Merging asymptotic expansions for semistable random variables

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Abstract. Merging asymptotic expansions are established for distribution functions from the domain of geometric partial attraction of a semistable law. The length of the expansion depends on the exponent of the semistable law, and on the characteristic function of the underlying distribution. We obtain sufficient conditions for the quantile function in order to get real infinite asymptotic expansion. The results are generalizations of the existing theory in the stable case.
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1 Introduction

Consider a semistable distribution function G_α of exponent $\alpha \in (0,2)$ on the real line $\mathbb{R}$, with characteristic function $\phi_\alpha(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} \mathrm{d}G_\alpha(x) = \mathrm{e}^{y_\alpha(t)}$. By the well-known characterization of semistable variables, we have the Lévy form ([9], p.70) of $y_\alpha(t)$:

$$y_\alpha(t) = \mathrm{i}t\theta + \int_{-\infty}^{0} \beta_t(x)\,\mathrm{d}L_\alpha(x) + \int_{0}^{\infty} \beta_t(x)\,\mathrm{d}R_\alpha(x)\,,$$

where

$$\beta_t(x) = \mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\,,$$

$\theta \in \mathbb{R}$, and there exist functions $M_L(\cdot)$ on $(-\infty,0)$ and $M_R(\cdot)$ on $(0,\infty)$, one of which has strictly positive infimum and the other one either has strictly positive infimum or is identically zero, such that $L_\alpha(x) = M_L(x)/|x|^\alpha$, $x<0$, is left-continuous and non-decreasing on $(-\infty,0)$ and $R_\alpha(x) = -M_R(x)/x^\alpha$, $x>0$, is right-continuous and non-decreasing on $(0,\infty)$ and $M_L(c^{1/\alpha}x) = M_L(x)$ for all $x<0$ and $M_R(c^{1/\alpha}x) = M_R(x)$ for all $x>0$, with the same period $c>1$. As in the general Lévy representation formula the constant θ and the functions L_α and R_α are uniquely determined. We need a variant of this formula by Csörgő, Haeusler and Mason [5], for y_α in connection with a probabilistic representation of the underlying random variable. We note that this representation is more general, it is applicable to distributions, which can be the limit of trimmed sums, but the representation itself is not needed here. If y_α is as above, then we have

$$y_\alpha(t) = \mathrm{i}t\theta + \int_0^\infty \beta_t(\psi_1^\alpha(u))\,\mathrm{d}u + \int_0^\infty \beta_t(-\psi_2^\alpha(u))\,\mathrm{d}u\,, \tag{1}$$

with

$$\psi_j^\alpha(s) = -\frac{M_j(s)}{s^{1/\alpha}}, \quad s > 0,\ j = 1, 2, \tag{2}$$

where M_1 and M_2 are non-negative, right-continuous functions on $(0, \infty)$, either identically zero or bounded away from both zero and infinity, such that at least one of them is not identically zero, the functions $\psi_j^\alpha(\cdot)$ are non-decreasing and the multiplicative periodicity property $M_j(cs) = M_j(s)$ holds for all $s > 0$, for some constant $c > 1$, $j = 1, 2$. (The superscript α in ψ_j^α is a label, not a power exponent.) Clearly, the two descriptions are equivalent, moreover the following inverse relations hold: $\psi_1^\alpha(s) = \inf\{x < 0 : L_\alpha(x) > s\}$ and $\psi_2^\alpha(s) = \inf\{x < 0 : -R_\alpha(-x) > s\}$, $s > 0$, and, conversely, $L_\alpha(x) = \inf\{s > 0 : \psi_1^\alpha(s) \geq x\}$, $x < 0$, and $R_\alpha(x) = -\inf\{s > 0 : \psi_2^\alpha(s) \geq -x\}$, $x > 0$.

Let $W(\psi_1^\alpha, \psi_2^\alpha, 0)$ denote the random variable, that has characteristic function (1) with $\theta = 0$. To keep complete accord with [8] as far as constants go, we also introduce $V(\psi_1^\alpha, \psi_2^\alpha, 0) = W(\psi_1^\alpha, \psi_2^\alpha, 0) + \theta(\psi_1^\alpha) - \theta(\psi_2^\alpha)$, where

$$\theta(\psi) = \int_0^1 \frac{\psi(s)}{1 + \psi^2(s)}\,\mathrm{d}s - \int_1^\infty \frac{\psi^3(s)}{1 + \psi^2(s)}\,\mathrm{d}s\,,$$

and for its distribution function we put

$$G_{\psi_1^\alpha, \psi_2^\alpha, 0}(x) = \mathbf{P}\big\{V(\psi_1^\alpha, \psi_2^\alpha, 0) \leq x\big\}, \quad x \in \mathbb{R}. \tag{3}$$

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with the common distribution function $F(\cdot)$ and let $V(\psi_1^\alpha, \psi_2^\alpha, 0)$ and $G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ be as in (3). Then F is in the domain of partial attraction of $G = G_{\psi_1^\alpha, \psi_2^\alpha, 0}$, written $F \in \mathbb{D}_{\mathrm{p}}(G)$, if for some centering and norming constants $c_{k_n} \in \mathbb{R}$ and $a_{k_n} > 0$ the convergence in distribution

$$\frac{1}{a_{k_n}}\left(\sum_{j=1}^{k_n} X_j - c_{k_n}\right) \xrightarrow{\mathcal{D}} V(\psi_1^\alpha, \psi_2^\alpha, 0), \tag{4}$$

holds along a subsequence $\{k_n\}_{n=1}^\infty \subset \mathbb{N} = \{1, 2, 3, \ldots\}$, where, and throughout the paper, all asymptotic relations are meant as $n \to \infty$ unless otherwise specified. The following theorem of Kruglov [11] highlights the importance of semistability; see [12] and [8] for further references. If (4) holds for some $F(\cdot)$ along some $\{k_n\}$ for which $\lim_{n\to\infty} k_{n+1}/k_n = c$ for some $c \in (1,\infty)$, then the limiting random variable is necessarily semistable, and when the exponent $\alpha < 2$, that is the limit distribution is non-normal, then the common multiplicative period of $M_1(\cdot)$ and $M_2(\cdot)$ in (2) is the c from the latter growth condition on $\{k_n\}$. Conversely, for an arbitrary semistable distribution $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ there exists a distribution function $F(\cdot)$ for which (4) holds along some $\{k_n\} \subset \mathbb{N}$ satisfying

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = c \quad \text{for some } c \in [1,\infty)\,. \tag{5}$$

We say that a distribution $F(\cdot)$ is in the *domain of geometric partial attraction of* G *with rank* $c \geq 1$, written $F \in \mathbb{D}_{\rm gp}^{(c)}(G)$, if (4) holds along a subsequence $\{k_n\}_{n=1}^\infty \subset \mathbb{N}$ satisfying (5). Clearly, if $\mathbb{D}_{\rm gp}(G) := \bigcup_{c\geq 1} \mathbb{D}_{\rm gp}^{(c)}(G) \neq \emptyset$ then G is semistable. Define $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) = \inf\{c > 1 : M_j(cs) = M_j(s),\ s > 0,\ j = 1,2\}$, the minimal common period of the functions $M_1,\ M_2$ in $\psi_1^\alpha,\ \psi_2^\alpha$ in (2), and $\mathbf{c}(G_{0,0,\sigma}) = 1$ for any $\sigma > 0$. Megyesi [12] showed that the entire domain $\mathbb{D}_{\rm gp}(G) = \bigcup_{c\geq 1} \mathbb{D}_{\rm gp}^{(c)}(G)$ of geometric partial attraction can be produced as $\mathbb{D}_{\rm gp}(G) = \mathbb{D}_{\rm gp}^{(\mathbf{c})}(G)$. Moreover, if $\mathbf{c}(G) = 1$ then the distribution G is necessarily stable.

