Merging of linear combinations to semistable laws

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We prove merge theorems along the entire sequence of natural numbers for the distribution functions of suitably centered and normed linear combinations of independent and identically distributed random variables from the domain of geometric partial attraction of any non-normal semistable law. Surprisingly, for some sequences of linear combinations, not too far from those with equal weights, the merge theorems reduce to ordinary asymptotic distributions with semistable limits. The proofs require to work out general conditions of merge in terms of characteristic functions.

KEY WORDS: semistable laws, domains of geometric partial attraction, merge theorems, pooling strategies, linear combinations

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1. INTRODUCTION

Let Y be an infinitely divisible real random variable with characteristic function $\phi(t) = \boldsymbol{E}(\mathrm{e}^{\mathrm{i}tY})$ in its Lévy form (Ref. 10, p. 70), given for each $t \in \mathbb{R}$ by

$$\phi(t) = \exp\left\{\mathrm{i}t\theta - \frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0} \beta_t(x)\,\mathrm{d}L(x) + \int_0^{\infty} \beta_t(x)\,\mathrm{d}R(x)\right\},$$

where

$$\beta_t(x) = \mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}$$

and where the constants $\theta \in \mathbb{R}$ and $\sigma \geq 0$ and the functions $L(\cdot)$ and $R(\cdot)$ are uniquely determined: $L(\cdot)$ is left-continuous and non-decreasing on $(-\infty, 0)$ with $L(-\infty) = 0$ and $R(\cdot)$ is right-continuous and non-decreasing on $(0, \infty)$ with $R(\infty) = 0$, such that $\int_{-\varepsilon}^{0} x^2 \mathrm{d}L(x) + \int_0^{\varepsilon} x^2 \mathrm{d}R(x) < \infty$ for every $\varepsilon > 0$. We need a variant of this formula for $\phi(\cdot)$ in connection with a probabilistic representation of Y in Ref. 4; the representation itself is not needed here. Let Ψ be the class of all non-positive, non-decreasing, right-continuous functions $\psi(\cdot)$, defined on $(0, \infty)$, such that $\int_{\varepsilon}^{\infty} \psi^2(s)\mathrm{d}s < \infty$ for each $\varepsilon > 0$. Then there is a one-to-one correspondence between the pairs of Lévy functions $L(\cdot)$ and $R(\cdot)$ and the pairs of functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ taken from Ψ if we put $\psi_1(s) = \inf\{x < 0 : L(x) > s\}$ and $\psi_2(s) =$

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$\inf\{x < 0 : -R(-x) > s\}$, $s > 0$, and, conversely, $L(x) = \inf\{s > 0 : \psi_1(s) \geq x\}$, $x < 0$, and $R(x) = -\inf\{s > 0 : \psi_2(s) \geq -x\}$, $x > 0$. Let $W(\psi_1, \psi_2, \sigma)$ be an infinitely divisible random variable with characteristic function

$$\begin{aligned}\boldsymbol{E}\Big(\mathrm{e}^{\mathrm{i}tW(\psi_1,\psi_2,\sigma)}\Big) &= \exp\Big\{-\frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0} \beta_t(x)\,\mathrm{d}L(x) + \int_0^\infty \beta_t(x)\,\mathrm{d}R(x)\Big\} \\ &= \exp\Big\{-\frac{\sigma^2}{2}t^2 + \int_0^\infty \beta_t(\psi_1(u))\,\mathrm{d}u + \int_0^\infty \beta_t(-\psi_2(u))\,\mathrm{d}u\Big\},\end{aligned} \tag{1}$$

where the second equality follows by Theorem 3 in Ref. 4. The uniqueness of $\sigma, L(\cdot), R(\cdot)$ and the one-to-one correspondence immediately implies the uniqueness of the triple $\sigma, \psi_1(\cdot), \psi_2(\cdot)$. A concrete version of $W(\psi_1, \psi_2, \sigma)$ is given in Ref. 6 and, to keep complete accord with Ref. 6 as far as constants go, we also introduce $V(\psi_1, \psi_2, \sigma) = W(\psi_1, \psi_2, \sigma) + \theta(\psi_1) - \theta(\psi_2)$, where

$$\theta(\psi) = \int_0^1 \frac{\psi(s)}{1+\psi^2(s)}\,\mathrm{d}s - \int_1^\infty \frac{\psi^3(s)}{1+\psi^2(s)}\,\mathrm{d}s, \quad \psi \in \Psi,$$

and for its distribution function we put

$$G_{\psi_1,\psi_2,\sigma}(x) = \boldsymbol{P}\big\{V(\psi_1,\psi_2,\sigma) \leq x\big\}, \quad x \in \mathbb{R}. \tag{2}$$

Referring to Refs. 13, 11, 14 and 6 for background, we describe semistable laws in the present framework as follows: an infinitely divisible law $G_{\psi_1,\psi_2,\sigma}$ is semistable if and only if either $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$ for some $\sigma > 0$, the normal distribution as a semistable distribution of exponent 2, or $(\psi_1, \psi_2, \sigma) = (\psi_1^\alpha, \psi_2^\alpha, 0)$, where

$$\psi_j^\alpha(s) = -\frac{M_j(s)}{s^{1/\alpha}}, \quad s > 0,\ j = 1, 2, \tag{3}$$

for some $\alpha \in (0, 2)$, defining a semistable law of exponent α, where $M_1(\cdot)$ and $M_2(\cdot)$ are non-negative, right-continuous functions on $(0, \infty)$, either identically zero or bounded away from both zero and infinity, such that at least one of them is not identically zero, the functions $\psi_j^\alpha(\cdot)$ are non-decreasing and the multiplicative periodicity property $M_j(cs) = M_j(s)$ holds for all $s > 0$, for some constant $c > 1$, $j = 1, 2$. (The superscript α in ψ_j^α is a label, not a power exponent.) For the Lévy form this means that there exist non-negative bounded functions $M_L(\cdot)$ on $(-\infty, 0)$ and $M_R(\cdot)$ on $(0, \infty)$, one of which has strictly positive infimum and the other one either has strictly positive infimum or is identically zero, such that $L(x) = M_L(x)/|x|^\alpha$, $x < 0$, is left-continuous and non-decreasing on $(-\infty, 0)$ and $R(x) = -M_R(x)/x^\alpha$, $x > 0$, is right-continuous and non-decreasing on $(0, \infty)$ and $M_L(c^{1/\alpha}x) = M_L(x)$ for all $x > 0$ and $M_R(c^{1/\alpha}x) = M_R(x)$ for all $x < 0$, with the same period $c > 1$. Clearly, the two descriptions are equivalent.

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with the common distribution function $F(\cdot)$ and let $V(\psi_1, \psi_2, \sigma)$ and $G_{\psi_1,\psi_2,\sigma}$ be as in (2). Then F is in the domain of partial attraction of $G = G_{\psi_1,\psi_2,\sigma}$, written $F \in \mathbb{D}_{\mathrm{p}}(G)$, if for some centering and norming constants $c_{k_n} \in \mathbb{R}$ and $a_{k_n} > 0$ the convergence in distribution

$$\frac{1}{a_{k_n}} \left(\sum_{j=1}^{k_n} X_j - c_{k_n} \right) \xrightarrow{\mathcal{D}} V(\psi_1, \psi_2, \sigma), \tag{4}$$

holds along a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N} = \{1, 2, 3, \ldots\}$, where, and throughout the paper, all asymptotic relations are meant as $n \to \infty$ unless otherwise specified. The following theorem of Kruglov(13) highlights the importance of semistability; see Refs. 14 and 6 for further references. If (4) holds for some $F(\cdot)$ along some $\{k_n\}$ for which $\lim_{n\to\infty} k_{n+1}/k_n = c$ for some $c \in (1, \infty)$, then $G_{\psi_1,\psi_2,\sigma}$ is necessarily semistable and, when the exponent $\alpha < 2$, the common multiplicative period of $M_1(\cdot)$ and $M_2(\cdot)$ in (3) is the c from the latter growth condition on $\{k_n\}$. Conversely, for an arbitrary semistable distribution $G_{\psi_1,\psi_2,\sigma}$ there exists a distribution function $F(\cdot)$ for which (4) holds along some $\{k_n\} \subset \mathbb{N}$ satisfying

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = c \qquad \text{for some } c \in [1, \infty). \tag{5}$$

We say that a distribution $F(\cdot)$ is in the *domain of geometric partial attraction of G with rank* $c \geq 1$, written $F \in \mathbb{D}_{\mathrm{gp}}^{(c)}(G)$, if (4) holds along a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ satisfying (5). Clearly, if $\mathbb{D}_{\mathrm{gp}}(G) := \bigcup_{c\geq 1} \mathbb{D}_{\mathrm{gp}}^{(c)}(G) \neq \emptyset$ then G is semistable. Define $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) = \inf\{c > 1 \colon M_j(cs) = M_j(s),\ s > 0,\ j = 1, 2\}$, the minimal common period of the functions $M_1,\ M_2$ in $\psi_1^\alpha,\ \psi_2^\alpha$ in (3), and $\mathbf{c}(G_{0,0,\sigma}) = 1$ for any $\sigma > 0$. Megyesi(14) showed that the entire domain $\mathbb{D}_{\mathrm{gp}}(G) = \bigcup_{c\geq 1} \mathbb{D}_{\mathrm{gp}}^{(c)}(G)$ of geometric partial attraction can be produced as $\mathbb{D}_{\mathrm{gp}}(G) = \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c})}(G)$. Moreover, if $\mathbf{c}(G) = 1$ then the distribution G is necessarily stable.

