Islands, lattices and trees, Mersenne numbers

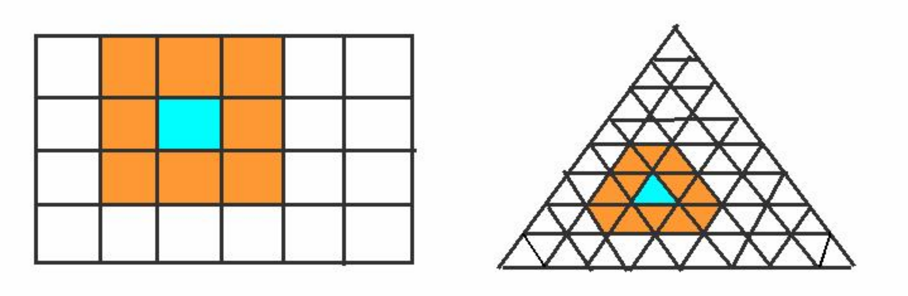
Eszter K. Horváth, Szeged

Coauthors: János Barát, Péter Hajnal, Zoltán Németh

SSAOS 2008, Třešt'

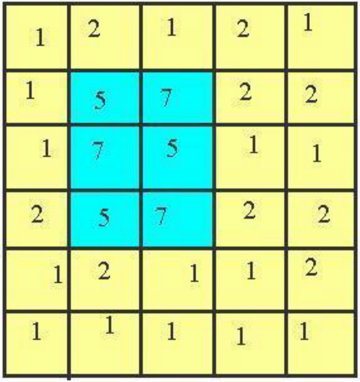
Definition

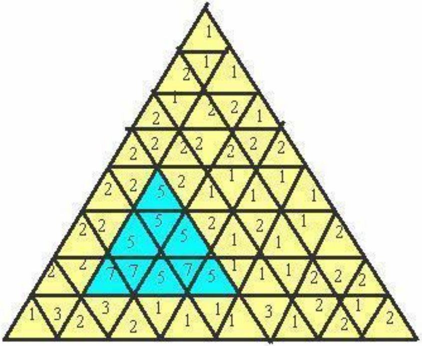
Grid, neighbourhood relation



Definition

We call a rectangle/triangle an *island*, if for the cell t, if we denote its height by a_t, then for each cell $\hat{t}$ neighbouring with a cell of the rectange/triangle T, the inequality $a_{\hat{t}} < min\{a_t : t \in T\}$ holds.





Preliminaries/1

Coding theory

S. Földes and N. M. Singhi: On instantaneous codes, J. of Combinatorics, Information and System Sci., 31 (2006), 317-326.

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Preliminaries/2

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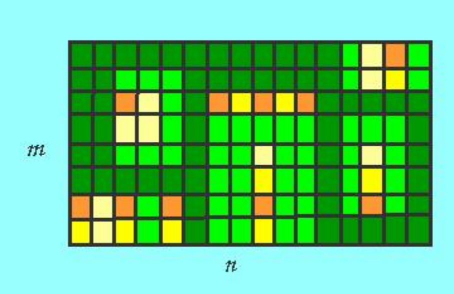
G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics., to appear.

The maximum number of rectangular islands in a $m \times n$ rectangular board on square grid:

$$f(m, n) = [\frac{mn + m + n - 1}{2}].$$



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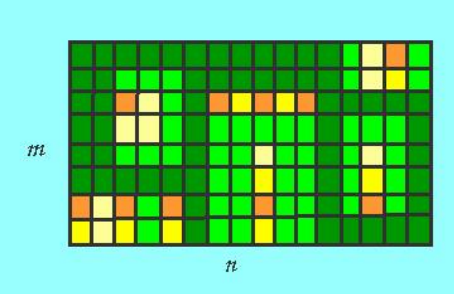

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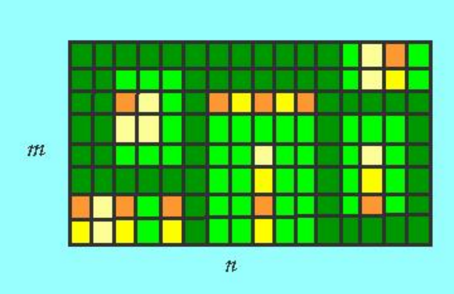

Preliminaries/3

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Preliminaries/4

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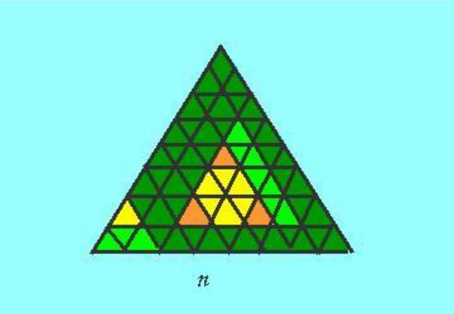
For the maximum number of triangular islands in an equilateral rectangle of side length n, $\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}$ holds.