Megyesi [12] characterized the domain of geometric partial attraction of the semistable laws for any subsequence satisfying (5). However, for the sake of simplicity, we will assume throughout the paper that our subsequence $\{k_n\}$ is as simple, as it can, that is $k_n \equiv \lfloor c^n \rfloor$ for $c = \mathbf{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) > 1$, where $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$ is an arbitrary non-normal semistable distribution and $\lfloor y \rfloor = \max\{k \in \mathbb{Z} : k \leq y\}$ is the usual integer part. In this special case (4) holds, if and only if

$$\begin{aligned} Q_+(s) &= -s^{-1/\alpha}\, l(s)\big[M_1(s) + h_1(s)\big] \quad \text{and} \\ Q(1-s) &= \ s^{-1/\alpha}\, l(s)\big[M_2(s) + h_2(s)\big] \quad \text{for all } \ s \in (0,1), \end{aligned} \tag{6}$$

where $Q(s) = \inf\{x : F(x) \geq s\}$, $s \in (0,1)$, is the quantile function of $F(\cdot)$, and $Q_+(\cdot)$ its right-continuous version, $l(\cdot)$ is a positive right-continuous function, slowly varying at zero, the functions M_j are from (2) and the error terms $h_1(\cdot)$, $h_2(\cdot)$ are right-continuous functions such that $\lim_{s\downarrow 0} h_j(s) = 0$ if M_j is continuous, while if M_j has discontinuities then $h_j(s)$ may not go to zero but $\lim_{n\to\infty} h_j(t/k_n) = 0$ for $t \in C(M_j)$, $j = 1,2$, where $C(f)$ stands for the set of continuity points of the function f. Conversely, if the $Q(\cdot)$ of $F(\cdot)$

satisfies (6), then $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ and

$$S_{k_n} := \frac{\sum_{j=1}^{k_n} X_j - k_n \int_{1/k_n}^{1-1/k_n} Q(u)\mathrm{d}u}{k_n^{1/\alpha}\, l(1/k_n)} \xrightarrow{\mathcal{D}} V(\psi_1^\alpha, \psi_2^\alpha, 0),$$

where $X_1, X_2, \ldots$ are independent with the common distribution function F.

For what follows we need to define the scaling transform of our semistable variable. Let $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$ be semistable with exponent $\alpha \in (0,2)$. For $\lambda > 0$, let ${}_\lambda\psi(s) = \psi(s/\lambda)$ and put $\psi_j^{\alpha,\lambda}(s) = \lambda^{-1/\alpha}{}_\lambda\psi_j^\alpha(s) = -M_j(s/\lambda)s^{-1/\alpha}$, $s > 0$, where the functions M_j are from (2), $j = 1, 2$. Introduce

$$V_{\alpha,\lambda}(M_1, M_2) = V(\psi_1^{\alpha,\lambda}, \psi_2^{\alpha,\lambda}, 0) \text{ and } \mathbf{E}(\mathrm{e}^{\mathrm{i}tV_{\alpha,\lambda}(M_1,M_2)}) = \mathrm{e}^{y_{\alpha,\lambda}(t)},\ t \in \mathbb{R}, \quad (7)$$

and notice the identity $V_{\alpha,\lambda}(M_1, M_2) = \lambda^{-1/\alpha}\, V({}_\lambda\psi_1^\alpha, {}_\lambda\psi_2^\alpha, 0)$. Put $G_{\alpha,\lambda}(x) = \mathbf{P}\{V_{\alpha,\lambda}(M_1, M_2) \le x\}$.

We restate a basic result from [8] in terms of what we call *circular convergence*. For a given $c > 1$ we say that the sequence $\{u_n\}_{n=1}^\infty \subset \mathbb{R}$ converges circularly to $u \in (c^{-1}, 1]$, written $u_n \xrightarrow{\mathrm{cir}} u$, if either $u \in (c^{-1}, 1)$ and $u_n \to u$, or $u = 1$ and the sequence $\{u_n\}$ has limit points c^{-1} or 1, or both. (For $c = 1$ the notion $u_n \xrightarrow{\mathrm{cir}} 1$ simply means that $u_n \to 1$.) Let the distribution function $F \in \mathbb{D}_{\mathrm{gp}}(G)$ be such that (4) holds along a subsequence $\{\lfloor c^n \rfloor\}_{n=1}^\infty$, where $c = \mathbf{c}(G)$. Part of the surprising result in Theorem 1 in [8] is that there are as many different limiting distributions as the continuum along different subsequences. Introduce the position parameter $\gamma_n = n/c^{\lceil \log_c n \rceil} \in (c^{-1}, 1]$, which describes the position of n between two consecutive powers of c. Then Theorem 1 in [8] says that if along a subsequence $\{n_r\}_{r=1}^\infty \subset \mathbb{N}$,

$$\frac{\sum_{j=1}^{n_r} X_j - c_{n_r}}{a_{n_r}} \xrightarrow{\mathcal{D}} W \quad \text{as} \quad r \to \infty$$

for a non-degenerate random variable W, then $\gamma_{n_r} \xrightarrow{\mathrm{cir}} \kappa \in (c^{-1}, 1]$ as $r \to \infty$, and the distribution of W is necessarily that of an affine linear transformation of $V_{\alpha,\kappa}(M_1, M_2)$. Conversely, if $\gamma_{n_r} \xrightarrow{\mathrm{cir}} \kappa \in (c^{-1}, 1]$ as $r \to \infty$, then the distributional convergence above holds with $c_{n_r} = n_r \int_{n_r^{-1}}^{1-n_r^{-1}} Q(s)\,\mathrm{d}s$, $a_{n_r} = n_r^{1/\alpha}\, l(1/n_r)$ and $W = V_{\alpha,\kappa}(M_1, M_2)$. This theorem leads to the following *merging theorem*, Theorem 2 in [8]:

$$\sup_{x\in\mathbb{R}} \left|\mathbf{P}\{S_n \le x\} - G_{\alpha,\gamma_n}(x)\right| \to 0\,.$$

The prototypes of distribution in the domain of geometric partial attraction of a semistable law with characteristic exponent $\alpha \in (0, 2)$, are the generalized St. Petersburg(α, p) games, $\alpha \in (0,2)$, $p \in (0,1)$. In this games for the gambler's winning X we have $\mathbf{P}\{X = r^{k/\alpha}\} = q^{k-1}p$, where $q = 1-p$ and $r = 1/q$. In this special case fast merge rates, depending upon the tail parameter α, were obtained by Csörgő [2]. Moreover, these rates were found to be optimal, guaranteed by the 3-term expansion in [3]. Motivated by these results Pap [14] obtained a sort of complete asymptotic expansion, the length of it is regulated by α: the closer α is to 0 or 2, the longer the expansion may be taken.

These asymptotic expansions depend on the existence of the mixed derivatives $G_{\alpha,\lambda}^{(k,j)}(x) = \partial^{k+j} G_{\alpha,\lambda}^{*u}(x)/\partial x^k \partial u^j|_{u=1}$, $k, j \in \mathbb{N}_0$, of the generalized convolution powers $G_{\alpha,\lambda}^{*u}(x)$ of a semistable distribution function, which will be introduced in the next section.

Asymptotic expansions in the usual sense, when the attracting G_α is nonnormal *stable* law, are also based on the existence and regularity of these mixed derivatives of G_α. We refer to Mitalauskas and Statulevičius [13], and to the monograph of Christoph and Wolf [1].

However, in the semistable case the regularity properties of the mixed derivatives was derived by Csörgő [4], only very recently. Actually, the problem treated in this paper was addressed already there: 'what expansions in the domain of attraction of a nonnormal stable law, as summarized in Chapts. 4 and 5 of [1], have merging analogues for distributions in the domain of geometric partial attraction of a nonnormal semistable law?'

Theorem 1 in the next section states that there is a certain length of asymptotic expansion, where, as usual, the length depends on the smoothness of the characteristic function near 0. If Cramer's continuity condition does not hold, which exactly the case in the St. Petersburg games, then the length of the expansion depends also on α: as in [14], the closer α is to 0 or 2, the longer expansion may be taken. Actually, Theorem 1 is the exact analogue of Theorem 4.11 in [1]. In Theorem 2 we give sufficient condition for an infinite expansion, in terms of the quantile function. All the proofs are placed in Section 3.

