Islands and proximity domains

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Islands

Discrete, "digitalized" islands, rectangular case

1	2	1	2	1
1	5	7	2	2
1	7	5	1	1
2	5	7	2	2
1	2	1	1	2
1	1	1	1	1

Island domain

$U \in \mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{P}(U)$

Let $h\colon U \to \mathbb{R}$ be a height function and let $S \in \mathcal{C}$ be a nonempty set.

We denote the cover relation of the poset $(\mathcal{K}, \subseteq)$ by $\prec$, and we write $K_1 \preceq K_2$ if $K_1 \prec K_2$ or $K_1 = K_2$.

We say that S is a *island* with respect to the triple $(\mathcal{C}, \mathcal{K}, h)$, if every $K \in \mathcal{K}$ with $S \prec K$ satisfies

$$h(u) < \min h(S) \text{ for all } u \in K \setminus S.$$

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,,Closeness" relation

$(\mathcal{C}, \mathcal{K})$

$\delta \subseteq \mathcal{C} \times \mathcal{C}$

$$A\delta B \Leftrightarrow \exists K \in \mathcal{K}: \quad A \preceq K \text{ and } K \cap B \neq \emptyset. \tag{1}$$

It is easy to verify that relation δ satisfies the following properties for all $A, B, C \in \mathcal{C}$ whenever $B \cup C \in \mathcal{C}$:

$$A\delta B \Rightarrow B \neq \emptyset;$$
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Distant families

We say that $A, B \in \mathcal{C}$ are *distant* if neither $A \delta B$ nor $B \delta A$ holds.

A nonempty family $\mathcal{H} \subseteq \mathcal{C}$ will be called a *distant family*, if any two incomparable members of $\mathcal{H}$ are distant.

Lemma If $\mathcal{H} \subseteq \mathcal{C}$ is a distant family, then $\mathcal{H}$ is CDW-independent. Moreover, if $U \in \mathcal{H}$, then U is admissible.

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CD-independent subsets in posets

Definitions

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set and $a, b \in P$. The elements a and b are called *disjoint* and we write $a \perp b$ if

either $\mathbb{P}$ has least element $0 \in P$ and $\inf\{a, b\} = 0$,

or if $\mathbb{P}$ is without 0, then a and b have no common lower bound.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$ or $x \perp y$ holds.

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CDW-independence

Definition

A family $\mathcal{H} \subseteq \mathcal{P}(U)$ is *weakly independent* if

$$H \subseteq \bigcup_{i \in I} H_i \implies \exists i \in I : H \subseteq H_i \tag{2}$$

holds for all $H \in \mathcal{H}, H_i \in \mathcal{H}\,(i \in I)$.

If $\mathcal{H}$ is both CD-independent and weakly independent, then we say that $\mathcal{H}$ is *CDW-independent.*

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Definition

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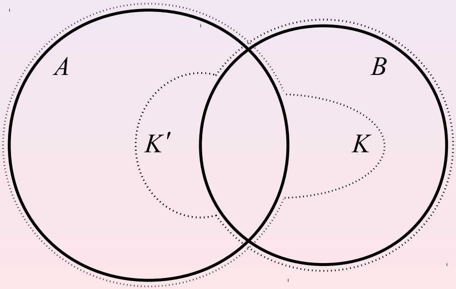
$$\exists H \in \mathcal{A}\ \forall K \in \mathcal{K} :\ H \subset K \implies K \nsubseteq \bigcup \mathcal{A}. \tag{3}$$

Connective island domains

Definition

A pair $(\mathcal{C}, \mathcal{K})$ is an *connective island domain* if

$$\forall A, B \in \mathcal{C}: \ (A \cap B \neq \emptyset \text{ and } B \nsubseteq A) \implies \exists K \in \mathcal{K} : A \subset K \subseteq A \cup B.$$



Connective island domains

Theorem

The following three conditions are equivalent for any pair $(\mathcal{C}, \mathcal{K})$:

(i) $(\mathcal{C}, \mathcal{K})$ is a connective island domain.

(ii) Every system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CD-independent.

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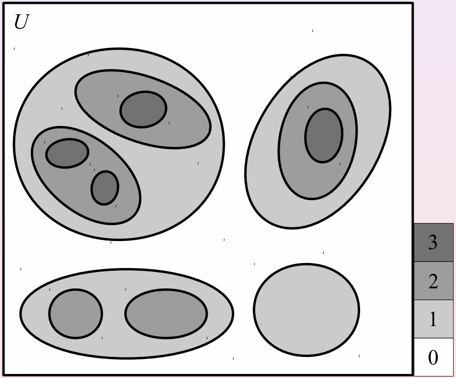
Standard height function

Let us consider a CD-independent family $\mathcal{H}$.

Clearly, for every $u \in U$, the set of members of $\mathcal{H}$ containing u is a finite chain.

The *standard height function* of $\mathcal{H}$ assigns to each element u the length of this chain, i.e., one less than the number of members of $\mathcal{H}$ that contain u.



