A lattice variant of thresholdness of Boolean functions

Eszter K. Horváth, Szeged

Co-authors: Branimir Šešelja, Andreja Tepavčević

Stará Lesná, September 10.

Threshold functions

A classical **threshold function** is a Boolean function $f : \{0,1\}^n \to \{0,1\}$ such that there exist real numbers $w_1, \ldots, w_n, t$, fulfilling

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \sum_{i=1}^{n} w_i \cdot x_i \geq t,$$

where w_i is called **weight** of x_i, for $i = 1, 2, \ldots, n$ and t is a constant called the **threshold value**.

modeling neurons

political decisions

electrical engineering

artifical intelligence

game theory

Threshold functions

– combinatorics (their number!!!)

– computer science

IN ALGEBRA

tolerance relation (B. Bódi et al.)

fundamental ideal of a groupring (B. Bódi et al.)

generalized clones (constraints) (S. Foldes, L. Hellerstein, M. Couceiro)
no superposition, not clone

invariance group

coalition lattice (conjecture)

Monotonicity and thresholdness

It is easy to see that threshold functions with positive weights and a threshold value are isotone.

However, an isotone Boolean function is not necessarily threshold, e.g. $f = x \cdot y \vee w \cdot z$ is isotone, but not a threshold function.

Threshold functions

$f = x \cdot y \vee w \cdot z$ is isotone, but not a threshold function because its invariance group is
$D8 = \{(), (1324), (12)(34), (1423), (12), (34), (13)(24), (14)(23)\}$

Theorem (1994.)
For every n-ary threshold function f there exists a partition C_f of the set of variables X such that the invariance group G of f consists of exactly those permuations of S_X which preserve each block of C_f.

I.e. the invariance groups of threshold functions are of the following form: direct product of symmetric groups.

Lattice-induced threshold functions

Let L be a complete lattice in which the bottom and the top are (also) denoted by 0 and 1 respectively; however, it is clear from the context whether 0 (1) is a component in some $(x_1, \ldots, x_n) \in \{0,1\}^n$, or it is from L.

For $x \in \{0,1\}$, and $w \in L$, we define a mapping $L \times \{0,1\}$ into L denoted by "$\cdot$", as follows:

$$w \cdot x := \begin{cases} w, & \text{if} \quad x = 1 \\ 0, & \text{if} \quad x = 0. \end{cases} \tag{1}$$

A function $f : \{0,1\}^n \to \{0,1\}$ is a **lattice-induced threshold function**, if there is a complete lattice L and $w_1, \ldots, w_n, t \in L$, such that

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \bigvee_{i=1}^{n} (w_i \cdot x_i) \geq t. \tag{2}$$

Lattice-induced threshold functions

Proposition

Every lattice-induced threshold function is isotone.

Theorem

Every isotone Boolean function is a lattice-induced threshold function.

Remark

The corresponding lattice in each case can be the free distributive lattice with n generators.

Sketch of the proof of the Theorem

We prove that for every $n \in \mathbb{N}$, there is a lattice L such that every isotone Boolean function is a lattice induced threshold function over L. Let $n \in \mathbb{N}$.

We take L to be a free distributive lattice with n generators $w_1, w_2, \ldots, w_n$.

Recall that every element in a free distributive lattice can be uniquely represented in a "conjunctive normal form" by means of generators (i.e., every element is a meet of elements of the type $\bigvee_{i \in J} w_j$, where $J \subseteq \{1, \ldots, n\}$.)

For $x, y \in L$, if $x = \bigwedge_{k=1}^{p} \bigvee_{j \in I_k} w_j$ and $y = \bigwedge_{k=1}^{l} \bigvee_{s \in J_k} w_s$, $x \leq y$ if and only if for every $u \in \{1, \ldots, l\}$ there is $k \in \{1, \ldots, p\}$ such that $I_k \subseteq J_u$. (*)

Sketch of the proof of the Theorem

Let $f : \{0,1\}^n \to \{0,1\}$ be an isotone Boolean function. Let F be the corresponding order semi-filter on $\{0,1\}^n$.

Further, let $m_1, \ldots, m_p$ be minimal elements of this semi-filter. Let $I_1, \ldots, I_p$ be subsets of $\{1, 2, \ldots, n\}$, i.e., sets of indices, such that $i \in I_k$ if and only if $i - th$ coordinate of m_k is equal to 1.