2 Results

As we promised, we begin with the existence and some basic properties of the mixed derivatives. Let G_α be a semistable distribution function, with exponent $\alpha \in (0,2)$. Consider for each $u > 0$ the infinitely divisible distribution function $G_\alpha(x\,; u)$, $x \in \mathbb{R}$, that has characteristic function $\boldsymbol{g}_\alpha(t\,; u) = \mathrm{e}^{u y_\alpha(t)}$, that is,

$$\boldsymbol{g}_\alpha(t\,;u) = \mathrm{e}^{u y_\alpha(t)} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}G_\alpha(x\,;u), \quad t\in\mathbb{R}.$$

It was shown in Lemma 2 in [4] that the partial derivatives

$$G_\alpha^{(k,j)}(x\,;u) = \frac{\partial^{k+j}\,G_\alpha(x\,;u)}{\partial x^k\,\partial u^j} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx}(-\mathrm{i}t)^{k-1}\big[y_\alpha(t)\big]^j\,\mathrm{e}^{u y_\alpha(t)}\,\mathrm{d}t$$

are well defined at all $x\in\mathbb{R}$ and $u>0$ for every $j\in\mathbb{N}_0=\{0,1,\ldots\}$ and $k\in\mathbb{N}$, so that

$$G_\alpha^{(k,j)}(x) = \frac{\partial^{k+j}\,G_\alpha(x\,;u)}{\partial x^k\,\partial u^j}\bigg|_{u=1}, \quad x\in\mathbb{R}, \quad \text{for} \quad j\in\mathbb{N}_0\,, k\in\mathbb{N}, \tag{8}$$

are all meaningful. Furthermore, by Lemma 3 in [4] we have the moment property

$$\int_{-\infty}^{\infty} |x|^\beta \big|G_\alpha^{(k+1,j)}(x)\big|\,\mathrm{d}x < \infty \quad 0\le\beta<\alpha \quad \text{for all} \quad j,k\in\mathbb{N}_0\,,$$

and the limit properties

$$G_\alpha^{(k+1,j)}(\pm\infty) = \lim_{x\to\pm\infty} G_\alpha^{(k+1,j)}(x) = 0 \quad \text{for all} \quad j,k\in\mathbb{N}_0\,. \tag{9}$$

Moreover, it was proved in [7] that the above remains true for every pair $(k+1,j)\neq(0,0)$, that is $G_\alpha^{(k,j)}(\pm\infty)=0$, if $k,j\in\mathbb{N}_0$, $k+j>0$. In particular, for every $j,k\in\mathbb{N}_0$ the function $G_\alpha^{(k+1,j)}(\cdot)$ is Lebesgue integrable on $\mathbb{R}$, and hence

$$G_\alpha^{(k,j)}(x) = \int_{-\infty}^{x} G_\alpha^{(k+1,j)}(v)\,\mathrm{d}v, \quad x\in\mathbb{R}, \tag{10}$$

is a function of bounded variation on the whole $\mathbb{R}$, with Fourier$-$Stieltjes transform

$$\begin{aligned}\boldsymbol{g}_\alpha^{(k,j)}(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}G_\alpha^{(k,j)}(x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,G_\alpha^{(k+1,j)}(x)\,\mathrm{d}x \\ &= (-\mathrm{i}t)^k\big[y_\alpha(t)\big]^j\boldsymbol{g}_\alpha(t) = (-\mathrm{i}t)^k\big[y_\alpha(t)\big]^j\,\mathrm{e}^{y_\alpha(t)}, \quad t\in\mathbb{R}.\end{aligned}$$

After these general results we turn back to the semistable random variables $V(\psi_1^\alpha,\psi_2^\alpha,0)$. Fix the functions $\psi_1^\alpha,\psi_2^\alpha$ in (2), and consider $V(\psi_1^\alpha,\psi_2^\alpha,0)$, defined above (3), with distribution function $G_\alpha = G_{\psi_1^\alpha,\psi_2^\alpha,0}$ and characteristic function $e^{y_\alpha(t)}$. For our purposes it is more convenient to write the exponent $y_\alpha(t)$ as in [4]. If $\alpha < 1$

$$\begin{aligned}
y_\alpha(t) &= \int_{-\infty}^0 \beta_t(x)\,\mathrm{d}L_\alpha(x) + \int_0^\infty \beta_t(x)\,\mathrm{d}R_\alpha(x) + \mathrm{i}t[\theta(\psi_1^\alpha) - \theta(\psi_2^\alpha)] \\
&= \int_{-\infty}^0 \left(e^{\mathrm{i}tx} - 1\right)\mathrm{d}L_\alpha(x) + \int_0^\infty \left(e^{\mathrm{i}tx} - 1\right)\mathrm{d}R_\alpha(x) \\
&\quad + \mathrm{i}t\left[\theta(\psi_1^\alpha) - \theta(\psi_2^\alpha) - \int_{-\infty}^0 \frac{x}{1+x^2}\mathrm{d}L_\alpha(x) - \int_0^\infty \frac{x}{1+x^2}\mathrm{d}R_\alpha(x)\right] \\
&= z_\alpha(t) + \mathrm{i}t\eta(\psi_1^\alpha, \psi_2^\alpha),
\end{aligned}$$

where $\eta(\psi_1^\alpha, \psi_2^\alpha) = \theta(\psi_1^\alpha) - \theta(\psi_2^\alpha) - \int_{-\infty}^0 \frac{x}{1+x^2}\mathrm{d}L_\alpha(x) - \int_0^\infty \frac{x}{1+x^2}\mathrm{d}R_\alpha(x)$. Using the definition of $\theta(\psi)$, and the integraltransformation $\int_{-\infty}^0 x/(1+x^2)\mathrm{d}L_\alpha(x) = \int_0^\infty \psi_1^\alpha(s)/(1+\psi_1^\alpha(s)^2)\mathrm{d}s$, and a similar for R_α, we obtain the nice form $\eta(\psi_1^\alpha, \psi_2^\alpha) = \int_1^\infty (\psi_2^\alpha(s) - \psi_1^\alpha(s))\mathrm{d}s$. For $\alpha > 1$ similar calculation shows that

$$\begin{aligned}
y_\alpha(t) &= \int_{-\infty}^0 \left(e^{\mathrm{i}tx} - 1 - \mathrm{i}tx\right)\mathrm{d}L_\alpha(x) + \int_0^\infty \left(e^{\mathrm{i}tx} - 1 - \mathrm{i}tx\right)\mathrm{d}R_\alpha(x) \\
&\quad + \mathrm{i}t \int_0^1 (\psi_1^\alpha(s) - \psi_2^\alpha(s))\mathrm{d}s \\
&= z_\alpha(t) + \mathrm{i}t\eta(\psi_1^\alpha, \psi_2^\alpha),
\end{aligned}$$

where now $\eta(\psi_1^\alpha, \psi_2^\alpha) = \int_0^1 (\psi_1^\alpha(s) - \psi_2^\alpha(s))\mathrm{d}s$. Using the inverse relation of L_α, R_α and $\psi_1^\alpha, \psi_2^\alpha$ we obtain that for $\alpha \neq 1$

$$\begin{aligned}
z_\alpha(t) &= \int_0^\infty \left[e^{\mathrm{i}t\psi_1^\alpha(s)} - 1 - I(\alpha > 1)\,\mathrm{i}t\psi_1^\alpha(s)\right]\mathrm{d}s \\
&\quad + \int_0^\infty \left[e^{-\mathrm{i}t\psi_2^\alpha(s)} - 1 + I(\alpha > 1)\,\mathrm{i}t\psi_2^\alpha(s)\right]\mathrm{d}s\,, \qquad (11)
\end{aligned}$$

where $I(\alpha > 1) = 1$ if $\alpha > 1$, 0 elsewhere.

For $\alpha = 1$ we do not have such a nice form, but actually we cannot hope, because the exponent in the characteristic function even in the stable case behaves strangely. For unifying the notations we also introduce $z_1(t) = y_1(t)$ and $\eta(\psi_1^1, \psi_2^1) = 0$. Csörgő proved in [4] Lemma 1 that

$$|z_\alpha(t)| \leq \begin{cases} C_\alpha |t|^\alpha, & \text{if } \alpha \neq 1, \\ C_1 |t|(5 + |\log|t||), & \text{if } \alpha = 1. \end{cases} \qquad (12)$$

It is important to note that for $\alpha \neq 1$ our z_α is exactly the same as in [4], however for $\alpha = 1$ the two definitions differ, and the difference is a constant factor of $\mathrm{i}t$, which can be built in C_1.

Now we compute $z_{\alpha,\lambda}(t)$, where of course $z_{\alpha,\lambda}$ belongs to $y_{\alpha,\lambda}$, as z_α belongs to y_α. Notice that $y_{\alpha,1} = y_\alpha$, and this hold also for z and ψ. The computations above implies

$$y_{\alpha,\lambda}(t) = z_{\alpha,\lambda}(t) + \mathrm{i}t\eta(\psi_1^{\alpha,\lambda}, \psi_2^{\alpha,\lambda}).$$

On the other hand by Lemma 1 in [6] (actually the exponent of λ in c_λ was miswritten there)

$$y_{\alpha,\lambda}(t) = \lambda y_{\alpha,1}(t/\lambda^{1/\alpha}) - \mathrm{i}tc_\lambda,$$

where

$$c_\lambda = \lambda^{(\alpha-1)/\alpha} \int_1^{1/\lambda} [\psi_2^\alpha(s) - \psi_1^\alpha(s)]\mathrm{d}s\,. \tag{13}$$