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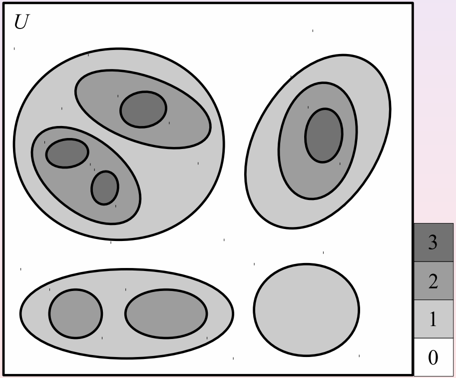

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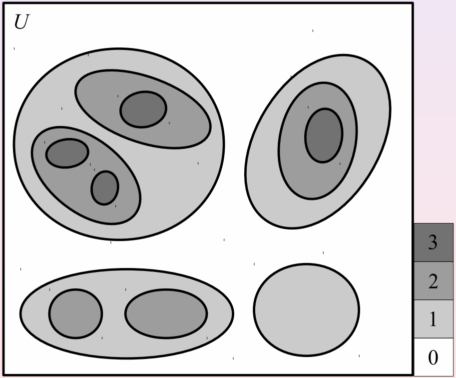

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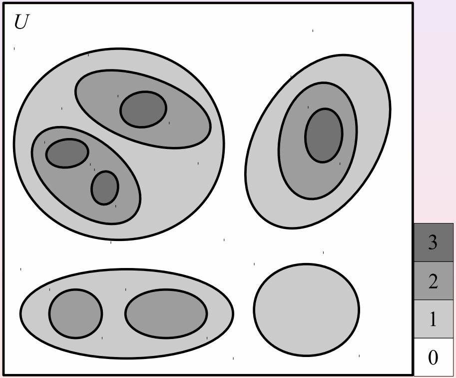

Distant families in connective island domains

Theorem

Let $(\mathcal{C}, \mathcal{K})$ be a connective island domain and let $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$ with $U \in \mathcal{H}$. If $\mathcal{H}$ is a distant family, then $\mathcal{H}$ is a system of islands; moreover, $\mathcal{H}$ is the system of islands corresponding to its standard height function.

The island domain $(\mathcal{C}, \mathcal{K})$ is called a *proximity domain*, if it is a connective island domain and the relation δ is symmetric for nonempty sets, that is

$$\forall A, B \in \mathcal{C} \setminus \{\emptyset\} : \; A\delta B \Leftrightarrow B\delta A. \tag{4}$$

If a relation δ defined on $\mathcal{P}(U)$ satisfies the mentioned three properties and δ is symmetric for nonempty sets, then (U, δ) is called a *proximity space*.

δ satisfies the following properties for all $A, B, C \in \mathcal{C}$ whenever $B \cup C \in \mathcal{C}$:

$$A\delta B \Rightarrow B \neq \emptyset;$$
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Islands and proximity domains

Proposition

If $(\mathcal{C}, \mathcal{K})$ is a proximity domain, then any system of islands corresponding to $(\mathcal{C}, \mathcal{K})$ is a distant system.

Proof

$$h(b) < \min\, h(A) \leq h(a)$$

$$h(a) < \min\, h(B) \leq h(b)$$

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Characterization for system of islands for proximity domains

Corollary

If $(\mathcal{C}, \mathcal{K})$ is a proximity domain, and $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$ with $U \in \mathcal{H}$, then $\mathcal{H}$ is a system of islands if and only if $\mathcal{H}$ is a distant family. Moreover, in this case $\mathcal{H}$ is the system of islands corresponding to its standard height function.

Pre-island

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Example

Let $A_1, \ldots, A_n$ be nonempty sets, and let $\mathcal{I} \subseteq A_1 \times \cdots \times A_n$. Let us define

$$\begin{aligned} U &= A_1 \times \cdots \times A_n, \\ \mathcal{K} &= \{B_1 \times \cdots \times B_n \colon \emptyset \neq B_i \subseteq A_i,\ 1 \leq i \leq n\} \\ \mathcal{C} &= \{C \in \mathcal{K} \colon C \subseteq \mathcal{I}\} \cup \{U\}, \end{aligned}$$

and let $h \colon U \longrightarrow \{0, 1\}$ be the height function given by

$$h(a_1, \ldots, a_n) := \begin{cases} 1, & \text{if } (a_1, \ldots, a_n) \in \mathcal{I}; \\ 0, & \text{if } (a_1, \ldots, a_n) \in U \setminus \mathcal{I}; \end{cases} \quad \text{for all } (a_1, \ldots, a_n) \in U.$$

It is easy to see that the pre-islands corresponding to the triple $(\mathcal{C}, \mathcal{K}, h)$ are exactly U and the maximal elements of the poset $(\mathcal{C} \setminus \{U\}, \subseteq)$.

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Canonical height function

Let $\mathcal{H} \subseteq \mathcal{C}$ be an admissible family of sets.