For the threshold $t \in L$ associated to the given function f we take

$$t = \bigwedge_{k=1}^{p} \bigvee_{j \in I_k} w_j.$$

Now, it is straightforward to show that

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \bigvee_{i=1}^{n} (w_i \cdot x_i) \geq t. \tag{3}$$

Lattice-valued Boolean functions, cuts

A function $f : \{0,1\}^n \to L$, where L is a complete lattice, is called a **lattice valued** (*L*-**valued**) **Boolean function**.

For $f : \{0,1\}^n \to L$ and $p \in L$, a cut set (cut) f_p is a subset of $\{0,1\}^n$:

$$f_p = \{x \in \{0,1\}^n \mid f(x) \geq p\}.$$

In other words, a p-cut of $\mu : B \to L$ is the inverse image of the principal filter $\uparrow p$, generated by $p \in L$:

$$\mu_p = \mu^{-1}(\uparrow p). \quad (4)$$

It is obvious that for $p, q \in L$,

from $p \leq q$ it follows that $\mu_q \subseteq \mu_p$.

Lattice-valued Boolean functions, cuts

Lemma

If $\mu : B \to L$ is an L-valued function on the set B, then the collection μ_L of all cuts of μ is a closure system on B under the set-inclusion.

Proposition

Let $\mathcal{F}$ be a closure system over a set B. Then there is a lattice L and an L-valued function $\mu : B \to L$, such that the collection μ_L of cuts of μ is $\mathcal{F}$.

Proof

It is straightforward to check that a required lattice L is the collection $\mathcal{F}$ ordered dually to the set-inclusion, and that $\mu : B \to L$ can be defined (as $\overline{x}$ in (12)):

$$\mu(x) = \bigcap \{f \in \mathcal{F} \mid x \in f\}. \tag{5}$$

Lattice-valued up-sets

An L-valued Boolean function $\mu : B \to L$ is called a **lattice valued** (**L-valued**) **up-set**, if from $x \leq y$ it follows that $\mu(x) \leq \mu(y)$.

Lemma

Let B be a Boolean lattice and $\mu : B \to L$ an L-valued Boolean function. Then μ is an L-valued up-set on B if and only if all the cuts of μ are up-sets (order-filters, semi-filters) on B.

Representation of lattice-valued up-sets by cuts

Let $B = (\{0,1\}^n, \leq)$, $n \in \mathbb{N}$, L_D a free distributive lattice with n generators $w_1, \ldots, w_n$ and $\overline{\beta} : B \to L_D$, an L_D-valued function on B defined in the following way: for $x = (x_1, \ldots, x_n) \in B$

$$\overline{\beta}(x) = \bigvee_{i=1}^{n} (w_i \cdot x_i), \tag{6}$$

where the function " $\cdot$" is defined by (1). By the definition, $\overline{\beta}$ is uniquely (up to a permutation of generators w_i) determined by a finite Boolean lattice $B = (\{0,1\}^n, \leq)$, i.e., by a positive integer n.

Observations

The L_D-valued function $\overline{\beta}$ defined by (6) is an L_D-valued up-set on B. Every cut of $\overline{\beta}$ is an up-set of a finite Boolean lattice $B = (\{0,1\}^n, \leq)$, $n \in \mathbb{N}$.

Representation of lattice-valued up-sets by cuts

Theorem

Every up-set of a finite Boolean lattice $B = (\{0,1\}^n, \leq)$, $n \in \mathbb{N}$, is a cut of $\overline{\beta}$.

Corollary

The collection of cuts of every L-valued up-set on B (for any L) is contained in the collection of cuts of $\overline{\beta}$.

Linear combinations

Let $B = (\{0,1\}^n, \leq)$ be a Boolean lattice, L a complete lattice, $x = (x_1, \dots, x_n) \in B$ and $w_1, \dots, w_n \in L$. Further, let the binary function "$\cdot$" which maps $L \times \{0,1\}$ into L be defined by (1). Then we call the term

$$\bigvee_{i=1}^{n} (w_i \cdot x_i), \tag{7}$$

a **linear combination** of elements $w_1, \dots, w_n$ from L.

Observe also that in the case of formula (6), the corresponding L_D-valued function is $\overline{\beta}$ and the following is obviously true: *the closure system consisting of all up-sets on B is the collection of cuts of $\overline{\beta}$.*

Generalization

Starting with a closure system $\mathcal{F}$ consisting of some up-sets on $B = (\{0,1\}^n, \leq)$, and we try to find a lattice L and $w_1, \dots, w_n \in L$, such that the family of cuts of the function

$$\bigvee_{i=1}^{n} (w_i \cdot x_i), \tag{8}$$

over this lattice (a linear combination of elements from L) coincides with $\mathcal{F}$.

