Pairwise comparable or disjoint elemets in a poset

Eszter K. Horváth, Szeged

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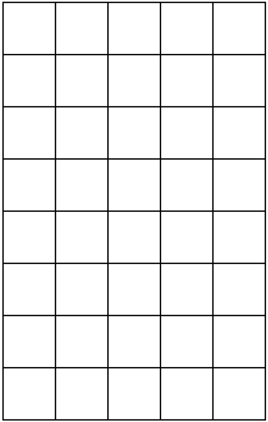
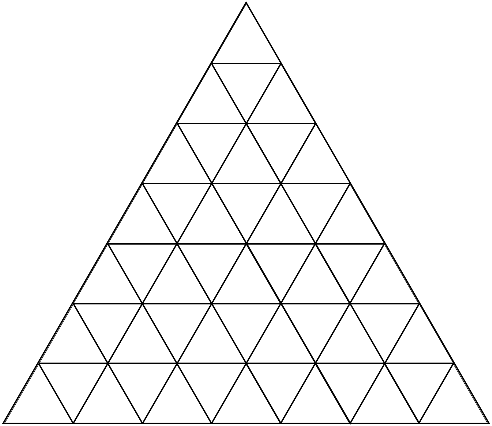
Island

Digital islands

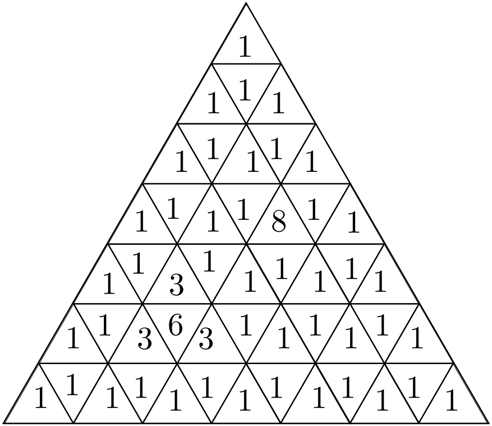




Grid

Digital islands

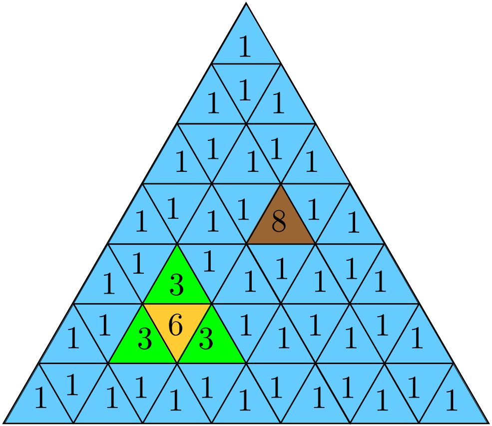
1	1	1	1	1
1	8	8	1	1
1	1	1	1	1
1	3	3	3	1
1	6	6	3	1
1	6	6	3	1
1	3	3	3	1
1	1	1	1	1



Grid, height function

Digital islands

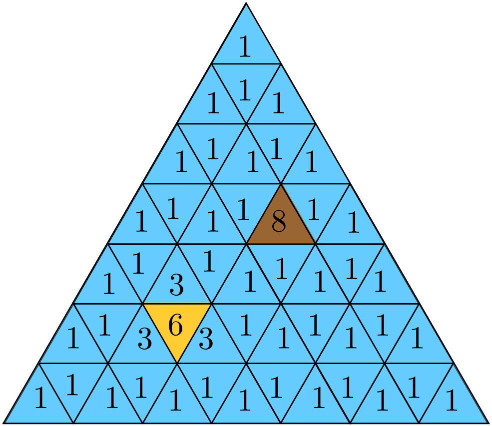
1	1	1	1	1
1	8	8	1	1
1	1	1	1	1
1	3	3	3	1
1	6	6	3	1
1	6	6	3	1
1	3	3	3	1
1	1	1	1	1