Applying this scaling law we get for $\alpha \neq 1$

$$\begin{aligned} y_{\alpha,\lambda}(t) &= \lambda y_{\alpha,1}(t/\lambda^{1/\alpha}) - \mathrm{i}tc_\lambda \\ &= \lambda \left(z_{\alpha,1}(t/\lambda^{1/\alpha}) + \mathrm{i}t\lambda^{-1/\alpha}\eta(\psi_1^\alpha, \psi_2^\alpha) \right) - \mathrm{i}tc_\lambda \\ &= \lambda z_{\alpha,1}(t/\lambda^{1/\alpha}) + \mathrm{i}t \left[\lambda^{1-1/\alpha}\eta(\psi_1^\alpha, \psi_2^\alpha) - c_\lambda \right]. \end{aligned}$$

Now, separating the cases $\alpha > 1$ and $\alpha < 1$, a somewhat long but straightforward calculation shows that $\eta(\psi_1^{\alpha,\lambda}, \psi_2^{\alpha,\lambda}) = \lambda^{(\alpha-1)/\alpha}\eta(\psi_1^\alpha, \psi_2^\alpha) - c_\lambda$, which implies the important equality

$$z_{\alpha,\lambda}(t) = \lambda z_{\alpha,1}(t\lambda^{-1/\alpha}). \tag{14}$$

For $\alpha = 1$ we only have Lemma 1 in [6], that is

$$z_{1,\lambda}(t) = \lambda z_{1,1}(t/\lambda) - \mathrm{i}t \int_1^{1/\lambda} \left(\psi_2^1(s) - \psi_1^1(s) \right) \mathrm{d}s.$$

We note that the multiplicative periodicity $\psi_j^{\alpha,\lambda}(t) \equiv \psi_j^{\alpha,c\lambda}(t)$ implies that $y_{\alpha,\lambda}(t) \equiv y_{\alpha,c\lambda}(t)$ and so $z_{\alpha,\lambda}(t) \equiv z_{\alpha,c\lambda}(t)$. Recalling that $\gamma_n = n/c^{\lceil \log_c n \rceil} \in (c^{-1}, 1]$ we may write $z_{\alpha,n}(t) = z_{\alpha,\gamma_n}(t)$. For what follows, we introduce the unifying notation

$$\eta_{\alpha,n}(t) = n z_{\alpha,1}(t/n^{1/\alpha}) = \begin{cases} z_{\alpha,\gamma_n}(t), & \text{if } \alpha \neq 1, \\ z_{1,\gamma_n}(t) + \mathrm{i}tc_n, & \text{if } \alpha = 1\,, \end{cases} \tag{15}$$

where the equation for $\alpha \neq 1$ holds because of (14) and the remark above, and c_n is from (13).

After these preliminaries we start to do expansions. We follow Christoph and Wolf [1]. First we carry out a formal expansion. Let $F \in \mathbb{D}_{gp}(G_\alpha)$ be a distribution function, and $\boldsymbol{f}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}F(x)$ its characteristic function. Let $X_1, X_2, \ldots$ iid random variables, with common characteristic function F, and put $V_n = \sum_{i=1}^n X_i$. Assume that we have the formal infinite expansion of the logarithm of the characteristic function

$$\log \boldsymbol{f}(t) = z_\alpha(t) + \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{\delta_{k,j}}{k!j!} (\mathrm{i}t)^k z_\alpha^j(t)\,, \tag{16}$$

with the coefficients

$$\delta_{0,0} = \delta_{0,1} = 0\,.$$

Without loss of generality, we may assume that $\delta_{1,0} = 0$, since if it is not 0, then we can consider the random variables $X_i - \delta_{1,0}$. Then for $\boldsymbol{f}_n(t) = \mathbf{E}(\mathrm{e}^{\mathrm{i}tV_n/n^{1/\alpha}}) = \boldsymbol{f}(t/n^{1/\alpha})^n$ by (16) we obtain

$$\boldsymbol{f}_n(t) = \mathrm{e}^{\eta_{\alpha,n}(t)} \exp\left\{ \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{\delta_{k,j}}{k!j!} (\mathrm{i}t)^k \eta_{\alpha,n}(t)^j n^{-(\frac{k}{\alpha}+j-1)} \right\}.$$

Define the polynomials $P_{u,v}(\omega,\nu)$ as the coefficients in the formal expansion

$$\exp\left\{ \sum_{k=0}^\infty \sum_{j=0}^\infty \frac{\delta_{k,j}}{k!j!} \omega^k \nu^j x^k y^{j-1} \right\} = 1 + \sum_{u=0}^\infty \sum_{v=-\lfloor u/2 \rfloor}^\infty P_{u,v}(\omega,\nu) x^u y^v\,, \tag{17}$$

where $P_{0,0} \equiv 0$. Clearly, the coefficients of the polynomials $P_{u,v}$ depend only on the coefficients $\delta_{k,j}$ in (16). Introduce the notations $d_{k,j} = \delta_{k,j}/(k!j!)$, $L_{m,u} = \max\{-\lfloor u/2 \rfloor, -m, m-u\}$ and put

$$w_{m,u,v} = \sum_{\substack{k_1+\cdots+k_m=u \\ s_1+\cdots+s_m=v+m}} d_{k_1,s_1} \cdot \ldots \cdot d_{k_m,s_m}$$

if $v \geq L_{m,u}$, and 0 otherwise. Then we have

$$P_{u,v}(\omega,\nu) = \sum_{m=\max\{1,-v\}}^{u+v} \frac{w_{m,u,v}}{m!} \omega^u \nu^{v+m}\,.$$

Since for $\alpha \neq 1$ by (15) $\eta_{\alpha,n}(t) = z_{\alpha,\gamma_n}(t)$, therefore substituting $\eta_{\alpha,n}$ and $\mathrm{i}t$ into $P_{u,v}$, we obtain the formal infinite expansion of the distribution function

$$\begin{aligned}\mathbf{P}\big\{V_n/n^{1/\alpha} \leq x\big\} &= H_{\alpha,\gamma_n}(x) \\ &+ \sum_{u=0}^\infty \sum_{v=-\lfloor u/2 \rfloor}^\infty \sum_{m=\max\{1,-v\}}^{u+v} \frac{w_{m,u,v}}{m!} (-1)^u H_{\alpha,\gamma_n}^{(u,v+m)}(x)\, n^{-u/\alpha-v}\end{aligned}$$

where $H_{\alpha,\lambda}(x)$ is the inverse Fourier–Stieltjes transform of $z_{\alpha,\lambda}(t)$. Notice that $H_{\alpha,\lambda}(x) = G_{\alpha,\lambda}(x-c)$, for some constant c depending on λ. Similar equality hold for $\alpha = 1$, which is more complicated due to the occurring constant c_n in (15).

In the following we make the computations above precise. The random variable X, or its distribution function F belongs to the class $v^r(z_\alpha)$ for $r \geq \alpha$ if for some $\varepsilon > 0$

$$\log \boldsymbol{f}(t) = z_\alpha(t) + \sum_{k+\alpha j \leq r} \frac{\delta_{k,j}}{k!j!} (\mathrm{i}t)^k z_\alpha^j(t) + u(t) \tag{18}$$

for $|t| < \varepsilon$, where $u(|t|) = o(|t|^r)$, $\delta_{0,0} = \delta_{0,1} = 0$ and $\delta_{k,j} \in \mathbb{R}$. We say that $F \in v^\infty(z_\alpha)$ if $F \in v^r(z_\alpha)$ for each $r > 0$. We also note that the assumption $\delta_{1,0} = 0$ in the theorem below is not a restricting condition. We may consider the random variable $X - \delta_{1,0}$, or what is the same, we expand the distribution function of the centered random variable $V_n/n^{1/\alpha} - n^{1-1/\alpha}\delta_{1,0}$.

A characteristic function $\boldsymbol{f}$ fulfils Cramér's continuity condition if

$$\limsup_{t\to\infty} |\boldsymbol{f}(t)| < 1. \tag{19}$$

This quite usual general condition holds, if the absolutely continuous part in the Lebesgue decomposition of the distribution function F, is not constant 0. This easily follows by an application of the Riemann–Lebesgue lemma. For lattice distributed random variables the condition fails. Moreover, which is more important in this context, the condition also fails for the generalized St. Petersburg(α, p) games, for each $\alpha \in (0, 2)$ and $p \in (0, 1)$. This was pointed out by Pap [14]. Therefore in Theorem 1 below we also investigate the case when (19) does not hold.