The following characterization, that refines the one in Ref. 11, of $\mathbb{D}_{\mathrm{gp}}(G)$ is also taken from Ref. 14. Fix a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ satisfying (5). If $c = 1$ then let $\gamma_x \equiv 1$, $x \geq 1$. If $c > 1$, then there exists an x_0 large enough such that for each $x > x_0$ there is a unique index $n^*(x)$ for which $k_{n^*(x)-1} < x \leq k_{n^*(x)}$. Then let $\gamma_x = x/k_{n^*(x)}$, for $x \in (x_0, \infty)$ and $\gamma_x = 1$ otherwise. We see by (5) that for any $\varepsilon > 0$ the inequality $c^{-1} - \varepsilon \leq \gamma_x \leq 1$ holds for all x large enough. We emphasize that γ_x depends on the subsequence $\{k_n\}_{n=1}^{\infty}$. For $s \in (0, 1)$ let $Q(s) = \inf\{x : F(x) \geq s\}$ be the quantile function of $F(\cdot)$, and let $Q_+(\cdot)$ denote its right-continuous version. Then (4) holds along the previously fixed subsequence

$\{k_n\}_{n=1}^{\infty}$ for an arbitrary non-normal semistable distribution $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$ if and only if

$$\begin{aligned} Q_+(s) &= -s^{-1/\alpha}\, l(s)\big[M_1(1/\gamma_{1/s}) + h_1(s)\big] \quad \text{and} \\ Q(1-s) &= s^{-1/\alpha}\, l(s)\big[M_2(1/\gamma_{1/s}) + h_2(s)\big] \quad \text{for all } \ s \in (0,1), \end{aligned} \tag{6}$$

where $l(\cdot)$ is a positive right-continuous function, slowly varying at zero, and the error terms $h_1(\cdot)$, $h_2(\cdot)$ are right-continuous functions such that $\lim_{s\downarrow 0} h_j(s) = 0$ if M_j is continuous, while if M_j has discontinuities then $h_j(s)$ may not go to zero but $\lim_{n\to\infty} h_j(t/k_n) = 0$ for $t \in C(M_j)$, $j = 1, 2$, where $C(f)$ stands for the set of continuity points of the function f. (The slightly different form of the quantile function here and in Ref. 14, p. 412, and Ref. 6 is due to the inverse relation between the two γ functions: instead of the $\gamma(\cdot)$ in Refs. 14 and 6, here we use $\gamma(s) = 1/\gamma_{1/s}$.) Conversely, if the $Q(\cdot)$ of $F(\cdot)$ satisfies (6), then $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ and

$$\frac{\sum_{j=1}^{k_n} X_j - k_n \int_{1/k_n}^{1-1/k_n} Q(u)\mathrm{d}u}{k_n^{1/\alpha}\, l(1/k_n)} \xrightarrow{\mathcal{D}} V(\psi_1^\alpha, \psi_2^\alpha, 0),$$

where $X_1, X_2, \ldots$ are independent with the common distribution function F.

The form (6) can be simplified for the simplest possible subsequence when (4) holds for $k_n \equiv \lfloor \mathbf{c}^n \rfloor$ for $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) > 1$. Then, as shown in Ref. 14,

$$\begin{aligned} Q_+(s) &= -\, s^{-1/\alpha}\, l(s)\big[M_1(s) + h_1(s)\big] \quad \text{and} \\ Q(1-s) &= s^{-1/\alpha}\, l(s)\big[M_2(s) + h_2(s)\big] \quad \text{for all } \ s \in (0,1), \end{aligned} \tag{7}$$

so we can just forget about the strange argument $1/\gamma_{1/s} = s\lfloor \mathbf{c}^{\lceil \log_{\mathbf{c}} \lceil 1/s \rceil \rceil} \rfloor$. Here $\lfloor y \rfloor = \max\{m \in \mathbb{Z} \colon m \le y\}$ and $\lceil y \rceil = \min\{m \in \mathbb{Z} \colon m \ge y\}$ denote the integer part and the ceiling of $y \in \mathbb{R}$ and $\log_{\mathbf{c}}$ stands for the logarithm to the base $\mathbf{c}$.

To introduce the problems in this paper, let $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ be a fixed distribution function, where $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ is an arbitrary non-normal semistable distribution with characteristic exponent $\alpha \in (0, 2)$. Let $X_1, X_2, \ldots$ be independent random variables with the common distribution function $F(\cdot)$. Then $X_1, X_2, \ldots, X_n$ may be viewed for each $n \in \mathbb{N}$ as the gains in ducats (losses when negative) of n gamblers Paul_1, Paul_2, ..., Paul_n, each playing one trial of the same game of chance. Our Pauls may not trust their own luck and, before they play, they may agree to use a *pooling strategy* $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \ldots, p_{n,n})$, where the components are non-negative and add to unity. Using this strategy, Paul_1 receives $p_{1,n}X_1 + p_{2,n}X_2 + \cdots + p_{n,n}X_n$ ducats, Paul_2 receives $p_{n,n}X_1 + p_{1,n}X_2 + \cdots + p_{n-1,n}X_n$ ducats, ..., and Paul_n receives $p_{2,n}X_1 + p_{3,n}X_2 + \cdots + p_{1,n}X_n$ ducats. Then all the individual winnings are pooled and this rotating system is fair to every Paul since their pooled winnings are equally distributed. The prototypes of such games are the generalized St. Petersburg(α, p) games, in which for a single gain X we have $\boldsymbol{P}\{X = r^{k/\alpha}\} = q^{k-1}p$, $k \in \mathbb{N}$, for $\alpha \in (0,2)$, $p \in (0,1)$, $q = 1-p$ and $r = 1/q$. The distribution of X is also the prototype of one in the domain of geometric partial attraction of a semistable law with characteristic exponent α; this is shown in Ref. 14 directly by (6). The motivating paradoxical result is that in St. Petersburg($1, p$) games there are strategies $\boldsymbol{p}_n$ which are better than the individualistic strategies, that is, each Paul expects more ducats from the pool than by holding on to their own personal winnings even though their total gain is the same $X_1 + \cdots + X_n$. This was proved for the classical case $p = 1/2$ in Ref. 8, and later in Ref. 12 in general. For $n \to \infty$, the asymptotic behavior of pooled winning of Paul$_1$ was investigated in Refs. 8 and 7 for $\alpha = 1$, and in Ref. 5 for arbitrary $\alpha \in (0,2)$ and $p \in (0,1)$.

Returning now to the general situation when $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$, our first main interest in this paper is the asymptotic distribution of the random variable

$$S_{\alpha, \boldsymbol{p}_n} = \sum_{j=1}^{n} \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} X_j - \sum_{j=1}^{n} \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} \int_{p_{j,n}}^{1-p_{j,n}} Q(s)\, \mathrm{d}s, \tag{8}$$

where the slowly varying function $l(\cdot)$ is from the representation (6) of the quantile function Q corresponding to F. We consider a sequence of strategies $\{\boldsymbol{p}_n\}$ that satisfies the asymptotic negligibility condition $\overline{p}_n = \max\{p_{j,n} : j = 1, 2, \ldots, n\} \to 0$.

The main result in this paper is Theorem 2.1 below, a merge theorem for $S_{\alpha, \boldsymbol{p}_n}$ in (8). The phenomenon of *merge* takes place when neither of two sequences of distributions converges weakly, but the Lévy or supremum distance between the n-th terms goes to zero as $n \to \infty$ along the entire sequence $\mathbb{N}$.

These linear combinations $S_{\alpha, \boldsymbol{p}_n}$ belong to a real pooling strategy only when $\alpha = 1$ and the slowly varying function $l(\cdot) \equiv 1$ in (6). The equivalent Theorem 2.2 contains a satisfactory version after a simple transformation. A surprising consequence is that for some sequences of strategies $\{\boldsymbol{p}_n\}$ ordinary asymptotic distributions of $S_{\alpha, \boldsymbol{p}_n}$ exist as $n \to \infty$ along the entire $\mathbb{N}$. In Section 3 we investigate merge on $\mathbb{R}$ in general and obtain necessary and sufficient Fourier–analytic conditions under weak assumptions. All the proofs are placed in Section 4.

2. MERGING SEMISTABLE APPROXIMATIONS OF LINEAR COMBINATIONS

Let $G = G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ be semistable with exponent $\alpha \in (0,2)$ as before. For $\psi \in \Psi$ and $\lambda > 0$, let ${}_\lambda\psi(s) = \psi(s/\lambda)$ and put $\psi_j^{\alpha,\lambda}(s) = \lambda^{-1/\alpha}\, {}_\lambda\psi_j^\alpha(s) = -M_j(s/\lambda) s^{-1/\alpha}$,

$s > 0$, where the functions M_j are from (3), $j = 1, 2$. Introduce

$$V_{\alpha,\lambda}(M_1, M_2) = V(\psi_1^{\alpha,\lambda}, \psi_2^{\alpha,\lambda}, 0) \quad \text{and} \quad \boldsymbol{E}(\mathrm{e}^{\mathrm{i}tV_{\alpha,\lambda}(M_1,M_2)}) = \mathrm{e}^{y_{\alpha,\lambda}(t)}, \quad t \in \mathbb{R}, \quad (9)$$

and notice the identity $V_{\alpha,\lambda}(M_1, M_2) = \lambda^{-1/\alpha} V(\lambda\psi_1^{\alpha}, \lambda\psi_2^{\alpha}, 0)$. The notation is the same as in Ref. 6 with two important exceptions. The random variable that belongs to λ here, belongs to λ^{-1} there (Ref. 6, p. 96). The other exception is the function γ_x mentioned before. The reason for the deviation is that for generalized St. Petersburg games our theorems here must reduce to the merge theorems in Refs. 3 and 5.