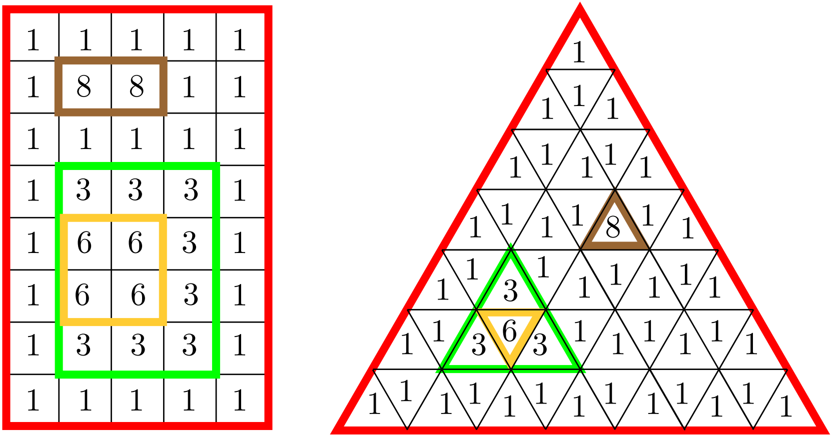
Grid, height function, water level: 2

Digital islands

1	1	1	1	1
1	8	8	1	1
1	1	1	1	1
1	3	3	3	1
1	6	6	3	1
1	6	6	3	1
1	3	3	3	1
1	1	1	1	1

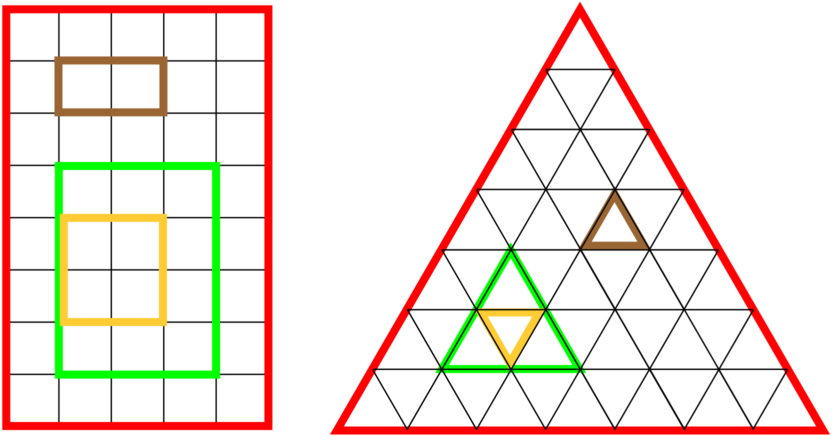


Grid, height function, water level: 4



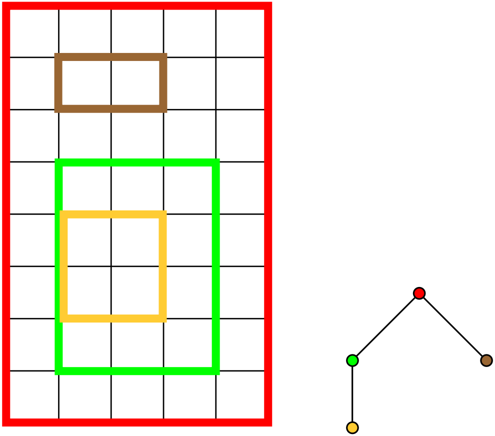
Grid, height function, island system

Digital islands



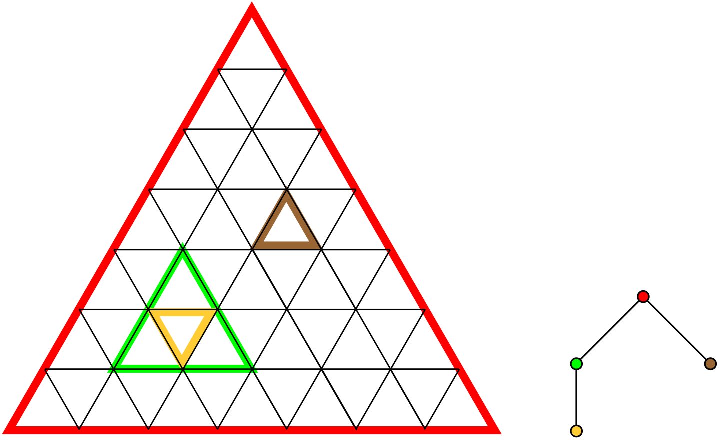
CD-independent: Comparable or Disjoint

Digital islands



CD-independent: Comparable or Disjoint

Tree



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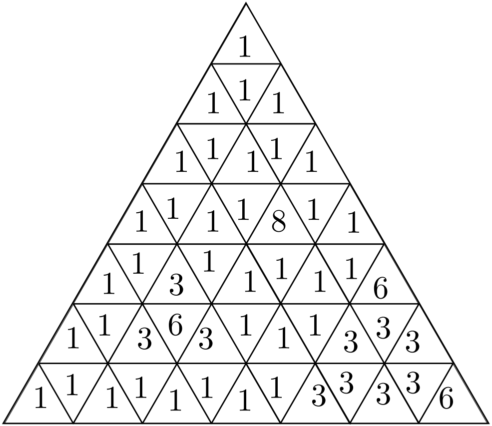

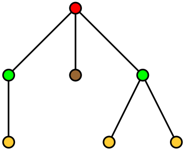

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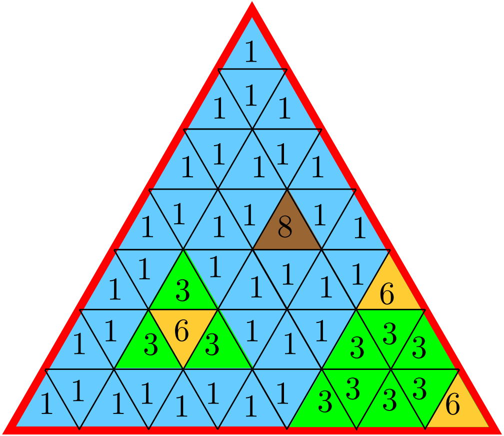

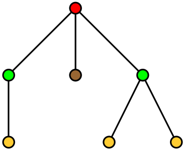

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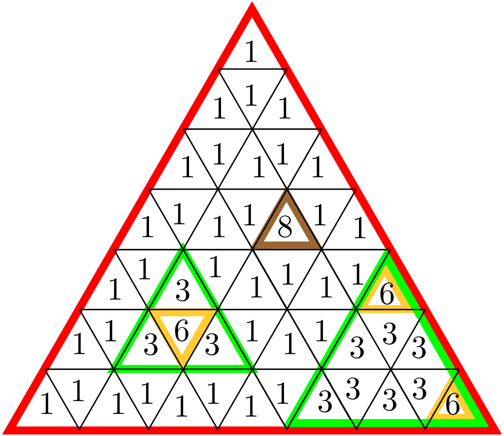

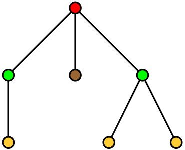