Let denote

$$\boldsymbol{g}_{n,r,\alpha}(t) = \mathrm{e}^{\eta_{\alpha,n}(t)} \left[1 + \sum_{u=0}^{\lfloor \frac{2(r-\alpha)}{2-\alpha} \rfloor} \sum_{v=-\lfloor u/2 \rfloor}^{\lfloor \frac{r-u-\alpha}{\alpha} \rfloor} P_{u,v}(\mathrm{i}t, \eta_{\alpha,n}(t)) n^{-\frac{u+\alpha v}{\alpha}} \right],$$

where the polynomials $P_{u,v}$ are defined in (17), and put $G_{n,r,\alpha}(x)$ the inverse Fourier–Stieltjes transform of $\boldsymbol{g}_{n,r,\alpha}(t)$. The existence, bounded variation on $\mathbb{R}$, and the limit relations $G_{n,r,\alpha}(\infty) = 1$, $G_{n,r,\alpha}(-\infty) = 0$ follow from the existence of mixed derivatives of $G_{\alpha,\lambda}$, in particular from (8) and (9). It is important to note that the number of terms in $\boldsymbol{g}_{n,r,\alpha}$, and so in $G_{n,r,\alpha}$ depend on the parameter α. We will see concrete examples after Theorem 2.

Now we can formulate our main theorem, which is the analogue of Theorem 4.11 in [1]. Here $o(1) \to 0$, as $n \to \infty$.

Theorem 1 *Assume that $F \in v_r(z_\alpha)$ and $\delta_{1,0} = 0$ in* (18)*. Then*

$$\sup_{x\in\mathbb{R}} \left| \mathbf{P}\{V_n/n^{1/\alpha} \leq x\} - G_{n,r,\alpha}(x) \right| \leq n^{-\frac{r-\alpha}{\alpha}} o(1) + Cn^{-1/\alpha}.$$

Moreover, if (19) *holds, then*

$$\sup_{x\in\mathbb{R}} \left|\mathbf{P}\{V_n/n^{1/\alpha} \le x\} - G_{n,r,\alpha}(x)\right| \le n^{-\frac{r-\alpha}{\alpha}} o(1).$$

It is important to note that if Cramér's continuity condition fails, then the approximating function $G_{n,r,\alpha}(x)$ may contain superfluous terms, which are of a smaller order than the remainder term $O(n^{-1/\alpha})$. To exclude these unnecessary terms we define the function $\widetilde{\boldsymbol{g}}_{n,r,\alpha}(t)$, which is the same as $\boldsymbol{g}_{n,r,\alpha}(t)$, the only different is that in the summation we only consider those terms for which $u + \alpha v < 1$. Put $\widetilde{G}_{n,r,\alpha}(x)$ the inverse Fourier–Stieltjes transform of $\widetilde{\boldsymbol{g}}_{n,r,\alpha}(t)$. With this notation the expansions may be simplified as follows.

Corollary. *Assume that* $F \in v_r(z_\alpha)$ *and* $\delta_{1,0} = 0$ *in* (18). *Then*

$$\sup_{x\in\mathbb{R}} \left|\mathbf{P}\{V_n/n^{1/\alpha} \le x\} - \widetilde{G}_{n,r,\alpha}(x)\right| \le n^{-\frac{r-\alpha}{\alpha}} o(1) + Cn^{-1/\alpha}.$$

Theorem 1 provides asymptotic expansions if the characteristic function is smooth enough. But what does this condition mean in terms of the distribution, or in terms of the quantile function. In the followings we investigate this problem, and we give sufficient condition for $F \in v^\infty(z_\alpha)$.

Clearly, even for the merging without any rate, it is necessary that $F \in \mathbb{D}_{gp}(V(\psi_1^\alpha, \psi_2^\alpha, 0))$, which is in terms of quantile functions nothing but (6). Additionally we assume that $l(s) \equiv 1$ and $k_n = \lfloor c^n \rfloor$. We need some further assumptions in

Theorem 2 *If* $F \in \mathbb{D}_{gp}(V(\psi_1^\alpha, \psi_2^\alpha, 0))$, *in the quantile function in* (6) *the slowly varying function* $l(s) \equiv 1$, *and for* $j = 1, 2$

(a) $M_j \neq 0$ *and for some* $h_0 > 0$ *we have* $h_j(s) = 0$ *for* $s < h_0$, *or*

(b) $M_j(s) \equiv 0$ *and* $h_j(s) = O(s^{1/\alpha})$,

then $F \in v^\infty(z_\alpha)$.

During the proof of the theorem, we will see that these conditions are natural, and at least technically, seems to be necessary.

Actually, in the proof of Theorem 2 we will show that $\boldsymbol{f}(t) - 1 = z_\alpha(t) + \sum_{j=1}^\infty \beta_j (\mathrm{i}t)^j / j!$, where β_j defined later in (23). Therefore we obtain the infinite expansion

$$\begin{aligned}
\log \boldsymbol{f}(t) &= \log\Big(1 + z_\alpha(t) + \sum_{j=1}^{\infty} \frac{(\mathrm{i}t)^j}{j!}\beta_j\Big) \\
&= z_\alpha(t) + \sum_{j=1}^{\infty} \frac{(\mathrm{i}t)^j}{j!}\beta_j - \frac{1}{2}\Big[z_\alpha(t) + \sum_{j=1}^{\infty} \frac{(\mathrm{i}t)^j}{j!}\beta_j\Big]^2 \\
&\quad + \frac{1}{3}\Big[z_\alpha(t) + \sum_{j=1}^{\infty} \frac{(\mathrm{i}t)^j}{j!}\beta_j\Big]^3 - \cdots \\
&= z_\alpha(t) + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{\delta_{k,j}}{k!j!}(\mathrm{i}t)^k z_\alpha(t)^j,
\end{aligned}$$

where the coefficients $\delta_{k,j}$ are polynomials of β_j. Clearly $\delta_{0,0} = \delta_{0,1} = 0$. Some of the first values are $\delta_{1,0} = \beta_1$,

$$\begin{aligned}
\delta_{2,0} &= -\beta_1^2 + \beta_2, \quad \delta_{1,1} = -\beta_1, \quad \delta_{0,2} = -1, \\
\delta_{3,0} &= 2\beta_1^3 - 3\beta_1\beta_2 + \beta_3, \quad \delta_{2,1} = 2\beta_1^2 - \beta_2, \quad \delta_{1,2} = 2\beta_1, \quad \delta_{0,3} = 2, \\
\delta_{4,0} &= -6\beta_1^4 + 12\beta_1^2\beta_2 - 3\beta_2^2 - 4\beta_1\beta_3 + \beta_4, \\
\delta_{3,1} &= -6\beta_1^3 + 6\beta_1\beta_2 - \beta_3, \quad \delta_{2,2} = -6\beta_1^2 + 2\beta_2, \quad \delta_{1,3} = -6\beta_1, \quad \delta_{0,4} = -6.
\end{aligned}$$

In the St. Petersburg case, simple computation shows that the constants $\beta_j = \mu_j^{\alpha,p} = p/(q^{j/\alpha} - q)$ are the virtual moments in [14].

We can also compute the approximating functions $G_{n,r,\alpha}$ for fixed $\alpha \in (0, 2)$ and $r > \alpha$. Here are some examples. Recall that $H_{\alpha,\lambda}(x)$ is the inverse Fourier–Stieltjes transform of $z_{\alpha,\lambda}(t)$. For $\alpha = 1/5$

$$\begin{aligned}
G_{n,\frac{2}{5},\frac{1}{5}}(x) &= H_{\alpha,\gamma_n}(x) - \frac{H^{(0,2)}_{\alpha,\gamma_n}(x)}{2n}, \\
G_{n,\frac{3}{5},\frac{1}{5}}(x) &= H_{\alpha,\gamma_n}(x) - \frac{H^{(0,2)}_{\alpha,\gamma_n}(x)}{2n} + \frac{8H^{(0,3)}_{\alpha,\gamma_n}(x) + 3H^{(0,4)}_{\alpha,\gamma_n}(x)}{24n^2}, \\
G_{n,\frac{4}{5},\frac{1}{5}}(x) &= H_{\alpha,\gamma_n}(x) - \frac{H^{(0,2)}_{\alpha,\gamma_n}(x)}{2n} + \frac{8H^{(0,3)}_{\alpha,\gamma_n}(x) + 3H^{(0,4)}_{\alpha,\gamma_n}(x)}{24n^2} \\
&\quad - \frac{12H^{(0,4)}_{\alpha,\gamma_n}(x) + 8H^{(0,5)}_{\alpha,\gamma_n}(x) + H^{(0,6)}_{\alpha,\gamma_n}(x)}{48n^3}.
\end{aligned}$$

These functions were given by Pap [14]. For $\alpha = 3/2$

$$\begin{aligned}
G_{n,2,\frac{3}{2}}(x) &= H_{\alpha,\gamma_n}(x) + H^{(2,0)}_{\alpha,\gamma_n}(x)\frac{\beta_2 - \beta_1^2}{2n^{1/3}}, \\
G_{n,\frac{5}{2},\frac{3}{2}}(x) &= H_{\alpha,\gamma_n}(x) + H^{(2,0)}_{\alpha,\gamma_n}(x)\frac{\beta_2 - \beta_1^2}{2n^{1/3}} + \frac{8\beta_1 H^{(1,1)}_{\alpha,\gamma_n}(x) + (\beta_1^2 - \beta_2)^2 H^{(4,0)}_{\alpha,\gamma_n}(x)}{8n^{2/3}}.
\end{aligned}$$

3 Proofs

The proof of Theorem 1 is based on Esseen's classical result (Theorem 5.2 in [15]), which we record here in a special case closest to our application.