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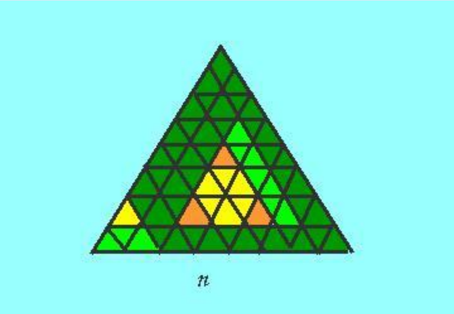

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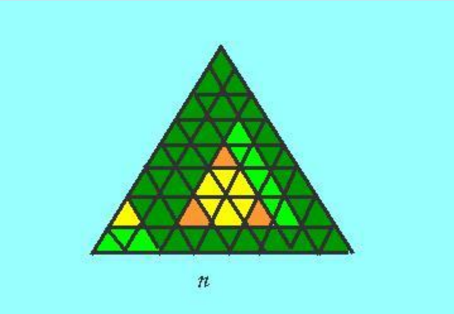

Lemma 1 (G. Czédli). Let $\mathcal{C}$ be the set of cells. Let $\mathcal{I}$ be a subset of $P(\mathcal{C})$. Then the following two conditions are equivalent:

(i) There exists mapping $A : \mathcal{C} \to \mathbb{R}$, $c \mapsto a_c$ such that $\mathcal{I} = \mathcal{I}_{\mathrm{rect/tri}}(A)$.

(ii) $\mathcal{C} \in \mathcal{I}$, and for any $I_1, I_2 \in \mathcal{I}$ either $I_1 \subseteq I_2$, or $I_2 \subseteq I_1$, or I_1 and I_2 are far from each other.

Subsets of $\mathcal{I}$ satisfying the equivalent conditions of Lemma 1 will be called *sytems of rectangular/triangular islands.*

Characterization of systems of islands without heights

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Proving methods/1

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G. Czédli, A. P. Huhn and E. T. Schmidt: Weakly independent subsets in lattices, Algebra Universalis 20 (1985), 194-196.

Any two weak bases of a finite distributive lattice have the same number of elements.

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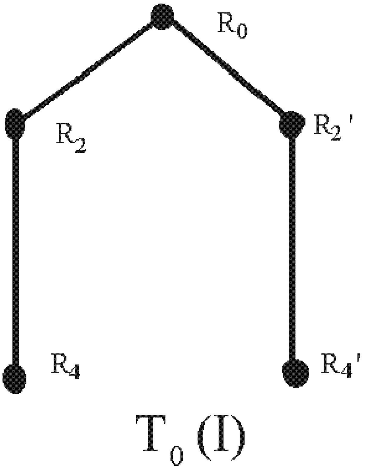

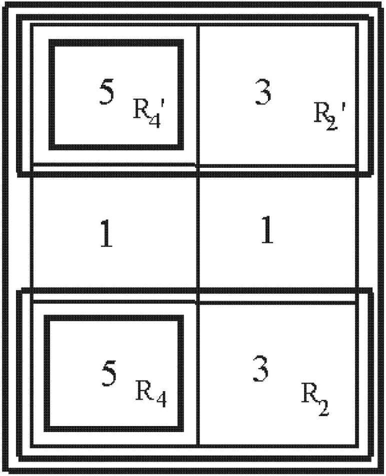

Proving methods/2

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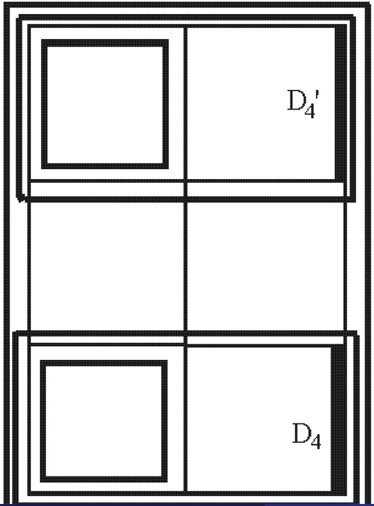

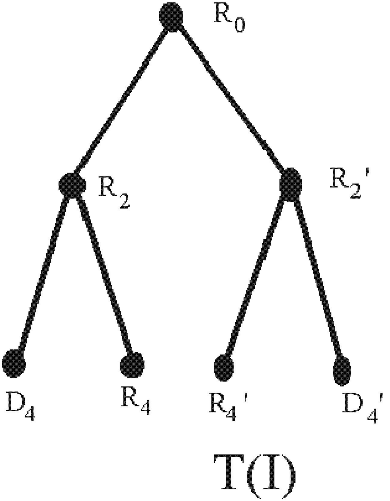

Proving methods/2

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Lemma 2 (folklore)

(i) Let T be a binary tree with ℓ leaves. Then the number of vertices of T depends only on ℓ, moreover $|V| = 2\ell - 1$.