We restate a basic result from Ref. 6 in terms of what we call *circular convergence*. For a given $\mathbf{c} > 1$ we say that the sequence $\{u_n\}_{n=1}^{\infty} \subset \mathbb{R}$ converges circularly to $u \in (\mathbf{c}^{-1}, 1]$, written $u_n \xrightarrow{\text{cir}} u$, if either $u \in (\mathbf{c}^{-1}, 1)$ and $u_n \to u$, or $u = 1$ and the sequence $\{u_n\}$ has limit points $\mathbf{c}^{-1}$ or 1, or both. (For $\mathbf{c} = 1$ the notion $u_n \xrightarrow{\text{cir}} 1$ simply means that $u_n \to 1$.) Let the distribution function $F \in \mathbb{D}_{\text{gp}}(G)$ be such that (4) holds along a subsequence $\{k_n\}_{n=1}^{\infty}$ satisfying (5), where $c = \mathbf{c}(G)$; this and nothing else is assumed for Theorems 2.1, 2.2 and the Corollary below. Part of the surprising result in Theorem 1 in Ref. 6 is that there are as many different limiting distributions as the continuum along different subsequences. In particular, if along a subsequence $\{n_r\}_{r=1}^{\infty} \subset \mathbb{N}$,

$$\frac{\sum_{j=1}^{n_r} X_j - c_{n_r}}{a_{n_r}} \xrightarrow{\mathcal{D}} W \quad \text{as} \quad r \to \infty \quad (10)$$

for a non-degenerate random variable W, then $\gamma_{n_r} \xrightarrow{\text{cir}} \kappa \in (\mathbf{c}^{-1}, 1]$ as $r \to \infty$, and the distribution of W is necessarily that of an affine linear transformation of $V_{\alpha,\kappa}(M_1, M_2)$. Conversely, if $\gamma_{n_r} \xrightarrow{\text{cir}} \kappa \in (\mathbf{c}^{-1}, 1]$ as $r \to \infty$, then (10) holds with $c_{n_r} = n_r \int_{n_r^{-1}}^{1-n_r^{-1}} Q(s)\,\mathrm{d}s$, $a_{n_r} = n_r^{1/\alpha}\, l(1/n_r)$ and $W = V_{\alpha,\kappa}(M_1, M_2)$.

Now let $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \dots, p_{n,n})$ be any strategy, so that $p_{1,n}, p_{2,n}, \dots, p_{n,n} \geq 0$ and $\sum_{j=1}^{n} p_{j,n} = 1$, and for simplicity put $\gamma_{j,n} = \gamma_{1/p_{j,n}}$ if $p_{j,n} > 0$, $j = 1, \dots, n$. The merging semistable approximation to the distribution functions of $S_{\alpha,\boldsymbol{p}_n}$ in (8) is given in the following main result by the distribution functions $G_{\alpha,\boldsymbol{p}_n}(x) = \boldsymbol{P}\{V_{\alpha,\boldsymbol{p}_n} \leq x\}$, $x \in \mathbb{R}$, of random variables $V_{\alpha,\boldsymbol{p}_n}$ that have characteristic functions

$$\boldsymbol{E}\big(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}}\big) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}G_{\alpha,\boldsymbol{p}_n}(x) = \exp\left\{\sum_{j=1}^{n} p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t)\right\}, \quad t \in \mathbb{R}, \quad (11)$$

where $y_{\alpha,\gamma_{j,n}}(\cdot)$ is the exponent function in the characteristic function of $V_{\alpha,\gamma_{j,n}}$ in (9), explicitly given in the proof of Lemma 1 below.

Theorem 2.1. For any sequence $\{\boldsymbol{p}_n\}_{n=1}^{\infty}$ of strategies such that $\overline{p}_n \to 0$,

$$\sup_{x \in \mathbb{R}} \big|\boldsymbol{P}\{S_{\alpha,\boldsymbol{p}_n} \leq x\} - G_{\alpha,\boldsymbol{p}_n}(x)\big| \to 0.$$

It follows from (11) that for the uniform strategies $\boldsymbol{p}_n^\diamond = (1/n, 1/n, \ldots, 1/n)$ the distributional equality $V_{\alpha,\boldsymbol{p}_n} \stackrel{\mathcal{D}}{=} V_{\alpha,\gamma_n}(M_1, M_2)$ holds, and hence Theorem 2.1 reduces to the most important special case of full sums in Theorem 2 in Ref. 6.

As noted before, there is real pooling of winnings only if $\alpha = 1$ and $l(\cdot) \equiv 1$ when the sum of the coefficients in (8) is 1. However, by a transformation we obtain a version of Theorem 2.1 that is satisfactory in this respect. This transformation is a generally implicit extension of that given in Ref. 5. The function $f(s) = s^{1/\alpha}/l(s)$ in (6) is regularly varying of order $1/\alpha$ at zero, and hence by general theory (Ref. 1, p. 23) it is asymptotically equivalent to a non-decreasing function. Therefore, to state Theorem 2.2 below, we may and do assume that $f(s) = s^{1/\alpha}/l(s)$ is itself non-decreasing and hence, by monotonicity, its inverse function $g(s)$ exists and it is also non-decreasing for s in a right neighborhood of zero. Then, if $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \ldots, p_{n,n})$ is an arbitrary strategy, consider

$$q_{j,n} = \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} \left(\sum_{k=1}^{n} \frac{p_{k,n}^{1/\alpha}}{l(p_{k,n})} \right)^{-1} = \frac{f(p_{j,n})}{\sum_{k=1}^{n} f(p_{k,n})}, \quad j = 1, 2, \ldots, n.$$

Then, clearly, $\boldsymbol{q}_n = (q_{1,n}, q_{2,n}, \ldots, q_{n,n})$ is a strategy. We need a one-to-one correspondence, that is, we have to determine $\boldsymbol{p}_n$ in terms of $\boldsymbol{q}_n$. Multiplying the defining equation by $\sum_{k=1}^{n} f(p_{k,n})$ and applying the inverse function $g(\cdot)$, we get the equation $g(q_{j,n} \sum_{k=1}^{n} f(p_{k,n})) = p_{j,n}$, so that $\sum_{j=1}^{n} g(q_{j,n} \sum_{k=1}^{n} f(p_{k,n})) = 1$. The monotonicity of $g(\cdot)$ implies that for a given strategy $\boldsymbol{q}_n$ there exists a *unique* constant $A_{\boldsymbol{q}_n} > 0$ for which $\sum_{j=1}^{n} g(q_{j,n} A_{\boldsymbol{q}_n}) = 1$, so that $A_{\boldsymbol{q}_n} = \sum_{k=1}^{n} f(p_{k,n})$. Thus $p_{j,n} = g(q_{j,n} A_{\boldsymbol{q}_n})$, $j = 1, 2, \ldots, n$, that is, the correspondence between $\boldsymbol{p}_n$ and $\boldsymbol{q}_n$ is one-to-one indeed. Now we can define the functions and random variables related to the strategy $\boldsymbol{q}_n$. Set $\nu_{k,n} = \gamma_{1/g(q_{k,n} A_{\boldsymbol{q}_n})}$, $k = 1, \ldots, n$, introduce

$$T_{\alpha,\boldsymbol{q}_n} = A_{\boldsymbol{q}_n} \sum_{k=1}^{n} q_{k,n} X_k - A_{\boldsymbol{q}_n} \sum_{k=1}^{n} q_{k,n} \int_{g(q_{k,n} A_{\boldsymbol{q}_n})}^{1-g(q_{k,n} A_{\boldsymbol{q}_n})} Q(s)\,\mathrm{d}s$$

and let $H_{\alpha,\boldsymbol{q}_n}(\cdot)$ be the semistable distribution function with characteristic function

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}H_{\alpha,\boldsymbol{q}_n}(x) = \exp\left\{ \sum_{k=1}^{n} g(q_{k,n} A_{\boldsymbol{q}_n})\, y_{\alpha,\nu_{k,n}}(t) \right\}.$$

Then a reformulated equivalent version of Theorem 2.1 is

Theorem 2.2. For any sequence $\{\boldsymbol{q}_n\}_{n=1}^{\infty}$ of strategies such that $g(\overline{q}_n A_{\boldsymbol{q}_n}) \to 0$,

$$\sup_{x\in\mathbb{R}} \left| \boldsymbol{P}\{T_{\alpha,\boldsymbol{q}_n} \le x\} - H_{\alpha,\boldsymbol{q}_n}(x) \right| \to 0.$$

The strange-looking assumption is needed because the relations $\overline{p}_n \to 0$ and $\overline{q}_n \to 0$ are independent in the sense that neither of them implies the other. This can be seen by easily constructed examples, even in the simplest case $l(\cdot) \equiv 1$.

Now we turn back to the setup in (8) and (11) and show that for special sequences $\{\boldsymbol{p}_n\}$ the merge in Theorem 2.1 reduces to ordinary limit theorems. Since for $\mathbf{c} = 1$ the approximating distribution is one and the same stable distribution already, we assume that $\mathbf{c} > 1$, in which case our conclusion is truly surprising.

Let $\{n_r\}_{r=1}^{\infty} \subset \mathbb{N}$ be an increasing subsequence and consider the sequence of strategies $\boldsymbol{p}_n = (1/n_r, 1/n_r, \ldots, 1/n_r, 0, 0, \ldots, 0)$ with n_r non-zero elements, where $n_r \le n < n_{r+1}$. This is the same situation as in (10), so there exists a limiting distribution for $\{\boldsymbol{p}_n\}_{n=1}^{\infty}$ if and only if it exists in (10) along $\{n_r\}_{r=1}^{\infty}$. There may be too many zero components in this type of strategies in the sense that in some of them the proportion of zeros is approximately $1 - \mathbf{c}^{-1}$ if $\lim_{r\to\infty} n_{r+1}/n_r = \mathbf{c}$. The following notion excludes such cases: we call a sequence $\{\boldsymbol{p}_n\}_{n=1}^{\infty}$ of strategies *balanced* if

$$\liminf_{n\to\infty} \frac{\min\{p_{j,n} : j = 1, 2, \ldots, n\}}{\max\{p_{j,n} : j = 1, 2, \ldots, n\}} > 0.$$

Roughly speaking this condition means that each component is important.

Classical theory says that if a limiting distribution exists for the uniform strategies $\boldsymbol{p}_n^{\diamond} = (1/n, 1/n, \ldots, 1/n)$, it must be stable. As an essence of semistability, the following corollary claims that semistable limiting distributions can be achieved by such balanced strategies that practically consist of only two different components.