Lemma 1 *Let F be a distribution function and G be a function of bounded variation on $\mathbb{R}$ with Fourier – Stieltjes transforms $\boldsymbol{f}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}F(x)$ and $\boldsymbol{g}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\mathrm{d}G(x)$, $t \in \mathbb{R}$, such that $G(-\infty) = \lim_{x\to-\infty} G(x) = 0 = F(-\infty)$ and the derivative G' of G exists and is bounded on the whole $\mathbb{R}$. Then*

$$\sup_{x\in\mathbb{R}} |F(x) - G(x)| \le \frac{b}{2\pi} \int_{-T}^{T} \left| \frac{\boldsymbol{f}(t) - \boldsymbol{g}(t)}{t} \right| \mathrm{d}t + c_b \frac{\sup_{x\in\mathbb{R}} |G'(x)|}{T}$$

for every choice of $T > 0$ and $b > 1$, where $c_b > 0$ is a constant depending only on b, which can be given as $c_b = 4bd_b^2/\pi$, where $d_b > 0$ is the unique root d of the equation $\frac{4}{\pi}\int_0^d \frac{\sin^2 u}{u^2}\mathrm{d}u = 1 + \frac{1}{b}$.

The next result is Lemma 3.3.1 in [10].

Lemma 2 *If $\boldsymbol{f}$ is a characteristic function of a non-lattice distribution, then for every $b > 0$ there exists a sequence $\rho_n \to \infty$ such that*

$$\int_b^{\rho(n)} \left| \frac{\boldsymbol{f}^n(t)}{t} \right| \mathrm{d}t = o(\mathrm{e}^{-\sqrt{n}/2}), \quad \text{as } n \to \infty\,. \tag{20}$$

The key to the proof of the theorem is the following lemma, which is an analogue of Lemma 4.30 in [1] (p. 107).

Lemma 3 *Assume that $F \in v^r(z_\alpha)$, and in (18) $\delta_{1,0} = 0$. Then there exists $\varepsilon > 0$ such that for $|t| < \varepsilon n^{1/\alpha}$*

$$|\boldsymbol{f}_n(t) - \boldsymbol{g}_{n,r,\alpha}(t)| \le d(n) n^{-(r-\alpha)/\alpha}(|t|^{a_1} + |t|^{a_2})\mathrm{e}^{-|t|^\alpha K_\alpha/2},$$

where $d(n) = o(1)$, $K_\alpha > 0$, $\boldsymbol{f}_n(t) = \boldsymbol{f}(t/n^{1/\alpha})^n$, $a_1, a_2 > 0$.

Proof. The proof is exactly the same as in [1], therefore we only emphasize the differences.

Kruglov [11] proved that if $\mathrm{e}^{y_\alpha(t)}$ is the characteristic function of a semi-stable law with exponent $\alpha \in (0, 2)$ then $\mathfrak{Re}\, y_\alpha(t) \le -K_\alpha|t|^\alpha$, where $K_\alpha > 0$. Thus by Lemma 1 in [6] we get

$$\mathfrak{Re}\, y_{\alpha,\lambda}(t) = \mathfrak{Re}\, \lambda y_{\alpha,1}(t/\lambda^{1/\alpha}) \le -\lambda K_\alpha |t|^\alpha \lambda^{-1} = -K_\alpha |t|^\alpha,$$

that is the constant in the estimation does not depend on λ. Therefore $|\mathrm{e}^{\eta_{\alpha,n}(t)}| \le \mathrm{e}^{-K_\alpha |t|^\alpha}$.

By (18) we have for $|t| < \varepsilon n^{1/\alpha}$

$$\begin{aligned}\log \boldsymbol{f}_n(t) &= \log \boldsymbol{f}(t/n^{1/\alpha})^n \\ &= \eta_{\alpha,n}(t) + \sum_{k+\alpha j \le r} \frac{\delta_{k,j}}{k!j!} (\mathrm{i}t)^k \eta_{\alpha,n}(t)^j n^{-(\frac{k}{\alpha}+j-1)} + nu(t/n^{1/\alpha}).\end{aligned}$$

Thus

$$\boldsymbol{f}_n(t) = \mathrm{e}^{\eta_{\alpha,n}(t)} \left\{ \mathrm{e}^W + \mathrm{e}^W \left(\mathrm{e}^{nu(t/n^{1/\alpha})} - 1 \right) \right\},$$

where

$$W = \sum_{k+\alpha j \le r} \frac{\delta_{k,j}}{k!j!} (\mathrm{i}t)^k \eta_{\alpha,n}(t)^j n^{-(\frac{k}{\alpha}+j-1)}.$$

The assumption $u(t) = o(|t|^r)$, implies $nu(t/n^{1/\alpha}) = |t|^r n^{-(r-\alpha)/\alpha} o(1)$. Using the inequality $|\mathrm{e}^u - 1| \le |u| \mathrm{e}^{|u|}$ we obtain for $|t| \le n^{1/\alpha}\varepsilon$

$$\left| \mathrm{e}^{nu(t/n^{1/\alpha})} - 1 \right| \le o(1) n^{-(r-\alpha)/\alpha} |t|^r \mathrm{e}^{K_\alpha |t|^\alpha/8}.$$

For arbitrary $\alpha \in (0,2)$ by (12) we have

$$|\eta_{\alpha,n}(t)| = |n z_{\alpha,1}(t/n^{1/\alpha})| \le \begin{cases} C_\alpha |t|^\alpha, & \text{if } \alpha \ne 1, \\ C_1 |t| (5 + |\log |t/n||), & \text{if } \alpha = 1. \end{cases} \tag{21}$$

Put $D = \max\{|\delta_{k,j}| : k+\alpha j \le r\}$, $l = \min\{2-\alpha, \alpha\}$, and $m = \lfloor (r-\alpha)/l \rfloor$. Then $k+\alpha(j-1) \ge l$ for every k, j such that $k+j \ge 2$. For $\alpha \ne 1$, $|t| \le \varepsilon n^{1/\alpha}$

$$\begin{aligned}|W| &\le \sum_{k+\alpha j \le r} \frac{D}{k!j!} |t|^{k+\alpha j} C_\alpha^j n^{-(\frac{k}{\alpha}+j-1)} \\ &\le |t|^\alpha \left(\frac{|t|}{n^{1/\alpha}} \right)^l D \sum_{k+\alpha j \le r} \frac{C_\alpha^j}{k!j!} \left(\frac{|t|}{n^{1/\alpha}} \right)^{k+\alpha(j-1)-l} \\ &\le |t|^\alpha \left(\frac{|t|}{n^{1/\alpha}} \right)^l D \mathrm{e}^{1+C_\alpha} \le \frac{K_\alpha}{8} |t|^\alpha,\end{aligned}$$

where the last inequality holds if ε small enough. Using the inequality $|\mathrm{e}^u - \sum_{k=0}^m u^k/k!| \le \mathrm{e}^{|u|} |u|^{m+1}/(m+1)!$, we obtain

$$\left| \mathrm{e}^W - \sum_{k=0}^m \frac{W^k}{k!} \right| \le \frac{|W|^{m+1}}{(m+1)!} \mathrm{e}^{|W|} \le |t|^{(l+\alpha)(m+1)} n^{-\frac{r-\alpha}{\alpha}} \mathrm{e}^{K_\alpha |t|^\alpha/8} d_1(n),$$

where

$$d_1(n) = \frac{\left(D\mathrm{e}^{1+C_\alpha}\right)^{m+1}}{(m+1)!} n^{\frac{r-\alpha-l(m+1)}{\alpha}}.$$

Note that $-\gamma_1 = [r-\alpha-l(m+1)]/\alpha < [r-\alpha-l(r-\alpha)/l]/\alpha = 0$

For $\alpha = 1$ the computations are similar. We have

$$\begin{aligned}
|W| &\le \sum_{k+j\le r} \frac{D}{k!j!} |t|^{k+j} C_1^j \left(5 + |\log|t/n||\right)^j n^{-(k+j-1)} \\
&\le \frac{|t|^2}{n} D \sum_{k+j\le r} \frac{C_1^j}{k!j!} \left(\frac{|t|}{n}\right)^{k+j-2} \left(5 + |\log|t/n||\right)^j \\
&\le \frac{|t|^2}{n} D\mathrm{e}^{1+C_1} \left(5 + |\log|t/n||\right)^{\lfloor r\rfloor} \le \frac{K_1}{8}|t|.
\end{aligned}$$

Thus

$$\left| \mathrm{e}^W - \sum_{k=0}^m \frac{W^k}{k!} \right| \le \frac{|W|^{m+1}}{(m+1)!} \mathrm{e}^{|W|} \le |t|^{2(m+1)-\frac{m+2-r}{2}} n^{-r+1} \mathrm{e}^{K_1|t|/8} d_2(n),$$

where

$$d_2(n) = \frac{\left(D\mathrm{e}^{C_1+1}\right)^{m+1}}{(m+1)!} n^{(r-m-2)/2} \sup_{|t|<\varepsilon n} \left(\frac{t}{n}\right)^{\frac{m+2-r}{2}} \left(5 + |\log|t/n||\right)^{\lfloor r\rfloor(m+1)}.$$

Again, $-\gamma_2 = (r-m-2)/2 < (r-1-(r-1))/2 = 0$. For every $k > 0$ $|t|(\log|t|)^k$ is bounded near 0, thus the supremum in the definition is finite.