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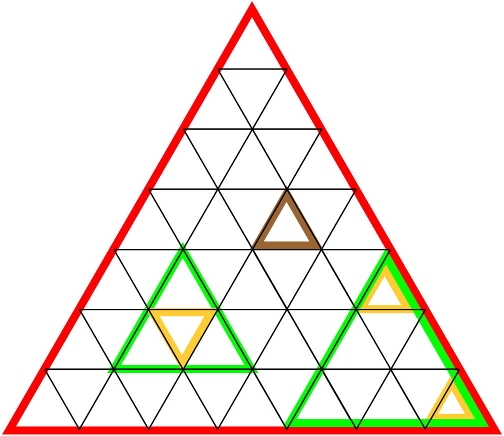

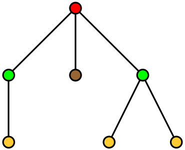

CD-independent: Comparable or Disjoint

CD-independent subsets in distributive lattices

G. Czédli, M. Hartmann and E. T. Schmidt: CD-independent subsets in distributive lattices, Publicationes Mathematicae Debrecen, 74/1-2 (2009).

Any two CD-bases of a finite distributive lattice have the same number of elements.

If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

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If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set, and let $a, b \in P$.

The elements a and b are called *disjoint* and we write $a \perp b$ if

either $\mathbb{P}$ has least element $0 \in P$ and $\inf\{a, b\} = 0$,
or $\mathbb{P}$ is without 0 and the elements a and b have no common lowerbound.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$, or $x \perp y$ holds.