(ii) Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T. Then $|V| \leq 2\ell - 1$.

We have $4s + 2d \leq (n+1)(m+1)$.
The number of leaves of $T(\mathcal{I})$ is $\ell = s + d$. Hence by Lemma 2 the number of islands is

$$|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n+1)(m+1) - 1.$$

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We define

$$\mu(R) = \mu(u, v) := (u + 1)(v + 1).$$

Now

$$f(m, n) = 1 + \sum_{R \in max\mathcal{I}} f(R) = 1 + \sum_{R \in max\mathcal{I}} ([\frac{(u+1)(v+1)}{2}] - 1)$$

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If $|max\mathcal{I}| \geq 2$, then the proof is ready. Case $|max\mathcal{I}| = 1$ is an easy excersise.

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Examples / Exact results

Peninsulas (semi islands) (J. Barát, P. Hajnal, E.K. Horváth):
$p(m, n) = f(m, n) = [(mn + m + n - 1)/2]$.

Cylindric board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_1(m, n) = [\frac{(m+1)n}{2}]$.

Cylindric board, cylindric and rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_2(m, n) = [\frac{(m+1)n}{2}] + [\frac{(m-1)}{2}]$.

Torus board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
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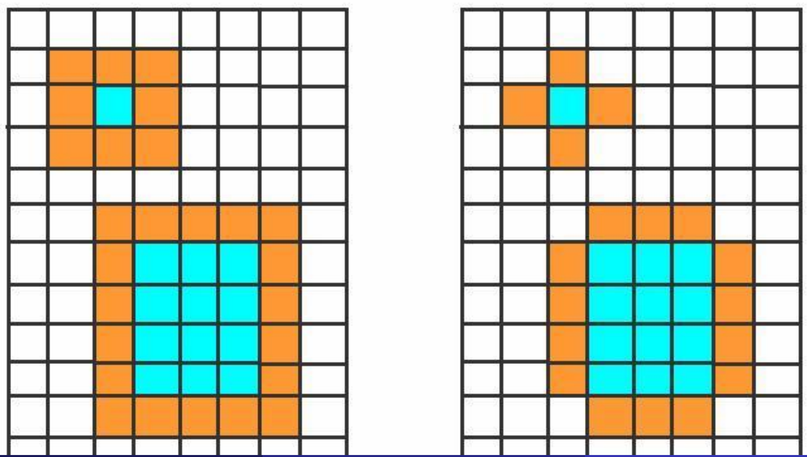
Examples / Exact results

Changing the neigbourhood relation of cells (J. Barát, P. Hajnal, E.K. Horváth): $f^*(m,n) = f(m,n) = [(mn+m+n-1)/2]$.



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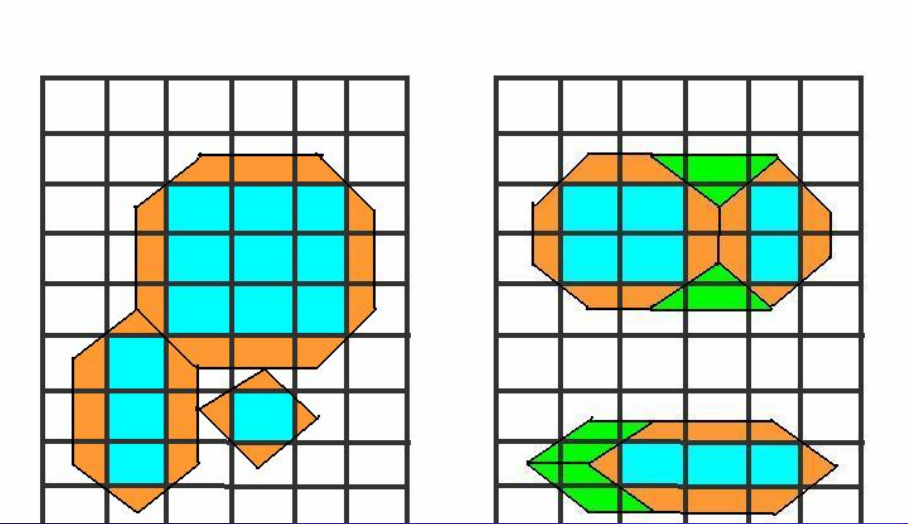
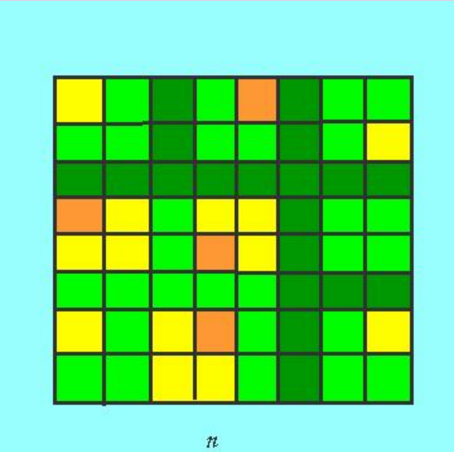