Finally

$$1 + \sum_{k=1}^m \frac{W^k}{k!} = 1 + \sum_{u=0}^{\lfloor \frac{2(r-\alpha)}{2-\alpha} \rfloor} \sum_{v=-\lfloor \frac{u}{2}\rfloor}^{\lfloor \frac{r-\alpha-u}{\alpha}\rfloor} P_{u,v}(\mathrm{i}t, \eta_{\alpha,n}(t)) n^{-\frac{u+\alpha v}{\alpha}} + R_n(t),$$

since if $u + \alpha v \le r - \alpha$, then the coefficient of $n^{-(u+\alpha v)/\alpha}$ in the finite sum in the left-hand side above and in the infinite sum is the same. Moreover, simple algebra shows that $u + \alpha v \le r - \alpha$ implies $u \le \lfloor 2(r-\alpha)/(2-\alpha)\rfloor$, $v \le \lfloor (r-u-\alpha)/\alpha \rfloor$. From this it follows that the terms in $R_n(t)$ are of order $n^{-\frac{u+\alpha v}{\alpha}}$, where $u + \alpha v > r - \alpha$. This is clear for $\alpha \ne 1$, and also for $\alpha = 1$, since it is easy to see that $c_n = O(\log n)$, in (13). Put $\gamma_3 = \min\{u + \alpha v - (r-\alpha)\}/(2\alpha) > 0$, where the min is taken over all pairs (u, v) with $u \ge 0$, $v \ge -\lfloor u/2 \rfloor$, for which $u + \alpha v > r - \alpha$, and for which the order $n^{-(u+\alpha v)/\alpha}$ occurs in the finite sum above. The latter property guarantees the finiteness of such pairs. After the estimation (21), the exponent of $|t|$ in each term in $R_n(t)$ is larger than r and smaller than mr. For $\alpha = 1$ there can also enter factors $(5+|\log|t/n||)^k$, where $k = 1, 2, \dots, m\lfloor r \rfloor$. As at the estimation of W these terms can be made bounded by multiplying with $|t/n|^{\gamma_3/2}$. We may choose γ_3 so small, to make $a_1 = r - \gamma_3/2 > 0$. Finally, we obtain

$$R_n(t) = d_3(n) n^{-(r-\alpha)/\alpha}(|t|^{a_1} + |t|^{a_2}),$$

where $d_3(n) = O(n^{-\gamma_3/2})$ and $a_2 = rm$.

Summing up these bounds we get

$$\begin{aligned} \left|\boldsymbol{f}_n(t) - \boldsymbol{g}_{n,r,\alpha}(t)\right| &= \left|e^{\eta_{\alpha,n}(t)}\right| \left\{ |R_n(t)| + \left|e^W \left(e^{nu(t/n^{1/\alpha})} - 1\right)\right| + \left|e^W - \sum_{k=0}^{m} \frac{W^k}{k!}\right| \right\} \\ &\le e^{-K_\alpha|t|^\alpha} \left\{ n^{-\frac{r-\alpha}{\alpha}} (|t|^{a_1} + |t|^{a_2}) o(1) + e^{\frac{K_\alpha|t|^\alpha}{4}} |t|^r n^{-\frac{r-\alpha}{\alpha}} o(1) \right. \\ &\quad \left. + e^{K_\alpha|t|^\alpha/8} |t|^{(l+\alpha)(m+1)} n^{-\frac{r-\alpha}{\alpha}} o(1) \right\} \\ &\le e^{-\frac{K_\alpha|t|^\alpha}{2}} n^{-\frac{r-\alpha}{\alpha}} (|t|^{a_1} + |t|^{a_2}) o(1), \end{aligned}$$

which proves our statement. ∎

Proof of Theorem 1. Since by (10) each term in $G'_{n,r,\alpha}$ is bounded uniformly in x, and there are finite number of terms, therefore for some $C > 0$ we have $\sup_{x\in\mathbb{R}} |G'_{n,r,\alpha}(x)| \le C$. So Esseen's lemma implies that

$$\sup_{x\in\mathbb{R}} |\mathbf{P}\{V_n/n^{1/\alpha} \le x\} - G_{n,r,\alpha}(x)| \le \frac{b}{2\pi} I_T + \frac{C}{T},$$

where

$$I_T = \int_{-T}^{T} \left| \frac{\boldsymbol{f}_n(t) - \boldsymbol{g}_{n,r,\alpha}(t)}{t} \right| \mathrm{d}t.$$

Let $T = \varepsilon n^{1/\alpha}$. From the lemma above the first statement of the theorem follows.

Now suppose that Cramér's condition also holds. In this case F is non-lattice so for any real $b > 0$ there exists a function $\rho(n) \to \infty$, such that (20) hold.

Depending on r put $T = \varepsilon n^{1/\alpha}$ for $r < 1+\alpha$, $T = \rho(n) n^{1/\alpha}$ for $r = 1+\alpha$, and $T = n^{r/\alpha}$ for $r > 1+\alpha$. Cutting up the integral we have $I_T \le I_1 + I_2 + I_3$, where

$$I_1 = \int_{|t|<\varepsilon n^{1/\alpha}} \left| \frac{\boldsymbol{f}_n(t) - \boldsymbol{g}_{n,r,\alpha}(t)}{t} \right| \mathrm{d}t,$$

$$I_2 = \int_{T \geq |t| \geq \varepsilon n^{1/\alpha}} \left| \frac{\boldsymbol{f}_n(t)}{t} \right| \mathrm{d}t, \quad \text{and } I_3 = \int_{|t| \geq \varepsilon n^{1/\alpha}} \left| \frac{\boldsymbol{g}_{n,r,\alpha}(t)}{t} \right| \mathrm{d}t.$$

As before $I_1 = o(n^{-(r-\alpha)/\alpha})$. It follows from (20) that in the case $r = 1 + \alpha$ we have $I_2 = o(n^{(r-\alpha)/\alpha})$. While, for $r > 1 + \alpha$ (19) implies $|\boldsymbol{f}(t)| \leq \mathrm{e}^{-K}$ for some $K > 0$, and $|t| \geq \varepsilon$. Thus $I_2 = O(\mathrm{e}^{-nK/2})$. Finally, for the third integral we easily obtain that $I_3 = O(\mathrm{e}^{-cn})$, for some appropriate $c > 0$. ■

Proof of Theorem 2. Clearly, we may assume that $h_0 = c^{-l}$, $l \in \mathbb{N}$. Then we have

$$\boldsymbol{f}(t) = \int_0^1 \mathrm{e}^{\mathrm{i}tQ(s)} \mathrm{d}s = \int_0^{c^{-l}} \mathrm{e}^{\mathrm{i}tQ(s)} \mathrm{d}s + \int_{c^{-l}}^{1-c^{-l}} \mathrm{e}^{\mathrm{i}tQ(s)} \mathrm{d}s + \int_0^{c^{-l}} \mathrm{e}^{\mathrm{i}tQ(1-s)} \mathrm{d}s. \quad (22)$$

Since $\log \boldsymbol{f}(t) = \log(1 + (\boldsymbol{f}(t) - 1))$, we have to expand $\boldsymbol{f}(t) - 1$. For the middle term we may write

$$\begin{aligned} \int_{c^{-l}}^{1-c^{-l}} \left(\mathrm{e}^{\mathrm{i}tQ(s)} - 1 \right) \mathrm{d}s &= \int_{c^{-l}}^{1-c^{-l}} \sum_{j=1}^{\infty} \frac{(\mathrm{i}tQ(s))^j}{j!} \mathrm{d}s \\ &= \sum_{j=1}^{\infty} \frac{(\mathrm{i}t)^j}{j!} \int_{c^{-l}}^{1-c^{-l}} Q(s)^j \mathrm{d}s. \end{aligned}$$

