Lemma 1. *Let $\mathcal{H}$ be a subset of $\mathcal{T}(n)$. Then the following two conditions are equivalent:*

(i) There exists mapping $A : T_0 \mapsto \mathbb{R},\ t \to a_t$ such that $\mathcal{H} = \mathcal{S}_{\rm tr}(A)$.

(ii) $T_0 \in \mathcal{H}$, and for any $T_1, T_2 \in \mathcal{H}$ either $T_1 \subseteq T_2$, or $T_2 \subseteq T_1$, or T_1 and T_2 are far fom each other.

Proof. Suppose $\mathcal{H} = \mathcal{S}_{\rm tr}(A)$ for some mapping A, according to (i). Then $T_0 \in \mathcal{H}$ is evident. If the rest of (ii) fails, then there exists a cell $\hat{t_2} \in T_2 \setminus T_1$ which is neighbouring with some cell of T_1. Since T_1 is a full triangular segment, we have $a_{\hat{t_2}} < \min\ \{a_{t_1} : t_1 \in T_1\}$. Similarly, there exists a cell $\hat{t_1} \in T_1 \setminus T_2$ which is neighbouring with some cell of T_2. Since T_2 is a full triangular segment, we have $a_{\hat{t_1}} < \min\ \{a_{t_2} : t_2 \in T_2\}$. Hence $a_{\hat{t_2}} < \min\ \{a_{t_1} : t_1 \in T_1\} \leq a_{\hat{t_1}} < \min\ \{a_{t_2} : t_2 \in T_2\}$ contradicts $\hat{t_2} \in T_2$. This proves (i) $\Rightarrow$ (ii).

The converse implication will be proved via induction on n. For $n = 1$ or $\mid \mathcal{H} \mid = 1$ everything is clear. Suppose $n > 1$, $\mid \mathcal{H} \mid > 1$ and (ii) holds for $\mathcal{H}$. Let $M_1, \ldots, M_k$ be the maximal elements of $\mathcal{H} \setminus \{T_0\}$. Clearly, $\mathcal{H}_i = \{T \in \mathcal{H} : T \subseteq M_i\}$ satisfies (ii) for the triangle M_i, $1 \leq i \leq k$. Hence, by the induction hypothesis, there is a mapping $A_i : M_i \mapsto \mathbb{R}$ such that $\mathcal{S}_{\rm tr}(A_i) = \mathcal{H}_i$ for each $i \in \{1, \ldots, k\}$ now chose an $r \in \mathbb{R}$ such that r is strictly less than the minimum of the image values of A_i for all $i \in \{1, \ldots, k\}$. Then the union of the A_i $1 \leq i \leq k$, and the constant mapping $T_0 \setminus (M_1 \cup \ldots \cup M_k) \to \{r\}$ is a $T_0 \mapsto \mathbb{R}$ mapping A. Since the M_i are pairwise far fom each other, obviously $\mathcal{H} \subseteq \mathcal{S}_{\rm tr}(A)$. But those cells that have images r belong only to the full triangular segment T_0. Q.E.D.

Subsets $\mathcal{H}$ satisfying the equivalent conditions of Lemma 1 will be called systems of full triangular segments.

Let $L = (L; \vee, \wedge)$ be a finite distributive lattice. A subset H of L is called *weakly independent* , if for any $k \in \mathbb{N}$ and $h,\ h_1, \ldots,\ h_k \in H$ which satisfy $h \leq h_1 \vee \ldots \vee h_k$ there exists an $i \in \{1, \ldots, k\}$ such that $h \leq h_i$. Maximal week independent subsets are called *weak bases* of L. The set $J_0(L)$ of join-irreducible elements and all maximal chains are weak bases of L.

Lemma 2 [Czédli, Huhn, Schmidt]. *Any two weak bases of a finite distributive lattice have the same number of elements.*

The lattice of all subsets of T_0 is a distributive lattice, which will be denoted by $\mathcal{P}(T_0) = (\mathcal{P}(T_0); \cup, \cap)$.