The answer to the above problem is not generally positive, as shown by the following example.

Example

Let $B = (\{0,1\}^2, \leq)$ be the four element Boolean lattice and

$\mathcal{F} = \{\{(1,1)\}, \{(1,1),(1,0)\}, \{(1,1),(1,0),(0,1)\}, \{(1,1),(1,0),(0,1),(0,0)$

a closure system consisting of some up-sets on B.
We show that there is no lattice L, hence neither there is an L-valued function $\nu : B \to L$, such that there are $w_1, w_2 \in L$ fulfilling that for all $x_1, x_2 \in \{0,1\}$

$$\nu(x_1, x_2) = (w_1 \cdot x_1) \vee (w_2 \cdot x_2)$$

and that the collections of cuts of ν is $\mathcal{F}$.

Example

Indeed, suppose that there is a lattice L and elements $w_1, w_2 \in L$, such that $\nu(x_1, x_2) = (w_1 \cdot x_1) \vee (w_2 \cdot x_2)$, for all $x_1, x_2 \in \{0, 1\}$.
Then, $\nu(0,0) = 0 \in L_1$, $\nu(0,1) = w_2$, $\nu(1,0) = w_1$ and $\nu(1,1) = w_1 \vee w_2$.
Now, since the cuts of ν are supposed to be elements from $\mathcal{F}$, and cuts are up-sets in B, we have that $\nu_{w_1 \vee w_2} = \{(1,1)\}$, and $w_1 \vee w_2$ would be the top element of the lattice L: otherwise the empty set would be a cut of this lattice valued function.

Lemma Let $\mu : B \to L$ be a lattice valued up-set, such that its collection of cuts is $\mathcal{F}$. If $\uparrow a \in \mathcal{F}$ and $\mu(a) = p$, then $\mu_p = \uparrow a$.

Now $\nu_{w_1} = \{(1,1), (1,0)\}$, $\nu_{w_2} = \{(1,1), (1,0), (0,1)\}$ and $\nu_0 = \{(1,1), (1,0), (0,1), (0,0)\}$. Since $(1,0) \in \nu_{w_2}$, we have that $\nu(1,0) \geq w_2$, i.e., $w_1 \geq w_2$. Hence, $w_1 \vee w_2 = w_1$, which contradicts $\nu_{w_1 \vee w_2} \neq \nu_{w_1}$.
Hence, the up-sets from the collection $\mathcal{F}$ cannot be represented as cuts of an L-valued function in the form (8). □

First problem

Find necessary and sufficient conditions under which a lattice valued up-set $\mu : B \to L$ on a finite Boolean lattice $B = (\{0,1\}^n, \leq)$ can be represented by the linear combination

$$\mu(x) = \bigvee_{i=1}^{n} (w_i \cdot x_i)$$

over L $(x = (x_1, \ldots, x_n) \in \{0,1\}^n, w_1, \ldots, w_n \in L$.

Definition

Starting with finite lattices M and L with the bottom elements 0_M and 0_L respectively, we say that a mapping $\mu : M \to L$ is a 0–$\vee$–**homomorphism**, if for all $x, y \in M$

$$\begin{aligned} \mu(x \vee y) &= \mu(x) \vee \mu(y) \quad \text{and} \\ \mu(0_M) &= 0_L. \end{aligned}$$

In particular, if μ maps a Boolean lattice $B = \{0,1\}^n$ into L, the condition that μ is a 0–$\vee$–homomorphism from B to L is equivalent with the following two conditions (observe that B is finite): for every collection A of some atoms in B

$$(i) \;\; \mu(\bigvee A) = \bigvee \mu(A) \quad \text{and} \quad (ii) \;\; \mu(0,\ldots,0) = 0. \tag{9}$$

Theorem (Characterization)

Let $B = (\{0,1\}^n, \leq)$ be a finite Boolean lattice and L an arbitrary complete lattice. Then an L-valued Boolean function $\mu : \{0,1\}^n \to L$ can be represented in the form

$$\mu(x) = \bigvee_{i=1}^{n} (w_i \cdot x_i)$$

for some elements $w_1, \ldots, w_n \in L$ if and only if μ as a mapping from B to L is a 0–$\vee$–homomorphism.

Next problem

Next we analyze the same problem (representability by linear combination), for closure systems of some up-sets on $B = (\{0,1\}^n, \leq)$.