Corollary. For an arbitrary $\kappa \in (\mathbf{c}^{-1}, 1]$ there exists a balanced sequence $\{\boldsymbol{p}_n\}_{n=1}^{\infty}$ of strategies such that $S_{\alpha,\boldsymbol{p}_n} \xrightarrow{\mathcal{D}} V_{\alpha,\kappa}(M_1, M_2)$, where $V_{\alpha,\kappa}(M_1, M_2)$ is as in (9). Moreover, for each $n \in \{2, 3, \ldots\}$ the strategy $\boldsymbol{p}_n = (p_{1,n}, p_{2,n}, \ldots, p_{n,n})$ can be constructed in such a way that there are at most two different values among its first $n - 1$ components.

It will be clear from the proof that the n-th component $p_{n,n}$, which can have a third different value, is just to make $\boldsymbol{p}_n$ a strategy, that is, to make $\sum_{j=1}^{n} p_{j,n} = 1$. Thus in fact there are only two different important components.

The difficulties of a closer description of the merging semistable random variables $V_{\alpha,\boldsymbol{p}_n}$ in (11) arise from the fact that the asymptotic equality $\gamma_{\mathbf{c}x} \sim \gamma_x$, as $x \to \infty$, for the function γ_x figuring in (6) does not reduce to true equality. Nevertheless, (7) says that for the special sequence $k_n \equiv \lfloor \mathbf{c}^n \rfloor$ we can define the function γ_x through the sequence $\mathbf{c}^n$ instead of $\lfloor \mathbf{c}^n \rfloor$ and obtain explicitly $\gamma_x = x/\mathbf{c}^{\lceil \log_{\mathbf{c}} x \rceil}$ for all $x > 0$. In this case, when $k_n \equiv \lfloor \mathbf{c}^n \rfloor$, let $V_{\alpha,1}, V_{\alpha,2}, \ldots, V_{\alpha,n}$ be independent copies of $V_{\alpha,1}(M_1, M_2)$. Then with $r_{j,n} = \lceil \log_{\mathbf{c}} p_{j,n}^{-1} \rceil$ and $\gamma_{j,n} = \gamma_{p_{j,n}^{-1}} = (p_{j,n} \mathbf{c}^{r_{j,n}})^{-1}$ as before, for any strategy $\boldsymbol{p}_n$ Lemmas 1 and 6 below imply the distributional equality

$$\sum_{j=1}^{n} p_{j,n}^{1/\alpha} V_{\alpha,j} - \sum_{j=1}^{n} \left(d_{-r_{j,n}} + p_{j,n}\, c_{\gamma_{j,n}} \right) \stackrel{\mathcal{D}}{=} V_{\alpha,\boldsymbol{p}_n}, \qquad (12)$$

where the constants c_λ, $\lambda > 0$, and d_m, $m \in \mathbb{Z}$, are also from those lemmas.

3. MERGE THEOREMS IN GENERAL

The systematic study of merge was initiated in Ref. 9 in the general setup of separable metric spaces. The study there did not get down to the characterization of merge in the Lévy distance on $\mathbb{R}$, and the aim of the present small section is exactly that. Of course, the deep and extended literature on Kolmogorov's uniform limit problem, highlighted by Arak's and Zaitsev's well-known results, deals with merge in the uniform distance ever since Prokhorov's first result in 1955. In our list here, Refs. 2 and 6 are also examples for merge in the uniform distance.

In this section $X, X_1, X_2, \ldots$, $Y, Y_1, Y_2, \ldots$ are real random variables with distribution and characteristic functions $F, F_1, F_2, \ldots$, $G, G_1, G_2, \ldots$ and $\phi, \phi_1, \phi_2, \ldots$, $\psi, \psi_1, \psi_2, \ldots$, respectively. If $F_n \Rightarrow G$ denotes weak convergence, that is, $F_n(x) \to G(x)$ at each $x \in C(G)$, where we recall that $C(G)$ is the set of continuity points of G, then of course $F_n \Rightarrow G$ is the definition of $X_n \xrightarrow{\mathcal{D}} Y$ used above, which is equivalent to $L(F_n, G) \to 0$, where $L(\cdot\,,\cdot)$ is Lévy's distance, given by $L(F, G) = \inf\{h > 0 \colon G(x-h) - h \leq F(x) \leq G(x+h) + h\}$. Extending this, we say that X_n and Y_n, or their distribution functions F_n and G_n, *merge together* if $L(F_n, G_n) \to 0$.

Here we give necessary and sufficient conditions for merge in terms of characteristic functions under the weak assumption that one of the sequences, $\{Y_n\}$ or equivalently $\{G_n\}$, say, is stochastically compact, meaning that for every subsequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ there is a further subsequence $\{n_{k_j}\}_{j=1}^{\infty} \subset \{n_k\}_{k=1}^{\infty}$ and a random variable Y, such that $Y_{n_{k_j}} \xrightarrow{\mathcal{D}} Y$, or equivalently $G_{n_{k_j}} \Rightarrow G$ as $j \to \infty$.

Theorem 3.1. If $\{G_n\}_{n=1}^{\infty}$ is stochastically compact, then $L(F_n, G_n) \to 0$ if and only if $\phi_n(t) - \psi_n(t) \to 0$ for every $t \in \mathbb{R}$.

The next theorem is the basic tool in the proof of Theorem 2.1. It says that if G_n is absolutely continuous for all $n \in \mathbb{N}$ and the corresponding density functions are uniformly bounded, then even uniform convergence holds under the same conditions.

Theorem 3.2. Assume that $\{G_n\}_{n=1}^{\infty}$ is stochastically compact and there is a constant $K > 0$ such that $\sup_{n\in\mathbb{N}} \sup_{x\in\mathbb{R}} |G_n'(x)| \leq K$. Then $F_n(x) - G_n(x) \to 0$ at every $x \in \mathbb{R}$ if and only if $\phi_n(t) - \psi_n(t) \to 0$ at every $t \in \mathbb{R}$. Moreover, if this holds, then in fact the convergence is uniform, so that $\sup_{x\in\mathbb{R}} |F_n(x) - G_n(x)| \to 0$.

4. PROOFS

Logic dictates to prove first the general theorems from the preceding section.

Proof of Theorem 3.1. Suppose first that $\phi_n(t) - \psi_n(t) \to 0$ for all $t \in \mathbb{R}$. Let $\{n_k\}_{k=1}^{\infty}$ be any subsequence of $\mathbb{N}$. By compactness there is a further subsequence

$\{n_{k_j}\}_{j=1}^{\infty} \subset \{n_k\}_{k=1}^{\infty}$ and a distribution function G such that $G_{n_{k_j}} \Rightarrow G$, so that $\psi_{n_{k_j}}(t) \to \psi(t)$, $t \in \mathbb{R}$, as $j \to \infty$ by continuity theorem. By the triangle inequality and the other direction in the continuity theorem, $F_{n_{k_j}} \Rightarrow G$, and so the triangle inequality for the Lévy metric yields $L(F_{n_{k_j}}, G_{n_{k_j}}) \to 0$ as $j \to \infty$. Since $\{n_k\}$ was arbitrary, it follows that $L(F_n, G_n) \to 0$. The proof of the converse is similar. ■

Proof of Theorem 3.2. Necessity is trivial, while the proof of sufficiency in the first statement is similar to the one above: using the uniform boundedness of G_n', one can show that the subsequential weak limits G are continuous, and so weak convergence implies convergence in each point.

To prove the stronger second statement, fix any $\varepsilon \in (0,1)$. Stochastic compactness is tightness, so there exists a $T > 0$ such that $G_n(x) > 1 - \varepsilon$ and $G_n(-x) < \varepsilon$ for all $x > T$ and $n \in \mathbb{N}$, and the uniform boundedness of the densities implies the existence of a subdivision $-T = x_0 < x_1 < \cdots < x_N = T$ such that $\sup_{1\le k\le N,\, n\in\mathbb{N}} |G_n(x_k) - G_n(x_{k-1})| < \varepsilon$. Since F_n and G_n merge together at each point, there is a threshold $n_0 \in \mathbb{N}$ such that $\max_{k=0,1,\ldots,N} |F_n(x_k) - G_n(x_k)| < \varepsilon$ if $n \ge n_0$. Then by easy calculation $\sup_{x\in\mathbb{R}} |F_n(x) - G_n(x)| < 2\varepsilon$ for all $n \ge n_0$. ■

Aiming at Theorem 2.1, first we prove six lemmas. The first is a scaling property that expresses the exponent function $y_{\alpha,\lambda}(\cdot)$ of the characteristic function in (9) in terms of $y_{\alpha,1}(\cdot)$, which was used for (12) and is needed for Lemmas 2 and 3.

Lemma 1. For every $\lambda > 0$ we have $y_{\alpha,\lambda}(t) = \lambda\, y_{\alpha,1}(t/\lambda^{1/\alpha}) - \mathrm{i}tc_\lambda$, $t \in \mathbb{R}$, where $c_\lambda = \lambda^{(1-\alpha)/\alpha} \int_1^{1/\lambda} \big[\psi_2^\alpha(s) - \psi_1^\alpha(s)\big]\,\mathrm{d}s$.

