infty} \frac{Y_0(t)}{t} = \infty \text{ a.s.}$$

Outline

Introduction

Setup

Motivation

Fix t, x

Existence

Tail

Growth rate

Continuous sampling

Discrete sampling

Spatial growth rate

Parabolic Anderson model

Existence

Growth

$$Y(t, x) = \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y) \Lambda(ds, dy) = \sum_{\tau_i \leq t} g(t-\tau_i, x-\eta_i) \zeta_i.$$

$$g(t, x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}$$

$$\eta(B) = \nu\left(\{(s, y, z) : s \leq t, g(s, y)z \in B\}\right)$$

Theorem

$Y(t, x)$ exists iff

$$\int_{(1, \infty)} (\log z)^{d/2} \lambda(dz) < \infty, \quad \begin{cases} \int_{(0,1]} z^2 \lambda(dz) < \infty & d = 1, \\ \int_{(0,1]} z^2 |\log z| \lambda(dz) < \infty & d = 2, \\ \int_{(0,1]} z^{1+2/d} \lambda(dz) < \infty & d \geq 3. \end{cases}$$

η is a Lévy measure and

$$\mathbf{E}[e^{i\theta Y(t,x)}] = \exp \left\{ i\theta A + \int_{(0, \infty)} \left(e^{i\theta u} - 1 - i\theta u \mathbf{1}(u \leq 1) \right) \eta(du) \right\}$$

Application of Rajput & Rosinski 1989

Introduction
○○○○○
○○○○○

Fix t, x
○○○
●○○○

Growth rate
○○○○○○○○○
○○○
○○○○○

Parabolic Anderson model
○○○
○○○○○○○○○

Tail

Outline

Introduction

Setup

Motivation

Fix t, x

Existence

Tail

Growth rate

Continuous sampling

Discrete sampling

Spatial growth rate

Parabolic Anderson model

Existence

Growth

$$\eta(B) = \nu\left(\{(s, y, z) : s \leq t, g(s, y)z \in B\}\right), \nu = \text{Leb} \times \lambda$$

Lemma

(i) If $m_{1+2/d}(\lambda) < \infty$

$$\bar{\eta}(r) \sim r^{-1-2/d} \frac{d^{d/2}}{2\pi(d+2)^{d/2+1}} m_{1+2/d}(\lambda), \quad r \rightarrow \infty.$$

(ii) If $\bar{\lambda}(r) = \ell(r)r^{-\alpha}$ for $\alpha \in (0, 1 + \frac{2}{d}]$,

$$\bar{\eta}(r) \sim \begin{cases} \ell(r)r^{-\alpha} \frac{D^{1+2/d-\alpha}}{2d\pi\alpha^{d/2}(1+\frac{2}{d}-\alpha)} & \text{if } \alpha < 1 + \frac{2}{d}, \\ L(r)r^{-1-2/d}(2d\pi(1+\frac{2}{d})^{d/2})^{-1} & \text{if } \alpha = 1 + \frac{2}{d}, \end{cases}$$

where $L(r) = \int_1^r \ell(u)u^{-1} du$, $D = (4\pi t)^{d/2}$.

Lemma

(iii) If $\bar{\lambda}(x) = \ell(x)$, then

$$\bar{\eta}(r) \sim L_0(r) \frac{D^{1+2/d}}{4\pi\Gamma(\frac{d}{2} + 1)(1 + \frac{2}{d})},$$

where

$$L_0(r) := \int_1^\infty \ell(ry)y^{-1}(\log y)^{d/2-1} dy$$

is slowly varying and $L_0(r)/\ell(r) \rightarrow \infty$ as $r \rightarrow \infty$.

$$\bar{Y}(t) = \sup_{\tau_i \leq t} g(t - \tau_i, \eta_i) \zeta_i$$

Theorem

- (i) $\bar{\eta}$ is subexponential.
(ii) As $r \rightarrow \infty$,

$$\mathbf{P}(Y(t, x) > r) \sim \mathbf{P}(\bar{Y}(t) > r) \sim \bar{\eta}(r).$$

- (iii) For $\alpha \in [0, 1 + \frac{2}{d})$, $\bar{\eta} \in \mathcal{RV}_{-\alpha}$ iff $\bar{\lambda} \in \mathcal{RV}_{-\alpha}$.