Lemma 3. *Let $\mathcal{H}$ be a system of triangular segments of T_0. Then $\mathcal{H}$ is a weakly independent subset of $\mathcal{P}(T_0)$. Consequently, $\mid \mathcal{H} \mid \leq n^2$.*

Proof. Suppose

$$T \subseteq T_1 \cup \ldots \cup T_k, \tag{1}$$

where $T,\ T_1 \cup \ldots \cup T_k \in \mathcal{H}$. We can assume that this inclusion is irredundant, i.e. no $i \in \{1, \ldots, k\}$ with $T \subseteq T_1 \cup \ldots \cup T_{i-1} \cup T_{i+1} \cup \ldots \cup T_k$. Then no T_i is disjoint from T and the T_i are pairwise incomparable. If $T \subseteq T_i$ for some i, then we are done. In the opposite case $k \geq 2$ and $T_i \subset T$ for all $i \in \{1, \ldots, k\}$. Then there is a cell c of T neighbouring with T_i. Since the T_j for $1 < j$ are far from T_1, c does not belong to $T_1 \cup \ldots \cup T_k$. This contradicts (1). Consequently, $\mathcal{H}$ is a weakly independent subset of $\mathcal{P}(T_0)$.

Now, we extend $\mathcal{H}$ to a weak basis $\mathcal{H}'$ of $\mathcal{P}(T_0)$, and consider a maximal chain $\mathcal{C}$ in $\mathcal{P}(T_0)$. Then C has $n^2 + 1$ elements and it is also a weak basis. Hence we obtain from Lemma 2 that $\mid \mathcal{H}' \mid = n^2 + 1$. On the other hand, the empty set belongs to every weak basis but not to $\mathcal{H}$, consequently $\mid \mathcal{H} \mid \leq \mid \mathcal{H}' \mid -1 = n^2$. Q.E.D.

5. Lower estimate

First, if we draw subtriangles into the big triangle in such a way that, that the sizes of the subtriangles are are ”as different as possible”, then we may use the recursive relation of Lemma 4 as follows

$$f(n) \geq f(n-3) + n + 1.$$

It is clear that the functions

$$g_c(n) := \frac{1}{6}n^2 + \frac{5}{6}n + c$$

satisfy the recursive relation

$$g_c(n) = g_c(n-3) + n + 1.$$

Moreover, if we choose $c := -\frac{1}{3}$, then then the relations for the initial values

$$g_c(1) \leq 1,\ \ g_c(2) \leq 2,\ \ g_c(3) \leq 4$$

are fulfilled, so, it is clear, that

$$f(n) \geq \frac{1}{6}n^2 + \frac{5}{6}n - \frac{1}{3},$$

so we may write

$$\frac{f(n)}{n^2} \geq \frac{1}{6}.$$

Remark. Finding inequalities between $f(n)$ and $f(n-k)$ where $k > 3$ instead of the one in Lemma 4 would probably result in a better estimation, but the obtaining inequality analogous to the one in Lemma 4 seems very complicated if possible at all.

In the second case we draw subtriangles into the big triangle in such a way that, the sizes of which are ”as same as possible”. This way we obtained the relations of Lemma 5 and Lemma 6, which are our starting points now:

$$f(2n+2) \geq 3f(n) + f(n+1) + 1$$

$$f(2n+1) \geq 3f(n) + f(n-1) + 1$$

Now we define quadratic functions

$$g_c(n) := \frac{3c+1}{6}n^2 + \frac{3c+1}{2}n + c.$$

Easy calculation gives that the equalities

$$g_c(2n+2) = 3g_c(n) + g_c(n+1) + 1,$$

$$g_c(2n+1) = 3g_c(n) + g_c(n-1) + 1$$

hold; moreover, if we choose $c := \frac{1}{18}$, then the initial conditions

$$g(1) \leq 1,\ \ g(2) \leq 2$$

are fulfilled.

Finally, it is clear, that

$$f(n) \geq \frac{7}{36}n^2 + \frac{7}{12}n + \frac{1}{18}.$$

We observe, that this estimate is slightly closer then the previous one.

$$
\begin{aligned}
d(n) = d(\mathcal{H}) &=| \mathcal{C} | \\
&= \sum_i^k | (\mathcal{C}_\mathrm{i}) | + | \mathrm{out}(\mathcal{H}) | -1 \\
&= \sum_i^k | (\mathcal{C}_\mathrm{i}) | + \mu(\mathrm{G}) - 3\mathrm{n} - \frac{5}{2} \\
&= -3n - \frac{5}{2} + \sum_i^k | (\mathcal{C}_\mathrm{i}) | + \sum_\mathrm{i}^\mathrm{k} \mu(\mathrm{T}_\mathrm{i}^* \setminus \mathrm{T}_\mathrm{i}) + \mu(\mathcal{E}) \\
&= -3n - \frac{5}{2} + \mu(\mathcal{E}) + \sum_\mathrm{i}^\mathrm{k} (| (\mathcal{C}_\mathrm{i}) | + \mu(\mathrm{T}_\mathrm{i}^* \setminus \mathrm{T}_\mathrm{i})) \\
&\geq -3n - \frac{5}{2} + \mu(\mathcal{E}) + \sum_\mathrm{i}^\mathrm{k} (\mathrm{d}(\mathrm{a}_\mathrm{i}) + 3\mathrm{a}_\mathrm{i} + \frac{3}{2}) \\
&\geq -3n - \frac{5}{2} + \mu(\mathcal{E}) + \sum_\mathrm{i}^\mathrm{k} [\mathrm{e}_{\mathrm{a}_\mathrm{i}} \mathrm{a}_\mathrm{i}^2 + (\mathrm{e}_{\mathrm{a}_\mathrm{i}} - 1)(3\mathrm{a}_\mathrm{i} + \frac{3}{2})],
\end{aligned}
$$