Maximal CD-independent sets (with respect to $\subseteq$) are called *CD-bases* in $\mathbb{P}$.

CD-independent subsets in posets

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Definition

A nonempty set D of nonzero elements of P is called a *set of pairwise disjoint element* in $\mathbb{P}$ if $x \perp y$ holds for all $x, y \in D$, $x \neq y$; if $\mathbb{P}$ has 0-element, then $\{0\}$ is considered to be a set of pairwise disjont elements, too.

D is a set of pairwise disjoint elements if and only if it is a CD-independent antichain in $\mathbb{P}$.

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Order ideals

Let $X \subseteq P$.

The order ideal $\{y \in P \mid y \leq x \text{ for some } x \in X\}$ is denoted by $\downarrow X$.

The order-ideals of any poset form a (distributive) lattice with respect to $\subseteq$.

So, the antichains of a poset can be ordered as follows:

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$\leqslant$ is a partial order.

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If A_1, A_2 are antichains in $\mathbb{P}$, then we say that A_1 *is dominated by* A_2, and we denote it by $A_1 \leqslant A_2$ if $\downarrow A_1 \subseteq \downarrow A_2$.

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Let $\mathcal{D}(\mathbb{P})$ denote the set of all sets of pairwise disjont elements of $\mathbb{P}$.

As sets of pairwise disjont elements of $\mathbb{P}$ are also antichains, restricting $\leqslant$ to $\mathcal{D}(\mathbb{P})$, we obtain a poset $(\mathcal{D}(\mathbb{P}), \leqslant)$.

The connection between CD-bases of a poset $\mathbb{P}$ and the poset $(\mathcal{D}(\mathbb{P}), \leqslant)$ is shown by the next theorem:

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Theorem (E. K. H., S. Radeleczki)

Let B *be a CD-base of a finite poset* $(P, \leq)$, *and* let $|B| = n$.

Then there exists a maximal chain $\{D_i\}_{1 \leq i \leq n}$ *in* $\mathcal{D}(P)$ *such that* $B = \bigcup_{i=1}^{n} D_i$.

Moreover, for any maximal chain $\{D_i\}_{1 \leq i \leq m}$ *in* $\mathcal{D}(P)$ *the set* $D = \bigcup_{i=1}^{m} D_i$ *is a CD-base in* $(P, \leq)$ *with* $|D| = m$.

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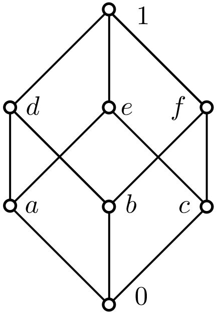
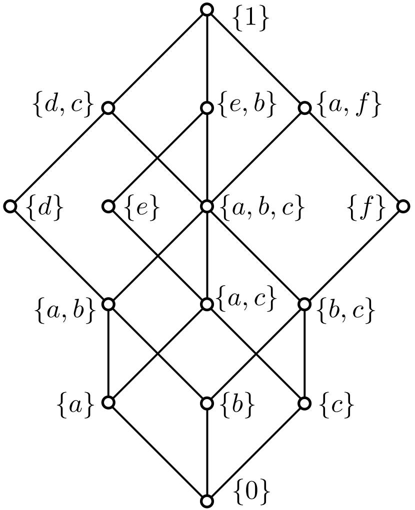
$\mathbb{P}_1$ és $\mathcal{D}(\mathbb{P}_1)$





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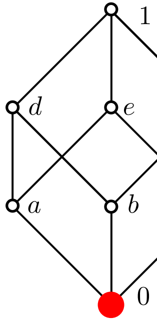