Embrechts, Goldie, Veraverbeke ('79), Pakes ('04)

Introduction
○○○○○
○○○○○

Fix t, x
○○○
○○○

Growth rate
●○○○○○○○○○
○○○
○○○
○○○○○

Parabolic Anderson model
○○○
○○○○○○○○○

Continuous sampling

Outline

Introduction

Setup

Motivation

Fix t, x

Existence

Tail

Growth rate

Continuous sampling

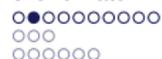
Discrete sampling

Spatial growth rate

Parabolic Anderson model

Existence

Growth



Simplification

Assume that $\lambda = \delta_1$, $\Lambda = N$ standard Poisson point process.

Theorem

Let $f : [0, \infty) \rightarrow [0, \infty)$ be a nondecreasing function. If

$$\int_1^{\infty} \frac{1}{f(t)} dt = \infty,$$

then

$$\limsup_{t \rightarrow \infty} \frac{Y_0(t)}{f(t)} = \infty \quad \text{a.s.}$$

Conversely, if the integral above is finite then

$$\limsup_{t \rightarrow \infty} \frac{Y_0(t)}{f(t)} = 0 \quad \text{a.s.}$$

Furthermore, $\liminf_{t \rightarrow \infty} Y_0(t)/t = 1$ a.s.

Remarks

$$\limsup_{t \rightarrow \infty} \frac{Y_0(t)}{t} = \limsup_{t \rightarrow \infty} \frac{Y_0(t)}{t \log(t)} = \infty, \quad \text{a.s.}$$

but

$$\limsup_{t \rightarrow \infty} \frac{Y_0(t)}{t(\log(t))^{1.1}} = 0 \quad \text{a.s.}$$

Gaussian case

Solution is locally a fractional Brownian motion with Hurst index $1/4$. (Lei, Nualart 2009)

Theorem

Suppose that $\dot{\Lambda}$ is a Gaussian space–time white noise in one spatial dimension. Then, a.s.

$$\limsup_{t \rightarrow \infty} \frac{Y_0(t)}{(2t/\pi)^{1/4} \sqrt{\log \log t}} = - \liminf_{t \rightarrow \infty} \frac{Y_0(t)}{(2t/\pi)^{1/4} \sqrt{\log \log t}} = 1.$$

In particular, the SLLN holds: $\lim_{t \rightarrow \infty} \frac{Y_0(t)}{t} = 0$ a.s.

Follows from Watanabe (1970).

Theorem

Let $f : [0, \infty) \rightarrow [0, \infty)$ be a nondecreasing function. If

$$\int_1^\infty \frac{1}{f(t)} dt = \infty,$$

then

$$\limsup_{t \rightarrow \infty} \frac{Y_0(t)}{f(t)} = \infty \quad \text{a.s.}$$

Conversely, if the integral above is finite then

$$\limsup_{t \rightarrow \infty} \frac{Y_0(t)}{f(t)} = 0 \quad \text{a.s.}$$

Proof I

For $x \in \mathbb{R}^d$, $g(t, x)$ is increasing on $[0, |x|^2/(2d)]$, decreasing on $[|x|^2/(2d), \infty)$, and its maximum is

$$g(|x|^2/(2d), x) = \left(\frac{d}{2\pi e}\right)^{d/2} |x|^{-d}.$$

Jump at x causes a peak of size $|x|^{-d}$.

Proof I

Assume first $\int_1^\infty 1/f(t)dt = \infty$. Let $K > 0$ be fix large.

$$A_n = \left\{ N \left([n, n+1] \times B([Kf(n+2)]^{-1/d}) \right) \geq 1 \right\}, \quad n \geq 0.$$

Proof I

Assume first $\int_1^\infty 1/f(t)dt = \infty$. Let $K > 0$ be fix large.

$$A_n = \left\{ N \left([n, n+1] \times B([Kf(n+2)]^{-1/d}) \right) \geq 1 \right\}, \quad n \geq 0.$$

Since N is a homogeneous Poisson process, we have

$$\mathbf{P}(A_n) = 1 - e^{-v_d/[Kf(n+2)]} \sim \frac{v_d}{Kf(n+2)},$$

Proof I

Assume first $\int_1^\infty 1/f(t)dt = \infty$. Let $K > 0$ be fix large.

$$A_n = \left\{ N \left([n, n+1] \times B([Kf(n+2)]^{-1/d}) \right) \geq 1 \right\}, \quad n \geq 0.$$

Since N is a homogeneous Poisson process, we have

$$\mathbf{P}(A_n) = 1 - e^{-v_d/[Kf(n+2)]} \sim \frac{v_d}{Kf(n+2)},$$

and thus $\sum_{n=1}^\infty \mathbf{P}(A_n) = \infty$. A_n 's are independent, by Borel–Cantelli A_n occurs infinitely many times.