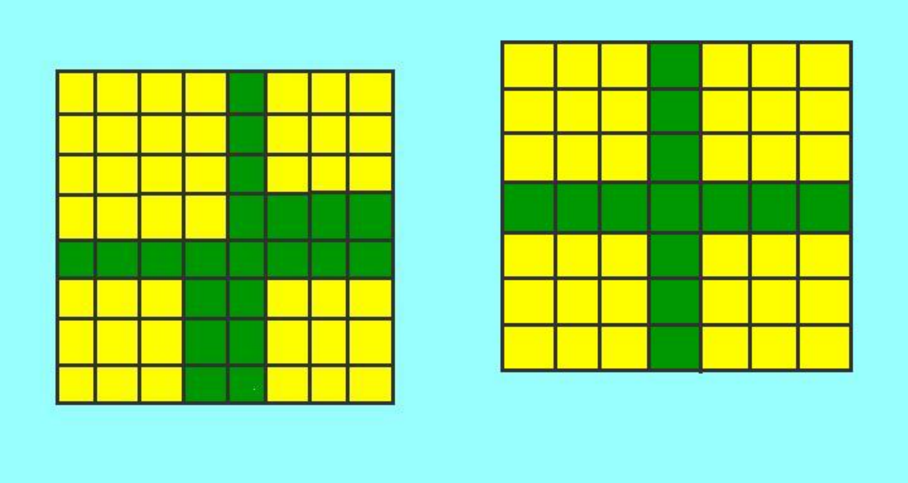
Examples / Square islands and Mersenne numbers

Square islands (E.K. Horváth, Z. Németh):
$\frac{2}{11}n^2 + n + \frac{2}{3} \leq f(n) \leq \frac{n^2+2n}{3}$. If n is a Mersenne number, i. e. $n = 2^k - 1$ for some $k \in \mathbb{N}$, then $f(n) = \frac{n^2+2n}{3}$.



Examples / Square islands and Mersenne numbers

$f(2k+2) \geq f(k+1) + 3f(k) + 1$; $f(2k+1) \geq 4f(k) + 1$;
or in unified form:
$f(n) \geq f[\frac{n}{2}] + 3f[\frac{n-1}{2}] + 1$.



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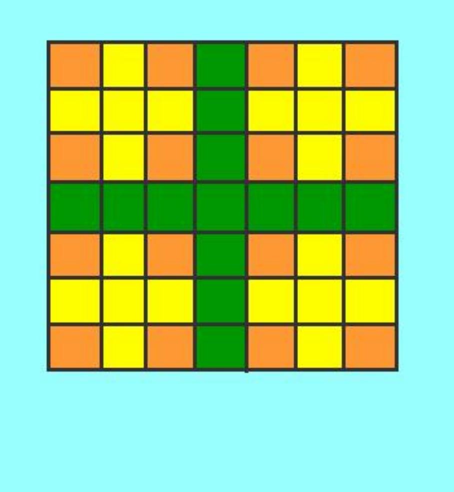
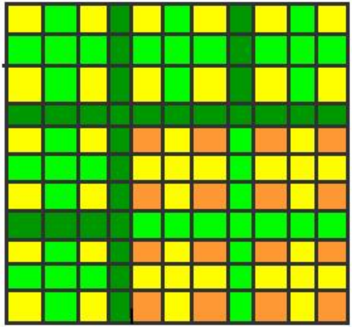


Open problems

Globe

Horváth - Németh conjecture

$$n+1 = 2^{\kappa_1} + 2^{\kappa_2} + \cdots + 2^{\kappa_k}, \qquad \kappa_1 > \kappa_2 > \ldots \kappa_k \geq 0.$$



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