If $\mathcal{F}$ is a closure system consisting of some up-sets on $B = (\{0,1\}^n, \leq)$, then for $x \in P$, we define

$$\overline{x} = \bigcap \{f \in \mathcal{F} \mid x \in f\}. \tag{10}$$

Proposition Let $\mathcal{F}$ be a closure system of some up-sets on B. If $\mathcal{F}$ is a family of cuts of an L-valued up-set μ on B represented by a linear combination over L, then the following holds: for all $x, y \in B$

$$\text{from } \overline{x} \subseteq \overline{y} \text{ it follows that } \overline{x \vee y} = \overline{x}. \tag{11}$$

For the proof of this Proposition: Properties of closure systems consisting of up-sets

Lemma

Let $\mathcal{F}$ be a closure system consisting of some up-sets on a poset $(P, \leq)$. For $x \in P$, denote

$$\overline{x} = \bigcap \{f \in \mathcal{F} \mid x \in f\}. \tag{12}$$

Then, for all $x, y, z \in P$, the following is true:

a) $x \leq y$ implies $\overline{y} \subseteq \overline{x}$.

b) $x \in \overline{x}$.

c) $\uparrow x \subseteq \overline{x}$.

d) If $z \in \overline{x}$ then $\overline{z} \subseteq \overline{x}$.

For the proof of this Proposition: Canonical representation

Let $\mu : B \to L$ be an L-valued Boolean function and $(\mu_L, \leq)$ the poset with $\mu_L = \{\mu_p \mid p \in L\}$ (the collection of cuts of μ) and the order $\leq$ is the inverse of the set-inclusion: for $\mu_p, \mu_q \in \mu_L$,

$$\mu_p \leq \mu_q \text{ if and only if } \mu_q \subseteq \mu_p.$$

Lemma

$(\mu_L, \leq)$ is a complete lattice.

We introduce the mapping $\widehat{\mu} : B \to \mu_L$ by the construction by

$$\widehat{\mu}(x) := \bigcap \{\mu_p \in \mu_L \mid x \in \mu_p\}. \tag{13}$$

We say that the lattice valued function $\widehat{\mu}$ is the **canonical representation** of μ.

For the proof of this Proposition: another Proposition

Proposition

If $\mu : B \to L$ is an L-valued function on the set B and $\mu(a) = \mu(b) \vee \mu(c)$ for some $a, b, c \in B$, then also for the canonical representation $\widehat{\mu}$ of μ, $\widehat{\mu}(a) = \widehat{\mu}(b) \vee \widehat{\mu}(c)$ analogously holds.

Remark

The equality $\widehat{\mu}(a) = \widehat{\mu}(b) \vee \widehat{\mu}(c)$ equivalently can be presented as $\widehat{\mu}(a) = \widehat{\mu}(b) \cap \widehat{\mu}(c)$, since the order in $(\mu_L, \leq)$ is dual to set-inclusion; therefore the join in this lattice is actually the set intersection.

Proof of this Proposition

By assumption, a closure system $\mathcal{F}$ on B consisting of some up-sets on B is a collection of cuts of a lattice valued function $\mu : \{0,1\}^n \to L$, for some lattice L, i.e., $\mu_L = \mathcal{F}$. We also assume that μ can be represented as a linear combination over L.

Now, if $\widehat{\mu} : B \to \mathcal{F}$ is the canonical representation of μ, then by (14), for every $x \in B$, $\widehat{\mu}(x) = \overline{x}$. Suppose that there are $x, y \in B$ such that $\overline{x} \subseteq \overline{y}$ and $\overline{x \vee y} \neq \overline{x}$.

Now, we have that $\widehat{\mu}(x) \vee \widehat{\mu}(y) = \widehat{\mu}(x) \cap \widehat{\mu}(y) = \overline{x} \cap \overline{y} = \overline{x} \neq \overline{x \vee y} = \widehat{\mu}(x \vee y)$. Applying Proposition, by contraposition we obtain that $\mu(x) \vee \mu(y) \neq \mu(x \vee y)$. Now, by Theorem, μ is not representable by a linear combination.

Remark

Canonical representation:
$\widehat{\mu} : B \to \mu_L$

$$\widehat{\mu}(x) := \bigcap\{\mu_p \in \mu_L \mid x \in \mu_p\}. \quad (14)$$

Proposition
If $\mu : B \to L$ is an L-valued function on B and $\mu(a) = \mu(b) \vee \mu(c)$ for some $a, b, c \in B$, then also for the canonical representation $\widehat{\mu}$ of μ, $\widehat{\mu}(a) = \widehat{\mu}(b) \vee \widehat{\mu}(c)$ analogously holds.