keeping in mind the monotonicity of e_n and observing the fact that the formula in the square brackets is equal to $e_{a_i}(a_i^2 + 3a_i + \frac{3}{2})$, we have

$$
\begin{aligned}
&\geq -3n - \frac{5}{2} + \mu(\mathcal{E}) + \mathrm{e}_{\mathrm{n}-1} \sum_\mathrm{i}^\mathrm{k} \mu(\mathrm{T}_\mathrm{i}^*) \\
&\geq -3n - \frac{5}{2} + (1 - e_{n-1})\mu(\mathcal{E}) + \mathrm{e}_{\mathrm{n}-1}(\mu(\mathcal{E}) + \sum_\mathrm{i}^\mathrm{k} \mu(\mathrm{T}_\mathrm{i}^*)) = (*),
\end{aligned}
$$

Observing the facts that $1 - e_{n-1} > 0$, $\mu(\mathcal{E}) \geq 0$ and

$$
\mu(\mathcal{E}) + \sum_\mathrm{i}^\mathrm{k} \mu(\mathrm{T}_\mathrm{i}^*) = \mu(\mathrm{T}_0^*) = \mathrm{n}^2 + 3\mathrm{n} + \frac{3}{2},
$$

we have

$$
\begin{aligned}
(*) &= -3n - \frac{5}{2} + e_{n-1}(n^2 + 3n + \frac{3}{2}) \\
&= e_{n-1}n^2 + (e_{n-1} - 1)(3n + \frac{3}{2}) - 1 \\
&= e_{n-1}(n^2 + 3n + \frac{3}{2}) - 3n - \frac{3}{2} - 1.
\end{aligned}
$$

If we put

$$
e_{n-1}(n^2 + 3n + \frac{3}{2}) - 3n - \frac{3}{2} - 1 := h(n) = e_n(n^2 + 3n + \frac{3}{2}) - 3n - \frac{3}{2},
$$

then we obtain the recursive relation

$$e_n = e_{n-1} - \frac{1}{n^2 + 3n + \frac{3}{2}},$$

which means that

$$e_n = \frac{9}{11} - \sum_2^\infty \frac{1}{k^2 + 3k + \frac{3}{2}}.$$

The series

$$\sum_2^\infty \frac{1}{k^2 + 3k + \frac{3}{2}}$$

converges, moreover, it is well known, that

$$\begin{aligned}
\sum_2^\infty \frac{1}{k^2 + 3k + \frac{3}{2}} &= -\frac{2}{3} - \frac{2}{11} + \sum_2^\infty \frac{1}{k^2 + 3k + \frac{3}{2}} \\
&= -\frac{2}{3} - \frac{2}{11} + \sum_2^\infty \frac{1}{\left[k + \left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right)\right]\left[k + \left(\frac{3}{2} + \frac{\sqrt{3}}{2}\right)\right]} \\
&= -\frac{2}{3} - \frac{2}{11} + \frac{1}{-\sqrt{3}}\left[\Psi\left(\frac{3}{2} + \frac{\sqrt{3}}{2}\right) - \Psi\left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right)\right],
\end{aligned}$$

here $\Psi(x)$ is the so-called digamma function, see, for example [PBM1.] Calculating the value $\lim e_n$ numerically gives

$$\lim e_n = 0.47\ldots,$$

which leads us to the upper estimate

$$\frac{f(n)}{n^2} \leq 0,52\ldots$$

for all n.

Remark. If we use $\mu(\mathcal{E}) \geq \frac{3}{2}$ instead of $\mu(\mathcal{E}) \geq 0$ (which is proven to be true), then in the same way as the above one, we can obtain the following estimation:

$$\begin{aligned}
e_1 &= \frac{9}{11} \\
e_n &= \frac{(2n^2 + 6n)e_{n-1} + 1}{2n^2 + 6n + 3}.
\end{aligned}$$

This is a better estimation then the previous one, so this must be also convergent.