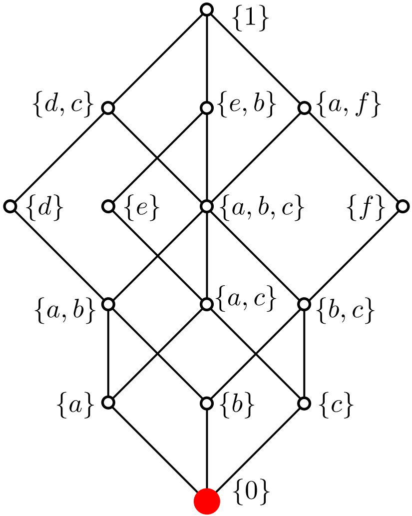

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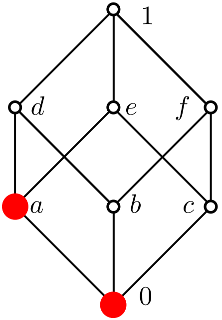

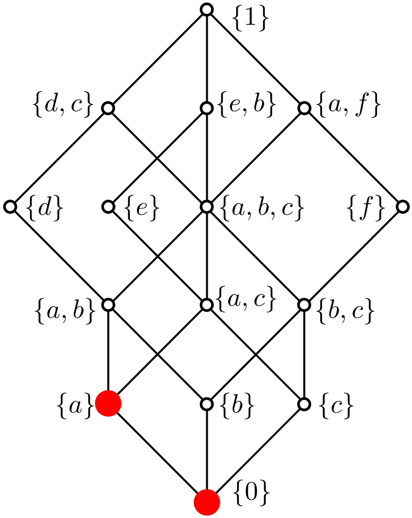

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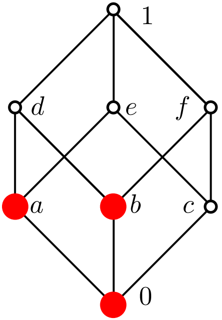

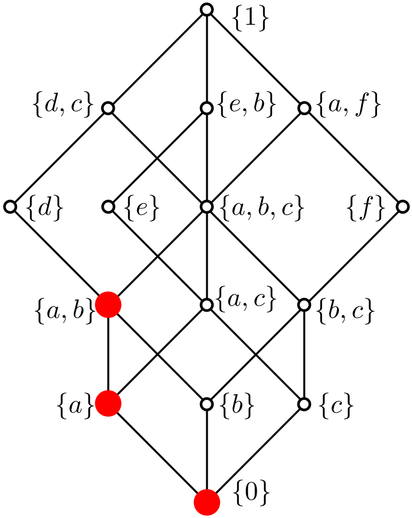

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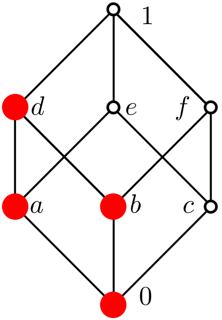

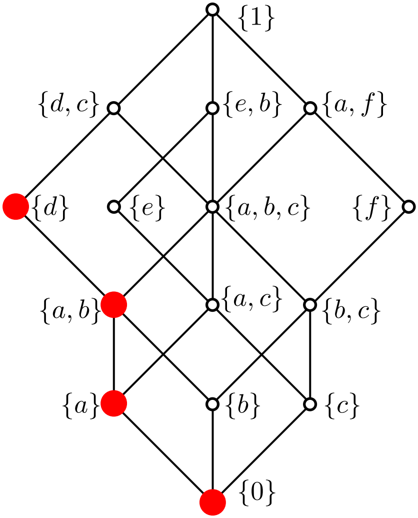

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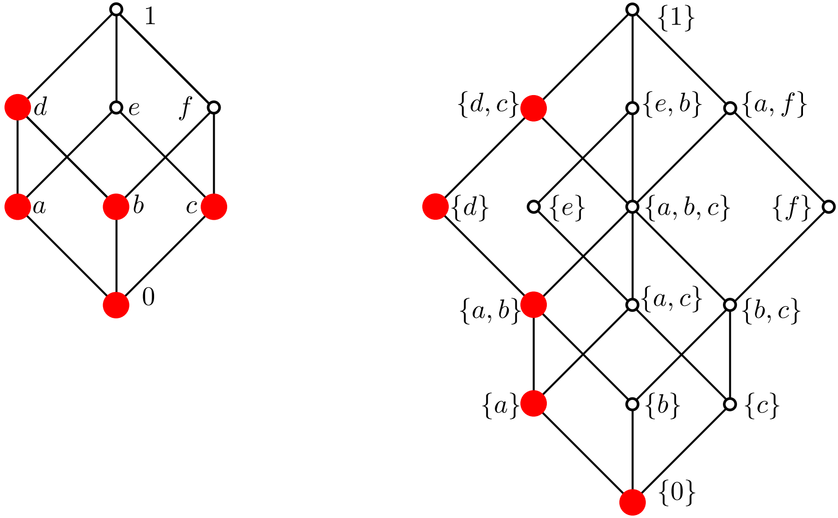