Proof I

$$A_n = \left\{ N \left([n, n+1] \times B([Kf(n+2)]^{-1/d}) \right) \geq 1 \right\}, \quad n \geq 0.$$

On A_n ,

$$\sup_{t \in [n, n+2]} Y_0(t) \geq (2\pi e/d)^{-d/2} Kf(n+2),$$

Proof I

$$A_n = \left\{ N \left([n, n+1] \times B([Kf(n+2)]^{-1/d}) \right) \geq 1 \right\}, \quad n \geq 0.$$

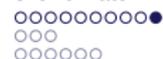
On A_n ,

$$\sup_{t \in [n, n+2]} Y_0(t) \geq (2\pi e/d)^{-d/2} Kf(n+2),$$

that is

$$\sup_{t \in [n, n+2]} \frac{Y_0(t)}{f(t)} > (2\pi e/d)^{-1/2} K.$$

Since A_n occurs infinitely often, and K is arbitrary large, the result follows.



Proof II

Now assume $\int_1^\infty 1/f(t)dt < \infty$. Let $K > 0$ be fix large.

Proof II

Now assume $\int_1^\infty 1/f(t)dt < \infty$. Let $K > 0$ be fix large. Let us fix $t \in [n, n+1]$. Introduce the events (recent close jumps)

$$A_n = \left\{ N([n, n+1] \times B([K/f(n)]^{1/d}) \geq 1 \right\}$$

$$B_n = \left\{ N([n, n+1] \times B([K/f(n)]^{1/d}, [(K \log n)/n]^{1/d})) \geq 2 \right\}$$

$$C_n = \left\{ N([n, n+1] \times B([(K \log n)/n]^{1/d}, 1) \geq 6 \log n \right\}$$

Almost surely only finitely many of these events occur.

Introduction
○○○○○
○○○○○

Fix t, x
○○○
○○○○

Growth rate
○○○○○○○○○
●○○
○○○○○

Parabolic Anderson model
○○○
○○○○○○○○○

Discrete sampling

Outline

Introduction

Setup

Motivation

Fix t, x

Existence

Tail

Growth rate

Continuous sampling

Discrete sampling

Spatial growth rate

Parabolic Anderson model

Existence

Growth

At which sequence we don't see the superlinear part?

Theorem

Let $t_n \uparrow \infty$. Then

$$\lim_{n \rightarrow \infty} \frac{Y_0(t_n)}{t_n} = 1 \quad \text{a.s.}$$

holds, if

$$\sum_{n=1}^{\infty} \frac{t_n^{-2/d} \wedge \Delta t_n}{t_n} < \infty.$$

Special case $t_n = n^p$

If $t_n = n^p$ for some $p > d/(d+2)$, we have

$$\lim_{n \rightarrow \infty} \frac{Y_0(t_n)}{t_n} = 1 \quad \text{a.s.},$$

while for $0 < p \leq d/(d+2)$, we have

$$\limsup_{n \rightarrow \infty} \frac{Y_0(t_n)}{t_n} = \infty, \quad \liminf_{n \rightarrow \infty} \frac{Y_0(t_n)}{t_n} = 1 \quad \text{a.s.}$$

Introduction
○○○○○
○○○○○

Fix t, x
○○○
○○○

Growth rate
○○○○○○○○○
○○○
○○○
●○○○○

Parabolic Anderson model
○○○
○○○○○○○○○

Spatial growth rate

Outline

Introduction

Setup

Motivation

Fix t, x

Existence

Tail

Growth rate

Continuous sampling

Discrete sampling

Spatial growth rate

Parabolic Anderson model

Existence

Growth



Fix t , and consider the behavior in x .