Proof. As in (1), let L_λ and R_λ denote the Lévy functions of the random variable $V({}_\lambda\psi_1^\alpha, {}_\lambda\psi_2^\alpha, 0)$ defined at (9). The inverse relation above (1) for the two representations shows that $L_\lambda(x) = \inf\{s\colon {}_\lambda\psi_1^\alpha(s) \ge x\} = \inf\{s\colon \psi_1^\alpha(s/\lambda) \ge x\} = \lambda L(x)$, $x < 0$, and similarly $R_\lambda(x) = \lambda R(x)$, $x > 0$, where $L(\cdot) = L_1(\cdot)$ and $R(\cdot) = R_1(\cdot)$. Thus, since $V(\psi_1,\psi_2,\sigma) = W(\psi_1,\psi_2,\sigma) + \theta(\psi_1) - \theta(\psi_2)$ in (2),

$$\mathrm{e}^{y_{\alpha,\lambda}(t)} = \boldsymbol{E}\Big(\mathrm{e}^{\mathrm{i}tV_{\alpha,\lambda}(M_1,M_2)}\Big) = \boldsymbol{E}\Big(\mathrm{e}^{\mathrm{i}\frac{t}{\lambda^{1/\alpha}}V({}_\lambda\psi_1^\alpha,{}_\lambda\psi_2^\alpha,0)}\Big) = \exp\left\{\mathrm{i}t\frac{\theta({}_\lambda\psi_1^\alpha) - \theta({}_\lambda\psi_2^\alpha)}{\lambda^{1/\alpha}}\right\}$$
$$\times \exp\left\{\lambda \int_{-\infty}^0 \beta_{\frac{t}{\lambda^{1/\alpha}}}(x)\,\mathrm{d}L(x) + \lambda \int_0^\infty \beta_{\frac{t}{\lambda^{1/\alpha}}}(x)\,\mathrm{d}R(x)\right\},$$

from which, forcing the exponent $y_{\alpha,1}(t\lambda^{-1/\alpha})$ in,

$$\mathrm{e}^{y_{\alpha,\lambda}(t)} = \exp\left\{-\,\mathrm{i}t\frac{\theta({}_\lambda\psi_2^\alpha) - \theta({}_\lambda\psi_1^\alpha)}{\lambda^{1/\alpha}} + \mathrm{i}t\lambda\,\frac{\theta(\psi_2^\alpha) - \theta(\psi_1^\alpha)}{\lambda^{1/\alpha}}\right\}$$
$$\times \exp\left\{\lambda\left[\mathrm{i}t\,\frac{\theta(\psi_1^\alpha) - \theta(\psi_2^\alpha)}{\lambda^{1/\alpha}} + \int_{-\infty}^0 \beta_{\frac{t}{\lambda^{1/\alpha}}}(x)\,\mathrm{d}L(x) + \int_0^\infty \beta_{\frac{t}{\lambda^{1/\alpha}}}(x)\,\mathrm{d}R(x)\right]\right\}$$

for all $t \in \mathbb{R}$, which is nothing but $\mathrm{e}^{y_{\alpha,\lambda}(t)} = \mathrm{e}^{-\mathrm{i}tc_\lambda}\mathrm{e}^{\lambda y_{\alpha,1}(t\lambda^{-1/\alpha})}$, where $c_\lambda = \lambda^{-1/\alpha}[\theta({}_\lambda\psi_2^\alpha) - \theta({}_\lambda\psi_1^\alpha) - \lambda\{\theta(\psi_2^\alpha) - \theta(\psi_1^\alpha)\}]$. Now, a somewhat long but straightforward calculation shows that $\theta({}_\lambda\psi) = \lambda\,\theta(\psi) + \lambda\int_1^{1/\lambda}\psi(t)\,\mathrm{d}t$. Further simple calculation then yields the stated form of c_λ. ■

Next, Lemmas 2 and 3 establish that the sequence $G_{\alpha,\boldsymbol{p}_n}$ in (11) has uniformly bounded densities and is stochastically compact, so that it meets the assumptions of Theorem 3.2. Here $\Gamma(u) = \int_0^\infty v^{u-1}\mathrm{e}^{-v}\,\mathrm{d}v$, $u > 0$, is the usual gamma function.

Lemma 2. For any strategy $\boldsymbol{p}_n$ the inequality

$$\sup_{x\in\mathbb{R}}\left|G'_{\alpha,\boldsymbol{p}_n}(x)\right| \le \frac{\Gamma(1/\alpha)}{\pi\alpha K_\alpha^{1/\alpha}}$$

holds, where the constant $K_\alpha > 0$ depends only on α.

Proof. It follows from a result of Kruglov[13] that $\mathfrak{Re}\, y_{\alpha,1}(t) \le -K_\alpha|t|^\alpha$, $t \in \mathbb{R}$. Then by Lemma 1, $\mathfrak{Re}\, y_{\alpha,\lambda}(t) = \lambda\mathfrak{Re}\, y_{\alpha,1}(t\lambda^{-1/\alpha}) \le -\lambda K_\alpha|t|^\alpha\lambda^{-1} = -K_\alpha|t|^\alpha$, for all $\lambda > 0$. Thus the distribution function of the variable in (9) and hence also $G_{\alpha,\boldsymbol{p}_n}(\cdot)$ in (11) is infinitely many times differentiable. In particular,

$$\begin{aligned}\left|G'_{\alpha,\boldsymbol{p}_n}(x)\right| &= \frac{1}{2\pi}\left|\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}tx}\boldsymbol{E}\big(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}}\big)\,\mathrm{d}t\right| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{\sum_{k=1}^{n}p_{k,n}\mathfrak{Re}\, y_{\alpha,\gamma_{k,n}}(t)\right\}\mathrm{d}t\\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{-K_\alpha|t|^\alpha\}\,\mathrm{d}t = \frac{\Gamma(1/\alpha)}{\pi\alpha K_\alpha^{1/\alpha}}\end{aligned}$$

for all $x \in \mathbb{R}$ by the density inversion formula, proving the lemma. ■

Lemma 3. For any sequence of strategies $\{\boldsymbol{p}_n\}_{n=1}^\infty$, the sequence of random variables $\{V_{\alpha,\boldsymbol{p}_n}\}_{n=1}^\infty$ is stochastically compact.

Proof. We rewrite the characteristic function in (11) in a form that was used in the St. Petersburg case in Ref. 8, p. 984. Setting $T_{\boldsymbol{p}_n}(\gamma) = \sum_{j=1}^n p_{j,n}I(\gamma_{j,n} \le \gamma)$, $0 < \gamma \le 1$, where $I(A)$ is the indicator of the event A, we have

$$\boldsymbol{E}\big(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}}\big) = \exp\left\{\sum_{j=1}^{n}p_{j,n}y_{\alpha,\gamma_{j,n}}(t)\right\} = \exp\left\{\int_0^1 y_{\alpha,\gamma}(t)\,\mathrm{d}T_{\boldsymbol{p}_n}(\gamma)\right\}.$$

By the multiplicative periodicity $y_{\alpha,\mathbf{c}\gamma}(t) = y_{\alpha,\gamma}(t)$ and by Lemma 1, $y_{\alpha,\gamma}(t)$ is a continuous and bounded function of γ in $(0,1]$ for each fixed $t \in \mathbb{R}$, while $T_{\boldsymbol{p}_n}$ is like an empirical distribution function with support contained in $[0,1]$. Since no mass can escape, the lemma follows by an application of the Helly selection theorem. ■

The following measure-theoretic lemma is also important in the proof of Theorem 2.1. It allows to pass on from subsequences to the entire sequence $\mathbb{N}$. Measurability and almost everywhere assumptions are meant in the usual Lebesgue sense and $\mathrm{mes}\{\cdot\}$ stands for Lebesgue measure and $\stackrel{\mathrm{mes}}{\longrightarrow}$ denotes convergence in measure.

Lemma 4. Let $q_n : I \to \mathbb{R}$ be sequence of measurable functions, $n \in \mathbb{N}$, and $\delta : \mathbb{N} \to \Lambda$ a sequence taking values in Λ, where $I \subset \mathbb{R}$ and $\Lambda \subset \mathbb{R}$ are compact intervals, and let $\nu_\lambda : I \to \mathbb{R}$ be a set of measurable functions, $\lambda \in \Lambda$. Suppose that if $\lim_{r\to\infty} \delta(n_r) = \lambda$ for a subsequence $\{n_r\}_{r=1}^\infty \subset \mathbb{N}$, then $q_{n_r}(s) - \nu_{\delta(n_r)}(s) \to 0$ for almost every $s \in I$ as $r \to \infty$. Then $q_n(\cdot) - \nu_{\delta(n)}(\cdot) \stackrel{\mathrm{mes}}{\longrightarrow} 0$, that is, $\mathrm{mes}\{s \in I : |q_n(s) - \nu_{\delta(n)}(s)| > \varepsilon\} \to 0$ for every $\varepsilon > 0$.

Proof. Fix any $\varepsilon > 0$ and let $A_n(\varepsilon) = \{s : |q_n(s) - \nu_{\delta(n)}(s)| > \varepsilon\}$. We have to prove that $\mathrm{mes}\{A_n(\varepsilon)\} \to 0$. Let $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ be any subsequence. Since Λ is compact, by the Bolzano–Weierstrass theorem there is a further subsequence $\{n_{k_l}\}_{l=1}^\infty \subset \{n_k\}_{k=1}^\infty$ such that $\delta(n_{k_l}) \to \lambda$ for some $\lambda \in \Lambda$ as $l \to \infty$. By assumption we have $q_{n_{k_l}}(s) - \nu_{\delta(n_{k_l})}(s) \to 0$ as $l \to \infty$ for almost all $s \in I$. Then by Egorov's theorem there exists a measurable set $E \subset I$ on which the convergence is uniform and $\mathrm{mes}\{I \setminus E\} < \varepsilon$. Thus $A_{n_{k_l}}(\varepsilon) \subset I \setminus E$ and so $\mathrm{mes}(A_{n_{k_l}}(\varepsilon)) < \varepsilon$ for all l large enough. Since $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ was arbitrary, the proof is complete. ∎

Lemma 4 will be used in a slightly different situation. The compact interval Λ will be the 'circle' $(\mathbf{c}^{-1}, 1]$ as the points $\mathbf{c}^{-1}$ and 1 are identified, and the convergence relation $\lim_{r\to\infty} \delta(n_r) = \lambda$ will be replaced by the corresponding $\delta(n_r) \stackrel{\mathrm{cir}}{\longrightarrow} \lambda$ as $r \to \infty$. Obviously, the lemma remains true in this setup.