The opposite implication to the one in this Proposition does not hold in general. Indeed, let $B = \{a, b, c, d\}$, and let L be the lattice given in Figure 1.

Remark

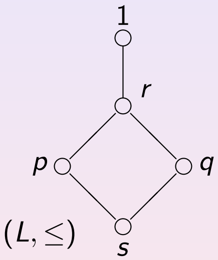

Figure 1

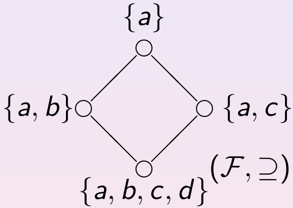

Figure 2

Remark

We define an L-valued function $\mu : B \to L$ as follows:

$$\mu = \begin{pmatrix} a & b & c & d \\ 1 & p & q & s \end{pmatrix}.$$

The cuts of μ are:
$\mu_L = \{\mu_1 = \mu_r = \{a\}, \mu_p = \{a, b\}, \mu_q = \{a, c\}, \ \mu_s = \{a, b, c, d\}\}$.
The lattice $(\mu_L, \supseteq)$ is depicted in Figure 2. The canonical representation of this lattice valued function is $\widehat{\mu} : B \to \mu_L$ and it is given by

$$\widehat{\mu} = \begin{pmatrix} a & b & c & d \\ \{a\} & \{a, b\} & \{a, c\} & \{a, b, c, d\} \end{pmatrix}.$$

Now, observe that $\widehat{\mu}(a) = \widehat{\mu}(b) \vee \widehat{\mu}(c)$. However, it is not true that $\mu(a) = \mu(b) \vee \mu(c)$. □

Example

Let $B = (\{0,1\}^2, \leq)$ be the four element Boolean lattice and

$\mathcal{F} = \{\{(1,1)\}, \{(1,1),(1,0)\}, \{(1,1),(1,0),(0,1)\}, \{(1,1),(1,0),(0,1),(0,0)$

a closure system consisting of some up-sets on B.

We already proved that this family is not the collection of cuts for a lattice valued function representable by a linear combination. If we define a mapping from B to $\mathcal{F}$ by

$$\overline{x} = \bigcap\{f \in \mathcal{F} \mid x \in f\}, \tag{15}$$

then the condition from Proposition that
for all $x, y \in B$

$$\text{from } \overline{x} \subseteq \overline{y} \text{ it follows that } \overline{x \vee y} = \overline{x}. \tag{16}$$

is not satisfied.

Theorem

Let $\mathcal{F}$ be a closure system of some up-sets on a Boolean algebra B and for $x \in B$, define $\overline{x}$ by (15):

$$\overline{x} = \bigcap \{ f \in \mathcal{F} \mid x \in f \}.$$

The following conditions are equivalent:

(i) for all $x, y \in B$

$$\text{from } \overline{x} \subseteq \overline{y} \text{ it follows that } \overline{x \vee y} = \overline{x}.$$

(ii) for all $x, y \in B$, $\overline{x \vee y} = \overline{x} \cap \overline{y}$.

(iii) There is a lattice L such that $\mathcal{F}$ is a family of cuts of an L-valued up-set on B which can be represented as a linear combination over L.

Last problem

Given a lattice valued up-set $\mu : B \to L$ on a finite Boolean lattice $B = \{0,1\}^n$, find a lattice L_1 and a lattice valued function $\nu : B \to L_1$ defined by the formula

$$\nu(x) = \bigvee_{i=1}^{n} (w_i \cdot x_i)$$

where $w_1, \ldots, w_n \in L_1$, such that the collections of cuts of μ and ν coincide.

Last problem solution

Corollary

For a lattice valued up-set $\mu : B \to L$ on a finite Boolean lattice $B = \{0, 1\}^n$, there is a lattice L_1 and a lattice valued function $\nu : B \to L_1$ defined by the formula

$$\nu(x) = \bigvee_{i=1}^{n} (w_i \cdot x_i)$$

such that the collections of cuts of μ and ν coincide if and only if $\overline{x \vee y} = \overline{x} \cap \overline{y}$ for $x, y \in B$, where the operator $\bar{}$ is defined by cuts of μ: define $\overline{x}$ by (15):

$$\overline{x} = \bigcap \{f \in \mathcal{F} \mid x \in f\}.$$

The publication/presentation is supported by the European Union and co-funded by the European Social Fund.
Project title: "Telemedicine-focused research activities on the field of Matematics, Informatics and Medical sciences"
Project number: TÁMOP-4.2.2.A-11/1/KONV-2012-0073