Theorem

Almost surely

$$\limsup_{x \rightarrow \infty} \frac{\sup_{|y| \leq x} Y(t, y)}{f(x)} = \infty \quad \text{or} \quad \limsup_{x \rightarrow \infty} \frac{\sup_{|y| \leq x} Y(t, y)}{f(x)} = 0,$$

according to whether the following integral diverges or converges:

$$\int_1^\infty r^{d-1} \bar{\tau}(f(r)) \, dr,$$

where $\tau(B) = (\text{Leb} \times \lambda) (\{(s, z) : (4\pi s)^{-d/2} z \in B, s \leq t\})$.

$$\tau(B) = (\text{Leb} \times \lambda) (\{(s, z) : (4\pi s)^{-d/2} z \in B, s \leq t\})$$

Lemma

(i) If $m_{2/d}(\lambda) < \infty$, then $\bar{\tau}(r) \sim (4\pi)^{-1} m_{2/d}(\lambda) r^{-2/d}$ as $r \rightarrow \infty$.

(ii) Assume that $\bar{\lambda}(r) = \ell(r)r^{-\alpha}$ for $\alpha \in [0, \frac{2}{d}]$, and further assume $\int_1^\infty \ell(u)u^{-1} du = \infty$ if $\alpha = \frac{2}{d}$. Then as $r \rightarrow \infty$

$$\bar{\tau}(r) \sim \begin{cases} \frac{2tD^{-\alpha}}{2-d\alpha} \ell(r)r^{-\alpha} & \text{if } \alpha < \frac{2}{d}, \\ \frac{1}{2\pi d} L(r)r^{-2/d} & \text{if } \alpha = \frac{2}{d}. \end{cases}$$

Special case

$$f(r) = r^p$$

$$\limsup_{x \rightarrow \infty} \frac{\sup_{|y| \leq x} Y(t, y)}{x^p} = \infty \quad \text{or } 0,$$

- ▶ if $m_{2/d}(\lambda) < \infty$: $p \leq d^2/2$ or $p > d^2/2$
- ▶ if $\bar{\lambda}(r) = r^{-\alpha}$, $\alpha \in (0, 2/d)$: $p \leq d/\alpha$ or $p > d/\alpha$.

Gaussian case (Khoshnevisan, Kim, Xiao 2017):

$$\limsup_{|x| \rightarrow \infty} \frac{Y(t, x)}{(\log |x|)^{1/2}} = \left(\frac{2t}{\pi}\right)^{1/4}$$

Introduction
○○○○○
○○○○○

Fix t, x
○○○
○○○

Growth rate
○○○○○○○○○
○○○
○○○
○○○○○

Parabolic Anderson model
●○○
○○○○○○○○○

Existence

Outline

Introduction

Setup

Motivation

Fix t, x

Existence

Tail

Growth rate

Continuous sampling

Discrete sampling

Spatial growth rate

Parabolic Anderson model

Existence

Growth

$$\begin{aligned}\partial_t Y(t, x) &= \Delta Y(t, x) + Y(t, x)\dot{\Lambda}(t, x), & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ Y(0, \cdot) &\equiv 1,\end{aligned}$$

Solution:

$$Y(t, x) = 1 + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y) Y(s, y) \Lambda(ds, dy)$$

Existence

$$Y(t, x) = 1 + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y) Y(s, y) \Lambda(ds, dy)$$

Berger & Chong & Lacoïn 2022: Solution exists if

$$\int_{(1, \infty)} (\log z)^{d/2} \lambda(dz) < \infty \quad \text{and}$$
$$\begin{cases} \int_{(0,1)} z^2 \lambda(dz) < \infty, & d = 1, \text{ (same as additive)} \\ \int_{(0,1)} z^2 |\log z| \lambda(dz) < \infty, & d = 2, \text{ (same as additive)} \\ \int_{(0,1)} z^{1+2/d} |\log z| \lambda(dz) < \infty, & d \geq 3, \text{ (log stronger than additive)} \end{cases}$$

Introduction
○○○○○
○○○○○

Fix t, x
○○○
○○○

Growth rate
○○○○○○○○○
○○○
○○○
○○○○○

Parabolic Anderson model
○○○
●○○○○○○○○○

Growth

Outline

Introduction

Setup

Motivation

Fix t, x

Existence

Tail

Growth rate

Continuous sampling

Discrete sampling

Spatial growth rate

Parabolic Anderson model

Existence

Growth

Gaussian case

Conus & Khoshnevisan 2012, Khoshnevisan & Kim & Xiao 2017

$$\limsup_{|x| \rightarrow \infty} \frac{Y(t, x)}{(\log |x|)^{1/2}} = \left(\frac{4t}{\pi}\right)^{\frac{1}{4}} \quad \text{additive,}$$

$$\limsup_{|x| \rightarrow \infty} \frac{\log Y(t, x)}{(\log |x|)^{2/3}} = \left(\frac{9t}{32}\right)^{\frac{1}{3}} \quad \text{multiplicative}$$

Assume $\bar{\lambda}(x) = x^{-\alpha}$, $x > 1$, $\lambda((0, 1)) = 0$.