Lemma 5. If $\{n_r\}_{r=1}^\infty \subset \mathbb{N}$ is a subsequence such that $\gamma_{n_r} \stackrel{\mathrm{cir}}{\longrightarrow} \kappa \in (\mathbf{c}^{-1}, 1]$ as $r \to \infty$, then

$$\frac{Q_+(s/n_r)}{n_r^{1/\alpha} l(1/n_r)} - \psi_1^{\alpha,\gamma_{n_r}}(s) \to 0, \quad s \in C(\psi_1^{\alpha,\kappa}),$$
$$-\frac{Q(1 - s/n_r)}{n_r^{1/\alpha} l(1/n_r)} - \psi_2^{\alpha,\gamma_{n_r}}(s) \to 0, \quad s \in C(\psi_2^{\alpha,\kappa})$$

as $r \to \infty$.

Proof. It is shown for the same $\{n_r\}$ in the proof of Theorem 1 in Ref. 6 that

$$\frac{Q_+(s/n_r)}{n_r^{1/\alpha} l(1/n_r)} - \psi_1^{\alpha,\kappa}(s) \to 0, \quad s \in C(\psi_1^{\alpha,\kappa}).$$

Since $\psi_1^{\alpha,1} \equiv \psi_1^{\alpha,\mathbf{c}^{-1}}$, the scaling property $\psi_1^{\alpha,\lambda}(s) = \lambda^{-1/\alpha}\psi_1^{\alpha,1}(s/\lambda)$ above (9) implies that $\psi_1^{\alpha,\kappa_n}(s) \to \psi_1^{\alpha,\kappa}(s)$, $s \in C(\psi_1^{\alpha,\kappa})$ whenever $\kappa_n \stackrel{\mathrm{cir}}{\longrightarrow} \kappa$. The two properties together give the desired result. The proof of the second statement is analogous. ∎

The following general lemma is in fact the semistable property, which is used in this paper only for the proof of (12). It goes back to Lévy, and the well-known proof is just patient calculation. (In fact, a certain converse is also true.)

Lemma 6. If $\mathrm{e}^{y_\alpha(\cdot)}$ is a semistable characteristic function of exponent $\alpha \in (0,2)$ and $c > 0$ is a multiplicative period of the functions M_1 and M_2 in (3), then $y_\alpha(c^{m/\alpha}t) = c^m y_\alpha(t) + \mathrm{i}td_m$, $t \in \mathbb{R}$, for every $m \in \mathbb{Z}$, where the constants $d_m \in \mathbb{R}$ depend on the distribution.

Proof of Theorem 2.1. By Lemmas 2 and 3 the sequence $\{V_{\alpha,\boldsymbol{p}_n}\}$ is stochastically compact and their densities are uniformly bounded. Thus by Theorem 3.2 it suffices to prove that $\Delta_{\alpha,\boldsymbol{p}_n}(t) := |\boldsymbol{E}(\mathrm{e}^{\mathrm{i}tS_{\alpha,\boldsymbol{p}_n}}) - \boldsymbol{E}(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}})| \to 0$ at each $t \in \mathbb{R}$.

Fixing $t \neq 0$ and setting

$$\mu(\boldsymbol{p}_n) = \sum_{j=1}^{n} \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} \int_{p_{j,n}}^{1-p_{j,n}} Q(s)\,\mathrm{d}s =: \sum_{j=1}^{n} \mu_{j,n}, \tag{13}$$

by (8) and (11) we can write

$$\begin{aligned}
\Delta_{\alpha,\boldsymbol{p}_n}(t) &= \left| \prod_{j=1}^{n} \boldsymbol{E}\left(\exp\left\{ \mathrm{i}t \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} X_j \right\} \right) \mathrm{e}^{-\mathrm{i}t\mu(\boldsymbol{p}_n)} - \exp\left\{ \sum_{j=1}^{n} p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) \right\} \right| \\
&= \left| \prod_{j=1}^{n} \big(1 + y_{j,n}(t)\big) - \exp\left\{ \sum_{j=1}^{n} p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) + \mathrm{i}t\mu(\boldsymbol{p}_n) \right\} \right| \\
&\le \left| \prod_{j=1}^{n} \big(1 + y_{j,n}(t)\big) - \exp\left\{ \sum_{j=1}^{n} y_{j,n}(t) \right\} \right| \\
&\quad + \left| \exp\left\{ \sum_{j=1}^{n} y_{j,n}(t) \right\} - \exp\left\{ \sum_{j=1}^{n} p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) + \mathrm{i}t\mu(\boldsymbol{p}_n) \right\} \right| \\
&\le \left| \exp\left\{ \sum_{j=1}^{n} \Big[\log\big(1 + y_{j,n}(t)\big) - y_{j,n}(t) \Big] \right\} - 1 \right| \\
&\quad + \left| \exp\left\{ \sum_{j=1}^{n} \Big[y_{j,n}(t) - p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) - \mathrm{i}t\mu_{j,n} \Big] \right\} - 1 \right|,
\end{aligned}$$

where

$$y_{j,n}(t) = \boldsymbol{E}\left(\exp\left\{ \mathrm{i}t \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} X_j \right\} - 1 \right) = \int_0^1 \left[\exp\left\{ \mathrm{i}t \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} Q(s) \right\} - 1 \right] \mathrm{d}s \tag{14}$$

Notice that $y_{j,n}(t) \to 0$ for all $j = 1, \ldots, n$ by the condition $\overline{p}_n \to 0$, and so the logarithms are well defined for all n large enough; in fact for our fixed $t \neq 0$ we will use a threshold $n_t \in \mathbb{N}$ such that $|y_{j,n}(t)| \le 1/2$, $j = 1, \dots, n$, for all $n \ge n_t$. We must prove that

$$\sum_{j=1}^{n} I_{j,n}(t) := \sum_{j=1}^{n} \big| \log\big(1 + y_{j,n}(t)\big) - y_{j,n}(t) \big| \to 0 \tag{15}$$

and

$$\sum_{j=1}^{n} \big[y_{j,n}(t) - p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) - \mathrm{i}t\mu_{j,n} \big] \to 0. \tag{16}$$

First we consider (15). Expanding the logarithm, for all $n \ge n_t$ we obtain

$$\begin{aligned} I_{j,n}(t) &= \left| \sum_{l=2}^{\infty} (-1)^{l+1} \frac{y_{j,n}^l(t)}{l} \right| \le \frac{|y_{j,n}(t)|^2}{2} \sum_{l=0}^{\infty} \big| y_{j,n}(t) \big|^l = \frac{|y_{j,n}(t)|^2}{2\{1 - |y_{j,n}(t)|\}} \\ &\le |y_{j,n}(t)|^2 \le p_{j,n} \left[\frac{1}{\sqrt{p_{j,n}}} \int_0^1 \left| \exp\left\{ \mathrm{i}t \frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})} Q(s) \right\} - 1 \right| \mathrm{d}s \right]^2 \end{aligned}$$

by (14). Since $\sum_{j=1}^{n} p_{j,n} = 1$, it is enough to show that

$$f_\alpha(x) := \frac{1}{\sqrt{x}} \int_0^1 \left| \mathrm{e}^{\mathrm{i}tQ(s)x^{1/\alpha}/l(x)} - 1 \right| \mathrm{d}s \to 0 \quad \text{as} \quad x \downarrow 0 \tag{17}$$

where $x \in (0,1)$ in general. Since $|\mathrm{e}^{\mathrm{i}u} - 1| \le \min\{2, u\}$, $u \in \mathbb{R}$, we see that

$$\int_0^1 \left| \mathrm{e}^{\mathrm{i}tQ(s)x^{1/\alpha}/l(x)} - 1 \right| \mathrm{d}s \le \int_0^x 2\, \mathrm{d}s + t\, \frac{x^{1/\alpha}}{l(x)} \int_x^{1-x} |Q(s)|\, \mathrm{d}s + \int_{1-x}^1 2\, \mathrm{d}s.$$

Megyesi[14], p. 423, proved that for h_0 small enough there exist constants $c_j > 0$ such that $\sup_{s \in (0,h_0]} |M_j(\gamma_{s^{-1}}^{-1}) + h_j(s)| \le c_j$, where $M_j(\cdot)$ and $h_j(\cdot)$ are from (6), and we choose c_j so large that the inequalities $\sup_{s \in (0,\infty)} M_j(s) \le c_j$ also hold, $j = 1, 2$. Further restrictions on h_0 will be introduced as we go along. Then by (6),

$$\begin{aligned} &\big|Q_+(s)\big| \le c_1 \frac{l(s)}{s^{1/\alpha}} \quad \text{and} \quad \big|Q(1-s)\big| \le c_2 \frac{l(s)}{s^{1/\alpha}}, \quad 0 < s \le h_0, \\ &\text{and} \quad \psi_j^{\alpha,\lambda}(s) \le \frac{c_j}{s^{1/\alpha}}, \quad s > 0, \quad j = 1, 2, \quad \text{for all} \quad \lambda > 0. \end{aligned} \tag{18}$$

Hence $\int_x^{h_0} \big|Q_+(s)\big|\, \mathrm{d}s \le c_1 \int_x^{h_0} l(s) s^{-1/\alpha}\, \mathrm{d}s$. Here we take $h_0 > 0$ be so small that $l(\cdot)$ is locally bounded on $(0, h_0)$, that is, $l(\cdot)$ is bounded on (ε, h_0) for each $\varepsilon > 0$. Note that $l(1/v)$, as a function of v, is slowly varying at infinity. We now apply Karamata's theorem (Ref. 1, pp. 26–27) and accordingly separate three cases of α.