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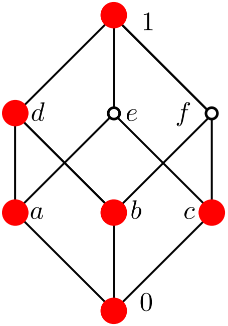

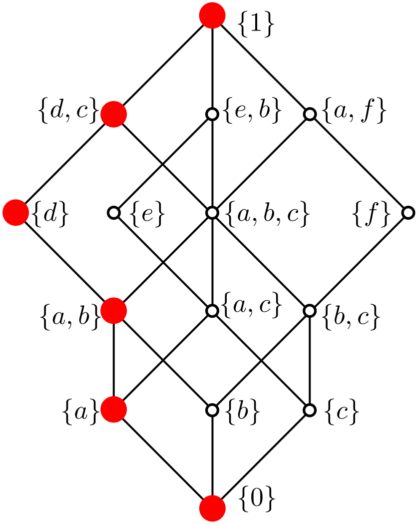

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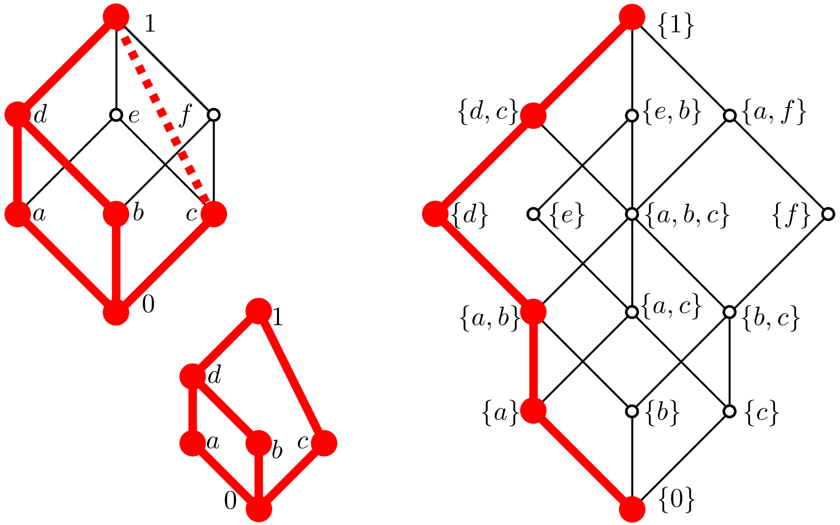

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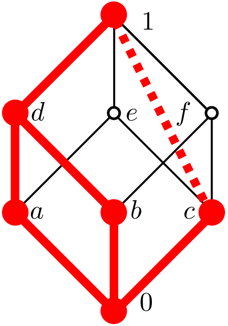