Theorem

Suppose $\alpha > \frac{2}{d}$. Let f be nondecreasing. Then for both additive and multiplicative noise

$$\limsup_{x \rightarrow \infty} \frac{\sup_{|y| \leq x} Y(t, y)}{f(x)} = \infty \text{ or } 0,$$

according to whether the integral

$$\int_1^{\infty} x^{d-1} f(x)^{-\frac{2}{d}} dx$$

diverges or converges.

Theorem

Suppose $\alpha < \frac{2}{d}$.

► *additive noise:*

$$\limsup_{x \rightarrow \infty} \frac{\sup_{|y| \leq x} Y(t, y)}{f(x)} = \infty \text{ or } 0,$$

depending on whether the following integral diverges or converges:

$$\int_1^{\infty} x^{d-1} f(x)^{-\alpha} dx.$$

Theorem

- *multiplicative noise: there are $0 < L_* \leq L^* < \infty$ such that for all $L > L^*$,*

$$\limsup_{x \rightarrow \infty} \frac{\sup_{|y| \leq x} Y(t, y)}{x^{d/\alpha} e^{L(\log x)^{1/(1+\theta_\alpha)}}} = 0 \quad \text{a.s.},$$

while for all $L < L_$,*

$$\limsup_{x \rightarrow \infty} \frac{\sup_{|y| \leq x} Y(t, y)}{x^{d/\alpha} e^{L(\log x)^{1/(1+\theta_\alpha)}}} = \infty \quad \text{a.s.}$$

where $\theta_\alpha = 1 - \frac{d}{2}(\alpha - 1)$.

Theorem

- If $\alpha > \frac{2}{d}$, in the additive case almost surely

$$\limsup_{x \rightarrow \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{f(x)} = \infty \quad \text{or } 0,$$

according to whether

$$\int_1^\infty x^{d-1} f(x)^{-[(1+2/d) \wedge \alpha]} dx.$$

diverges or converges.

Theorem

- If $\alpha \in (\frac{2}{d}, 1 + \frac{2}{d})$ in the multiplicative case there are $0 < M_* \leq M^* < \infty$ such that for all $M > M^*$,

$$\limsup_{x \rightarrow \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{x^{d/\alpha} e^{M(\log x)^{1/(1+\theta_\alpha)}}} = 0 \quad \text{a.s.},$$

while for $M < M_*$,

$$\limsup_{x \rightarrow \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{x^{d/\alpha} e^{M(\log x)^{1/(1+\theta_\alpha)}}} = \infty \quad \text{a.s.}$$

Theorem

- If $\alpha \geq 1 + \frac{2}{d}$ in the multiplicative case there are $0 < M_* \leq M^* < \infty$ such that for $M > M^*$,

$$\limsup_{x \rightarrow \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{x^{d^2/(2+d)} e^{M(\log x)(\log \log \log x)/\log \log x}} = 0 \quad \text{a.s.}$$

while for $M < M_*$,

$$\limsup_{x \rightarrow \infty} \frac{\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)}{x^{d^2/(2+d)} e^{M(\log x)(\log \log \log x)/\log \log x}} = \infty \quad \text{a.s.}$$

Theorem

- ▶ If $\alpha < \frac{2}{d}$, then in both cases $\sup_{y \in \mathbb{Z}^d, |y| \leq x} Y(t, y)$ has the same asymptotic as $\sup_{y \in \mathbb{R}^d, |y| \leq x} Y(t, y)$

References

- ▶ Chong, C. and Kevei, P. The almost-sure asymptotic behavior of the solution to the stochastic heat equation with Lévy noise. AoP, 2020.
- ▶ Chong, C. and Kevei, P. Extremes of the stochastic heat equation with additive Lévy noise. EJP, 2022.
- ▶ Chong, C. and Kevei, P. A landscape of peaks: the intermittency islands of the stochastic heat equation with Lévy noise. AoP, to appear.
- ▶ Khoshnevisan, D. Analysis of Stochastic Partial Differential Equations, AMS, 2014