If $\alpha < 1$ then $\frac{1}{\alpha} - 2 > -1$, and so we have the asymptotic inequality

$$\int_x^{h_0} \frac{l(s)}{s^{1/\alpha}}\,\mathrm{d}s = \int_{1/h_0}^{1/x} v^{\frac{1}{\alpha}-2}\, l(1/v)\,\mathrm{d}v \sim \frac{\alpha}{1-\alpha}\, x^{1-\frac{1}{\alpha}}\, l(x) \quad \text{as} \quad x \downarrow 0,$$

where we write $f(u) \sim g(u)$ if $\lim_{u\to\infty} f(u)/g(u) = 1$, and hence, as $x \downarrow 0$,

$$\begin{aligned} f_\alpha(x) &\le 4\sqrt{x} + t(c_1+c_2)\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)} \int_x^{h_0} \frac{l(s)}{s^{1/\alpha}}\,\mathrm{d}s + t\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)} \int_{h_0}^{1-h_0} |Q(s)|\,\mathrm{d}s \\ &= 4\sqrt{x} + t\frac{(c_1+c_2)\alpha}{1-\alpha}\sqrt{x}\,\big(1+o(1)\big) + t\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)} \int_{h_0}^{1-h_0} |Q(s)|\,\mathrm{d}s \to 0. \end{aligned}$$

If $\alpha = 1$ then $\frac{1}{\alpha} - 2 = -1$, in which case the function $l^*(x) = \int_{1/h_0}^{1/x} v^{-1} l(1/v)\,\mathrm{d}v$ is slowly varying at 0, so that, as $x \downarrow 0$,

$$\begin{aligned} f_1(x) &\le 4\sqrt{x} + t\frac{\sqrt{x}}{l(x)}\,(c_1+c_2)\int_x^{h_0} \frac{l(s)}{s}\,\mathrm{d}s + t\frac{\sqrt{x}}{l(x)} \int_{h_0}^{1-h_0} |Q(s)|\,\mathrm{d}s \\ &= 4\sqrt{x} + t(c_1+c_2)\sqrt{x}\,\frac{l^*(x)}{l(x)} + t\frac{\sqrt{x}}{l(x)} \int_{h_0}^{1-h_0} |Q(s)|\,\mathrm{d}s \to 0. \end{aligned}$$

Finally if $\alpha > 1$ then $2 - \frac{1}{\alpha} > 1$, so that $c_3 := \int_{1/h_0}^{\infty} v^{\frac{1}{\alpha}-2}\, l(1/v)\,\mathrm{d}v < \infty$ and

$$\begin{aligned} f_\alpha(x) &\le 4\sqrt{x} + t(c_1+c_2)\,\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)} \int_x^{h_0} \frac{l(s)}{s^{1/\alpha}}\,\mathrm{d}s + t\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)} \int_{h_0}^{1-h_0} |Q(s)|\,\mathrm{d}s \\ &= 4\sqrt{x} + t(c_1+c_2)c_3\,\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)} + t\frac{x^{\frac{1}{\alpha}-\frac{1}{2}}}{l(x)} \int_{h_0}^{1-h_0} |Q(s)|\,\mathrm{d}s \to 0, \end{aligned}$$

as $x \downarrow 0$. Thus (17) and, therefore, (15) is completely proved.

Now we turn to (16). For each $j = 1, 2, \ldots, n$ using the change of variables $s = up_{j,n}$ in (13) and in (14), we see that

$$\mu_{j,n} = p_{j,n} \int_1^{\frac{1}{p_{j,n}}-1} Q(up_{j,n})\frac{p_{j,n}^{1/\alpha}}{l(p_{j,n})}\,\mathrm{d}u \tag{19}$$

and

$$\begin{aligned} y_{j,n}(t) &= p_{j,n} \int_0^{1/p_{j,n}} \Big(\exp\Big\{\mathrm{i}tQ(up_{j,n})p_{j,n}^{1/\alpha}/l(p_{j,n})\Big\} - 1\Big)\,\mathrm{d}u \\ &= p_{j,n}\bigg\{ \int_0^{h_0/p_{j,n}} \Big(\exp\Big\{\mathrm{i}tQ(sp_{j,n})p_{j,n}^{1/\alpha}/l(p_{j,n})\Big\} - 1\Big)\,\mathrm{d}s \\ &\quad + \int_{h_0/p_{j,n}}^{(1-h_0)/p_{j,n}} \Big(\exp\Big\{\mathrm{i}tQ(sp_{j,n})p_{j,n}^{1/\alpha}/l(p_{j,n})\Big\} - 1\Big)\,\mathrm{d}s \\ &\quad + \int_0^{h_0/p_{j,n}} \Big(\exp\Big\{\mathrm{i}tQ(1-sp_{j,n})p_{j,n}^{1/\alpha}/l(p_{j,n})\Big\} - 1\Big)\,\mathrm{d}s \bigg\}. \end{aligned} \tag{20}$$

Therefore, (16) to be proved is equivalent to $\sum_{j=1}^{n} p_{j,n} J_{j,n}(t) \to 0$, where

$$
\begin{aligned}
J_{j,n}(t) = &\int_0^{\frac{1}{p_{j,n}}} \left[\exp\left\{ \mathrm{i}t \frac{Q(sp_{j,n}) p_{j,n}^{1/\alpha}}{l(p_{j,n})} \right\} - 1 \right] \mathrm{d}s - y_{\alpha,\gamma_{j,n}}(t) \\
&- \mathrm{i}t \int_1^{\frac{1}{p_{j,n}}-1} \frac{Q(sp_{j,n}) p_{j,n}^{1/\alpha}}{l(p_{j,n})} \,\mathrm{d}s
\end{aligned}
$$

Since $\sum_{j=1}^{n} p_{j,n} = 1$ and $\overline{p}_n \to 0$, it suffices to show that

$$
h_\alpha(x) \to 0 \quad \text{as} \quad x \downarrow 0, \tag{21}
$$

where

$$
h_\alpha(x) = \int_0^{\frac{1}{x}} \left[\exp\left\{ \mathrm{i}t Q(sx) \frac{x^{1/\alpha}}{l(x)} \right\} - 1 \right] \mathrm{d}s - y_{\alpha,\gamma_{1/x}}(t) - \mathrm{i}t \int_1^{\frac{1}{x}-1} Q(sx) \frac{x^{1/\alpha}}{l(x)} \,\mathrm{d}s.
$$

Now we rewrite the characteristic function of $G_{\alpha,\boldsymbol{p}_n}(\cdot)$ in the theorem. By (1),

$$
\begin{aligned}
\int_0^\infty \beta_t(\psi_1(s))\,\mathrm{d}s = &\int_0^1 \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1 \right] \mathrm{d}s - \mathrm{i}t \int_0^1 \frac{\psi_1^{\alpha,\lambda}(s)}{1 + \left\{\psi_1^{\alpha,\lambda}(s)\right\}^2} \,\mathrm{d}s \\
&+ \int_1^\infty \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1 - \mathrm{i}t\psi_1^{\alpha,\lambda}(s) \right] \mathrm{d}s \\
&+ \mathrm{i}t \int_1^\infty \left[\psi_1^{\alpha,\lambda}(s) - \frac{\psi_1^{\alpha,\lambda}(s)}{1 + \left\{\psi_1^{\alpha,\lambda}(s)\right\}^2} \right] \mathrm{d}s \\
= &\int_0^1 \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1 \right] \mathrm{d}s + \int_1^\infty \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1 - \mathrm{i}t\psi_1^{\alpha,\lambda}(s) \right] \mathrm{d}s \\
&- \mathrm{i}t\theta\left(\psi_1^{\alpha,\lambda}\right),
\end{aligned}
$$

where $\theta(\psi)$ as above (2). With the analogous form of other integral we finally get

$$
\begin{aligned}
y_{\alpha,\lambda}(t) = &\int_0^1 \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1 \right] \mathrm{d}s + \int_1^\infty \left[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\lambda}(s)} - 1 - \mathrm{i}t\psi_1^{\alpha,\lambda}(s) \right] \mathrm{d}s \\
&+ \int_0^1 \left[\mathrm{e}^{\mathrm{i}t\{-\psi_2^{\alpha,\lambda}(s)\}} - 1 \right] \mathrm{d}s + \int_1^\infty \left[\mathrm{e}^{\mathrm{i}t\{-\psi_2^{\alpha,\lambda}(s)\}} - 1 - \mathrm{i}t\{-\psi_2^{\alpha,\lambda}(s)\} \right] \mathrm{d}s.
\end{aligned}
$$

Using this, (19) and (20), we obtain

$$\begin{aligned}
h_\alpha(x) = &\int_0^1 \Big[\Big(\exp\Big\{\mathrm{i}tQ(sx)\frac{x^{1/\alpha}}{l(x)}\Big\}-1\Big)-\Big(\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s)}-1\Big)\Big]\,\mathrm{d}s \\
&+\int_1^{h_0/x}\Big[\exp\Big\{\mathrm{i}tQ(sx)\frac{x^{1/\alpha}}{l(x)}\Big\}-1-\mathrm{i}tQ(sx)\frac{x^{1/\alpha}}{l(x)} \\
&\qquad -\Big(\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s)}-1-\mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s)\Big)\Big]\,\mathrm{d}s \\
&+\int_{h_0/x}^{(1-h_0)/x}\Big[\exp\Big\{\mathrm{i}tQ(sx)\frac{x^{1/\alpha}}{l(x)}\Big\}-1-\mathrm{i}tQ(sx)\frac{x^{1/\alpha}}{l(x)}\Big]\,\mathrm{d}s \\
&+\int_0^1 \Big[\Big(\exp\Big\{\mathrm{i}tQ(1-sx)\frac{x^{1/\alpha}}{l(x)}\Big\}-1\Big)-\Big(\mathrm{e}^{-\mathrm{i}t\psi_2^{\alpha,\gamma_{1/x}}(s)}-1\Big)\Big]\,\mathrm{d}s \\
&+\int_1^{h_0/x}\Big[\exp\Big\{\mathrm{i}tQ(1-sx)\frac{x^{1/\alpha}}{l(x)}\Big\}-1-\mathrm{i}tQ(1-sx)\frac{x^{1/\alpha}}{l(x)} \\
&\qquad -\Big(\mathrm{e}^{-\mathrm{i}t\psi_2^{\alpha,\gamma_{1/x}}(s)}-1+\mathrm{i}t\psi_2^{\alpha,\gamma_{1/x}}(s)\Big)\Big]\,\mathrm{d}s \\
&-\int_{h_0/x}^\infty\Big[\mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s)}-1-\mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s)+\mathrm{e}^{-\mathrm{i}t\psi_2^{\alpha,\gamma_{1/x}}(s)}-1+\mathrm{i}t\psi_2^{\alpha,\gamma_{1/x}}(s)\Big]\,\mathrm{d}s \\
=:\,& h_{\alpha,1}(x)+h_{\alpha,2}(x)+h_{\alpha,3}(x)+h_{\alpha,4}(x)+h_{\alpha,5}(x)-h_{\alpha,6}(x).
\end{aligned}$$