Since $Q(s)$ is bounded on $(c^{-l}, 1 - c^{-l})$ we could use Fubini's theorem. The first and third term in (22) can be handled similarly, so we investigate the first. First let $\alpha \neq 1$. Put $I(\alpha > 1) = 1$ if $\alpha > 1$, 0 elsewhere. Then

$$\begin{aligned} \int_0^{c^{-l}} \left[\mathrm{e}^{\mathrm{i}tQ(s)} - 1 \right] \mathrm{d}s = & \int_0^{\infty} \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha}(s)} - 1 - I(\alpha > 1)\, \mathrm{i}t\, \psi_1^{\alpha}(s) \right] \mathrm{d}s \\ & - \int_{c^{-l}}^{\infty} \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha}(s)} - 1 - I(\alpha > 1)\, \mathrm{i}t\, \psi_1^{\alpha}(s) \right] \mathrm{d}s \\ & + \int_0^{c^{-l}} \left[\mathrm{e}^{\mathrm{i}tQ(s)} - \mathrm{e}^{\mathrm{i}t\psi_1^{\alpha}(s)} \right] \mathrm{d}s + I(\alpha > 1)\, \mathrm{i}t \int_0^{c^{-l}} \psi_1^{\alpha}(s) \mathrm{d}s. \end{aligned}$$

We recall that $\psi_1^\alpha(s) = -M_1(s)/s^{1/\alpha}$, and so

$$\begin{aligned}
&\int_{c^{-l}}^{\infty} \left[\mathrm{e}^{\mathrm{i}t\psi_1^\alpha(s)} - 1 - I(\alpha>1)\mathrm{i}t\psi_1^\alpha(s)\right] \mathrm{d}s \\
&\qquad = \int_{c^{-l}}^{\infty} \sum_{j=1+I(\alpha>1)}^{\infty} \frac{(\mathrm{i}t\psi_1^\alpha(s))^j}{j!} \,\mathrm{d}s \\
&\qquad = \sum_{j=1+I(\alpha>1)}^{\infty} \frac{(-\mathrm{i}t)^j}{j!} \int_{c^{-l}}^{\infty} \left(\frac{M_1(s)}{s^{1/\alpha}}\right)^j \mathrm{d}s \\
&\qquad = \sum_{j=1+I(\alpha>1)}^{\infty} \frac{(-\mathrm{i}t)^j}{j!} \sum_{k=0}^{\infty} \int_{c^{-l+k}}^{c^{-l+k+1}} \frac{M_1(s)^j}{s^{j/\alpha}} \,\mathrm{d}s \\
&\qquad = \sum_{j=1+I(\alpha>1)}^{\infty} \frac{(-\mathrm{i}t)^j}{j!} \sum_{k=0}^{\infty} c^{l(j/\alpha-1)} c^{-k(j/\alpha-1)} \int_1^c \frac{M_1(t)^j}{t^{j/\alpha}} \,\mathrm{d}t \\
&\qquad = \sum_{j=1+I(\alpha>1)}^{\infty} \frac{(-\mathrm{i}t)^j}{j!} \frac{c^{l(j-\alpha)/\alpha}}{1-c^{-(j-\alpha)/\alpha}} \int_1^c \frac{M_1(t)^j}{t^{j/\alpha}} \,\mathrm{d}t \,,
\end{aligned}$$

where in the fourth equality we used the multiplicative periodicity of M_j.

Now we have to expand $\int_0^{c^{-l}} [\mathrm{e}^{\mathrm{i}tQ(s)} - \mathrm{e}^{\mathrm{i}t\psi_1^\alpha(s)}]\mathrm{d}s$. In case (a) this term is 0. In case (b) let $j=1$, that is $M_1 \equiv 0$. Then

$$\begin{aligned}
\int_0^{c^{-l}} \left(\mathrm{e}^{-\mathrm{i}th_1(s)/s^{1/\alpha}} - 1\right) \mathrm{d}s &= \int_0^{c^{-l}} \sum_{j=1}^{\infty} \frac{(-\mathrm{i}th_1(s))^j}{s^{j/\alpha} j!} \,\mathrm{d}s \\
&= \sum_{j=1}^{\infty} \frac{(-\mathrm{i}t)^j}{j!} \int_0^{c^{-l}} \left(\frac{h_1(s)}{s^{1/\alpha}}\right)^j \mathrm{d}s \,.
\end{aligned}$$

Summing up we have

$$
\begin{aligned}
\boldsymbol{f}(t)-1 =& \int_0^\infty \left[\mathrm{e}^{\mathrm{i}t\psi_1^\alpha(s)} - 1 - I(\alpha>1)\,\mathrm{i}t\psi_1^\alpha(s)\right] \mathrm{d}s \\
&- \sum_{j=1+I(\alpha>1)}^\infty \frac{(-\mathrm{i}t)^j}{j!} \frac{c^{l(j-\alpha)/\alpha}}{1-c^{-(j-\alpha)/\alpha}} \int_1^c \frac{M_1(t)^j}{t^{j/\alpha}}\,\mathrm{d}t \\
&+ \sum_{j=1}^\infty \frac{(-\mathrm{i}t)^j}{j!} \int_0^{c^{-l}} \left(\frac{h_1(s)}{s^{1/\alpha}}\right)^j \mathrm{d}s + I(\alpha>1)\,\mathrm{i}t \int_0^{c^{-l}} \psi_1^\alpha(s)\,\mathrm{d}s \\
&+ \sum_{j=1}^\infty \frac{(\mathrm{i}t)^j}{j!} \int_{c^{-l}}^{1-c^{-l}} Q(s)^j\,\mathrm{d}s \\
&+ \int_0^\infty \left[\mathrm{e}^{-\mathrm{i}t\psi_2^\alpha(s)} - 1 + I(\alpha>1)\,\mathrm{i}t\psi_2^\alpha(s)\right] \mathrm{d}s \\
&- \sum_{j=1+I(\alpha>1)}^\infty \frac{(\mathrm{i}t)^j}{j!} \frac{c^{l(j-\alpha)/\alpha}}{1-c^{-(j-\alpha)/\alpha}} \int_1^c \frac{M_2(t)^j}{t^{j/\alpha}}\,\mathrm{d}t \\
&+ \sum_{j=1}^\infty \frac{(\mathrm{i}t)^j}{j!} \int_0^{c^{-l}} \left(\frac{h_2(s)}{s^{1/\alpha}}\right)^j \mathrm{d}s - I(\alpha>1)\,\mathrm{i}t \int_0^{c^{-l}} \psi_2^\alpha(s)\,\mathrm{d}s \\
=& z_\alpha(t) + \sum_{j=1}^\infty \frac{(\mathrm{i}t)^j}{j!}\beta_j,
\end{aligned}
$$

where for $j \geq 2$

$$
\begin{aligned}
\beta_j =& -\frac{c^{l(j/\alpha-1)}}{1-c^{-(j-\alpha)/\alpha}} \int_1^c \frac{(-M_1(t))^j + M_2(t)^j}{t^{j/\alpha}}\,\mathrm{d}t \\
&+ \int_0^{c^{-l}} \frac{(-h_1(s))^j + h_2(s)^j}{s^{j/\alpha}} \mathrm{d}s + \int_{c^{-l}}^{1-c^{-l}} Q(s)^j\,\mathrm{d}s,
\end{aligned} \tag{23}
$$

and for $j = 1$

$$
\begin{aligned}
\beta_1 =& (1-I(\alpha>1))\frac{c^{l(1/\alpha-1)}}{1-c^{-(1-\alpha)/\alpha}} \int_1^c \frac{M_1(t)-M_2(t)}{t^{1/\alpha}}\,\mathrm{d}t + \int_{c^{-l}}^{1-c^{-l}} Q(s)\,\mathrm{d}s \\
&+ I(\alpha>1)\mathrm{i}t \int_0^{c^{-l}} [\psi_1^\alpha(s) - \psi_2^\alpha(s)]\,\mathrm{d}s + \int_0^{c^{-l}} \frac{h_2(s)-h_2(s)}{s^{1/\alpha}}\,\mathrm{d}s.
\end{aligned}
$$

It is clear that for some constant C large enough $|\beta_j| \leq C^j$, that is the infinite series converges absolutely. This immediately implies that $F \in v^r(z_\alpha)$ for each $r > \alpha$, that is $F \in v^\infty(z_\alpha)$. ■

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