We define subfamilies $\mathcal{H}^{(i)} \subseteq \mathcal{H}$ $(i = 0, 1, 2, \ldots)$ recursively as follows.

Let $\mathcal{H}^{(0)} = \{U\}$.

For $i > 0$, if $\mathcal{H} \neq \mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(i-1)}$, then let $\mathcal{H}^{(i)}$ consist of all those sets $H \in \mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(i-1)})$ that have the following property:

$$\forall K \in \mathcal{K}: \; H \subset K \implies K \not\subseteq \bigcup \left(\mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(i-1)})\right). \tag{6}$$

Since $\mathcal{H}$ is finite and admissible, after finitely many steps we obtain a partition $\mathcal{H} = \mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(r)}$.

The *canonical height function corresponding to* $\mathcal{H}$ is the function $h_{\mathcal{H}}: U \to \mathbb{N}$ defined by

$$h_{\mathcal{H}}(x) := \max\left\{ i \in \{1, \ldots, r\} : x \in \bigcup \mathcal{H}^{(i)} \right\} \text{ for all } x \in U. \tag{7}$$

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Let $\mathcal{H} \subseteq \mathcal{C}$ be an admissible family of sets.

We define subfamilies $\mathcal{H}^{(i)} \subseteq \mathcal{H}$ $(i = 0, 1, 2, \ldots)$ recursively as follows.

Let $\mathcal{H}^{(0)} = \{U\}$.

For $i > 0$, if $\mathcal{H} \neq \mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(i-1)}$, then let $\mathcal{H}^{(i)}$ consist of all those sets $H \in \mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(i-1)})$ that have the following property:

$$\forall K \in \mathcal{K}: \; H \subset K \implies K \not\subseteq \bigcup \left(\mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(i-1)})\right). \quad (6)$$

Since $\mathcal{H}$ is finite and admissible, after finitely many steps we obtain a partition $\mathcal{H} = \mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(r)}$.

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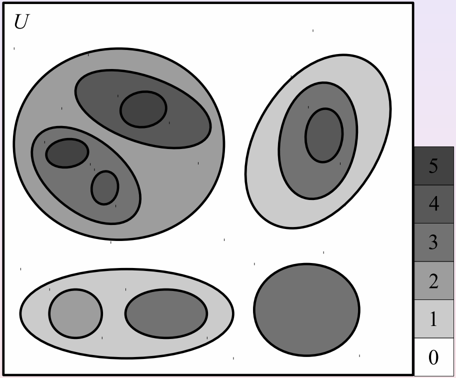
Canonical height functions



Pre-islands and admissible systems

Proposition

If $\mathcal{H} \subseteq \mathcal{C}$ is an admissible family of sets and $h_{\mathcal{H}}$ is the corresponding canonical height function, then every member of $\mathcal{H}$ is a pre-island with respect to $(\mathcal{C}, \mathcal{K}, h_{\mathcal{H}})$.

Theorem

A subfamily of $\mathcal{C}$ is a maximal system of pre-islands if and only if it is a maximal admissible family.

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Subsets of pre-island systems

The following two conditions are equivalent for any pair $(\mathcal{C}, \mathcal{K})$:

Any subset of a system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$ is also a system of pre-islands.

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Finally, let us consider the following condition on $(\mathcal{C}, \mathcal{K})$, which is stronger than that of being a connective island domain:

$$\forall K_1, K_2 \in \mathcal{K}: \; K_1 \cap K_2 \neq \emptyset \implies K_1 \cup K_2 \in \mathcal{K}. \tag{8}$$

Theorem

Suppose that $(\mathcal{C}, \mathcal{K})$ satisfies the above condition, and assume that for all $C \in \mathcal{C}$, $K \in \mathcal{K}$ with $C \prec K$ we have $|K \setminus C| = 1$. Then $(\mathcal{C}, \mathcal{K})$ is a proximity domain; pre-islands and islands corresponding to $(\mathcal{C}, \mathcal{K})$ coincide. Therefore, if $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$ and $U \in \mathcal{H}$, then $\mathcal{H}$ is a system of (pre-) islands if and only if $\mathcal{H}$ is a distant family. Moreover, in this case $\mathcal{H}$ is the system of (pre-) islands corresponding to its standard height function.

Example

Let $G = (U, E)$ be a connected simple graph with vertex set U and edge set E; let $\mathcal{K}$ consist of the connected subsets of U, and let $\mathcal{C} \subseteq \mathcal{K}$ such that $U \in \mathcal{C}$. Let $\mathcal{C}$ consist of the connected convex sets of vertices.

Corollary

Let G be a graph with vertex set U; let $(\mathcal{C}, \mathcal{K})$ be a connective island domain corresponding to $(\mathcal{C}, \mathcal{K})$, and let $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$ with $U \in \mathcal{H}$. Then $\mathcal{H}$ is a system of (pre-) islands if and only if $\mathcal{H}$ is distant; moreover, in this case $\mathcal{H}$ is the system of (pre-) islands corresponding to its standard height function.

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ÚJ SZÉCHENYI TERV