Using the inequality $|\mathrm{e}^{\mathrm{i}u}-1-\mathrm{i}u|\le u^2/2$, $u\in\mathbb{R}$, and then the bounds $\{\psi_j^{\alpha,\gamma_{1/x}}(s)\}^2\le c_j^2/s^{2/\alpha}$, $j=1,2$, established in (18), we see that $|h_{\alpha,6}(x)|\le 2^{-1}(c_1^2+c_2^2)\,t^2\int_{h_0/x}^\infty s^{-2/\alpha}\,\mathrm{d}s\to 0$ as $x\downarrow 0$. Also, with the substitution $sx=y$,

$$|h_{\alpha,3}(x)|\le\int_{h_0/x}^{(1-h_0)/x}\frac{t^2Q^2(sx)x^{2/\alpha}}{l^2(x)}\,\mathrm{d}s=\frac{x^{\frac{2}{\alpha}-1}}{l^2(x)}\,t^2\int_{h_0}^{1-h_0}Q^2(y)\,\mathrm{d}y\to 0\quad\text{as}\quad x\downarrow 0.$$

Clearly, $h_{\alpha,1}(\cdot)$ and $h_{\alpha,4}(\cdot)$ behave analogously and can be handled the same way, and $h_{\alpha,2}(\cdot)$ and $h_{\alpha,5}(\cdot)$ can also be handled the same way. Hence we deal only with $h_{\alpha,1}(\cdot)$ and $h_{\alpha,2}(\cdot)$. First note that Lemmas 4 and 5 together imply

$$\mathrm{mes}\Big\{0\le s\le N\colon\Big|\frac{Q_+(s/n)}{n^{1/\alpha}l(1/n)}-\psi_1^{\alpha,\gamma_n}(s)\Big|>\varepsilon\Big\}\to 0\quad\text{for all}\quad\varepsilon>0,$$

convergence in measure on $[0,N]$ for each $N>0$. Using the monotonicity of $\psi_1^{\alpha,\gamma_{1/x}}(\cdot)$ and $Q(\cdot)$, we show that in this convergence $n^{-1}\downarrow 0$ can be extended to $x\downarrow 0$. To this end, consider any $x_n\downarrow 0$ such that $\gamma_{1/x_n}\xrightarrow{\text{cir}}\kappa\in(\mathbf{c}^{-1},1]$. Then also $\gamma_{\lfloor 1/x_n\rfloor}\xrightarrow{\text{cir}}\kappa$ and $\gamma_{\lceil 1/x_n\rceil}\xrightarrow{\text{cir}}\kappa$, so that, according to the proof of Lemma 5, $Q_+(s/y_n)/\{y_n^{1/\alpha}l(1/y_n)\}\to\psi_1^{\alpha,\kappa}(s)$ and $\psi_1^{\alpha,\gamma_{y_n}}(s)\to\psi_1^{\alpha,\kappa}(s)$, $s\in C(\psi_1^{\alpha,\kappa})$, where

y_n can be chosen in both convergence relations as $1/x_n$, $\lceil 1/x_n \rceil$ and $\lfloor 1/x_n \rfloor$. Using that $Q_+(s/\lceil 1/x_n \rceil) \le Q_+(sx_n) \le Q_+(s/\lfloor 1/x_n \rfloor)$, $l(1/\lfloor 1/x_n \rfloor)/l(x_n) \to 1$ and $l(1/\lceil 1/x_n \rceil)/l(x_n) \to 1$, we get $\{Q_+(sx_n)x_n^{1/\alpha}/l(x_n)\} - \psi_1^{\alpha,\gamma_{1/x_n}}(s) \to 0$ for all $s \in C(\psi_1^{\alpha,\kappa})$ by standard manipulation. This implies by Lemma 4 that

$$\text{mes}\left\{0 \le s \le N\colon \left| \frac{Q_+(sx)\, x^{1/\alpha}}{l(x)} - \psi_1^{\alpha,\gamma_{1/x}}(s) \right| > \varepsilon \right\} \to 0 \quad \text{for all} \quad \varepsilon > 0,$$

as $x \downarrow 0$. We note that if the functions ψ_j^α, $j = 1, 2$, in (3) are continuous, then Lemma 4 is needless because convergence holds pointwise.

Thus, towards the proof of (21), we showed that in the integrands in $h_{\alpha,1}(\cdot)$ and $h_{\alpha,2}(\cdot)$ go to 0 in measure as $x \downarrow 0$ on each interval $[0, N]$. Thus, it suffices to find common integrable bounds. For the first integral the function 2 does the job, so that $h_{\alpha,1}(x) \to 0$ and $h_{\alpha,4}(x) \to 0$ as $x \downarrow 0$. For the second, by (18) we have

$$\left| \exp\left\{ \frac{\mathrm{i}tQ(sx)x^{1/\alpha}}{l(x)} \right\} - 1 - \frac{\mathrm{i}tQ(s\,x)x^{1/\alpha}}{l(x)} \right| + \left| \mathrm{e}^{\mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s)} - 1 - \mathrm{i}t\psi_1^{\alpha,\gamma_{1/x}}(s) \right|$$
$$\le t^2 \frac{Q^2(sx)x^{2/\alpha}}{l^2(x)} + t^2\{\psi_1^{\alpha,\gamma_{1/x}}(s)\}^2 \le t^2 \frac{Q^2(sx)x^{2/\alpha}}{l^2(x)} + t^2 \frac{c_1^2}{s^{2/\alpha}},$$

and the second term is integrable on $[1, \infty)$. For the first term we need Potter's theorem (Ref. 1, p. 25), which for the function $l_\infty(y) = l(1/y)$, $y \ge 1$, slowly varying at infinity, states that for each $\delta > 0$ and $A > 1$ there is a $K = K(A, \delta)$ such that

$$\frac{l_\infty(y)}{l_\infty(z)} \le A \max\left\{ \left(\frac{y}{z}\right)^\delta, \left(\frac{z}{y}\right)^\delta \right\}, \quad y, z > K.$$

Take $A = 2$ and $\delta = (2\alpha)^{-1} - 4^{-1}$ and let $h_0 < 1/K(2, \delta)$. Then for $x < h_0$ and $s \in [1, h_0/x]$ we have $\{l(sx)/l(x)\} \le 2 \max\{s^\delta, s^{-\delta}\} = 2s^\delta$, and so, first by (18),

$$\left| \frac{Q_+^2(sx)x^{2/\alpha}}{l^2(x)} \right| \le c_1^2 \frac{l^2(sx)}{(sx)^{2/\alpha}}\, \frac{x^{2/\alpha}}{l^2(x)} = c_1^2 s^{-\frac{1}{2} - \frac{1}{\alpha}} \left(\frac{l(sx)}{l(x)s^\delta} \right)^2 \le 4c_1^2 s^{-\frac{1}{2} - \frac{1}{\alpha}},$$

which is integrable on $[1, \infty)$. Therefore, $h_{\alpha,2}(x) \to 0$ and $h_{\alpha,5}(x) \to 0$ as $x \downarrow 0$, proving (16) and hence the theorem. ∎

Proof of the Corollary. We construct a strategy $\boldsymbol{p}_n$ such that $\gamma_{j,n} = \kappa$ for all $j = 1, 2, \ldots, n-1$, and $p_{n,n} \to 0$. Then for the characteristic function

$$\boldsymbol{E}(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}}) = \exp\left\{ \sum_{j=1}^n p_{j,n}\, y_{\alpha,\gamma_{j,n}}(t) \right\} = \mathrm{e}^{y_{\alpha,\kappa}(t)}\, \mathrm{e}^{p_{n,n}\,[y_{\alpha,\gamma_{n,n}}(t) - y_{\alpha,\kappa}(t)]},$$

so that $\boldsymbol{E}(\mathrm{e}^{\mathrm{i}tV_{\alpha,\boldsymbol{p}_n}}) \to \mathrm{e}^{y_{\alpha,\kappa}(t)}$, $t \in \mathbb{R}$. Since $S_{\alpha,\boldsymbol{p}_n}$ and $V_{\alpha,\boldsymbol{p}_n}$ merge together by Theorem 2.1, we get $S_{\alpha,\boldsymbol{p}_n} \xrightarrow{\mathcal{D}} V_{\alpha,\kappa}(M_1, M_2)$. So it is enough to find such a strategy.

Fix $n \in \mathbb{N}$ sufficiently large to have $k_{n^*-1} < n \le k_{n^*}$ for $n^* = n^*(n)$, as before (6), and put $x_0 = \kappa k_{n^*}$, $x_{-1} = \kappa k_{n^*-1}$ and $x_{+1} = \kappa k_{n^*+1}$. Clearly, $\gamma_{x_j} = \kappa$, $j = 0, \pm 1$. If $x_0 = n$, then the uniform strategy $\boldsymbol{p}_n = (1/n, 1/n, \ldots, 1/n)$ is suitable.

If $x_0 \neq n$, we begin by equating each component to $1/x_0$. Suppose that $x_0 > n$. Then, starting with the first component, we proceed step by step and substitute $1/x_0$ by $1/x_{-1}$, so that the sum of the components is increased at each step. We do this until the sum is still less than 1. Since $n/x_{-1} > 1$, we will not change all components. Finally, increase the last $1/x_0$ to some $p_{n,n} \in (1/x_0, 1/x_{-1})$ that makes the sum 1, and the construction is complete.

For $x_0 < n$ the proof is similar, only we decrease $1/x_0$ by $1/x_{+1}$ at each step. ■

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