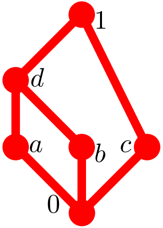

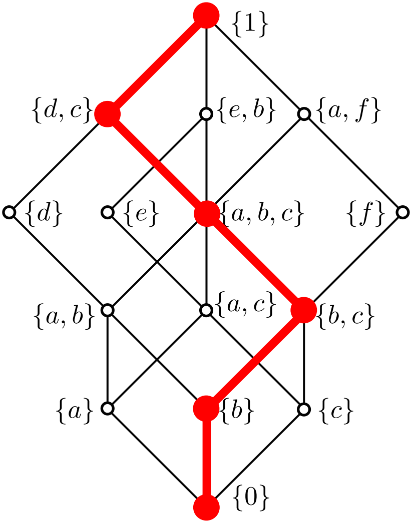

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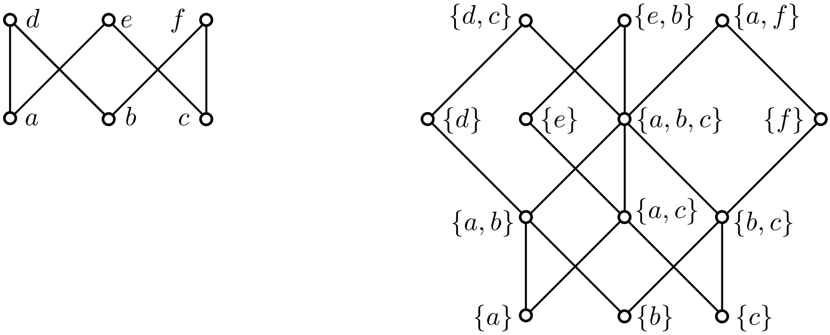
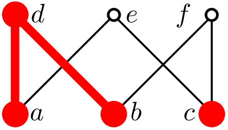
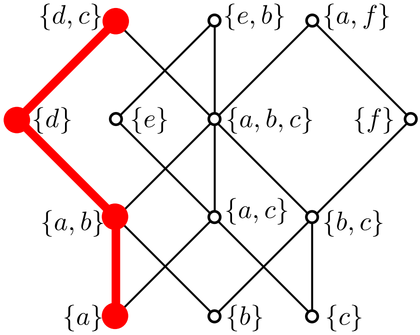
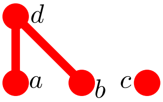


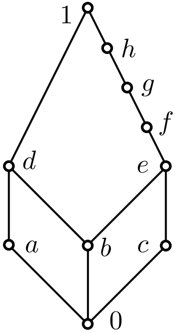
$\mathbb{P}_2$ és $\mathcal{D}(\mathbb{P}_2)$, maximális lánc $\mathcal{D}(\mathbb{P}_2)$-ben

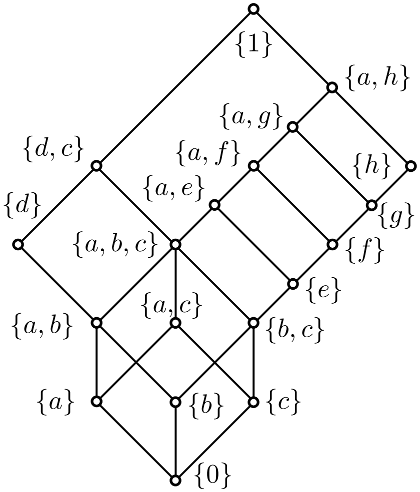




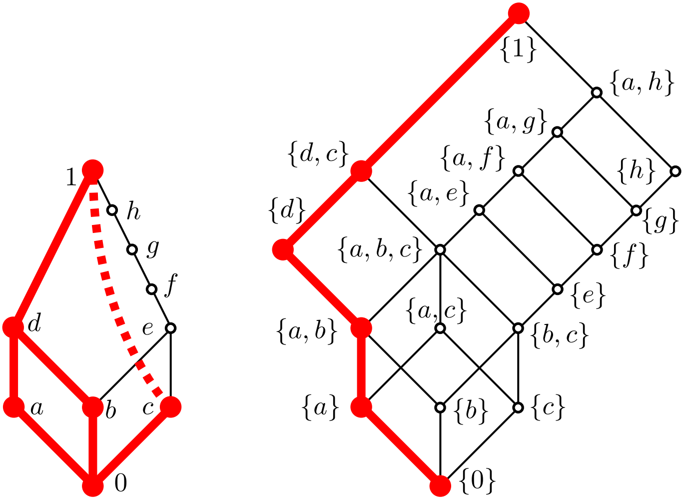


$\mathbb{P}_3$ and $\mathcal{D}(\mathbb{P}_3)$





$\mathbb{P}_3$ és $\mathcal{D}(\mathbb{P}_3)$, maximális lánc $\mathcal{D}(\mathbb{P}_3)$-ban



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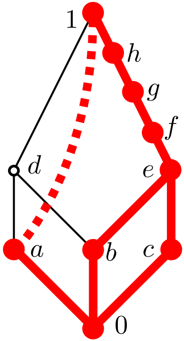

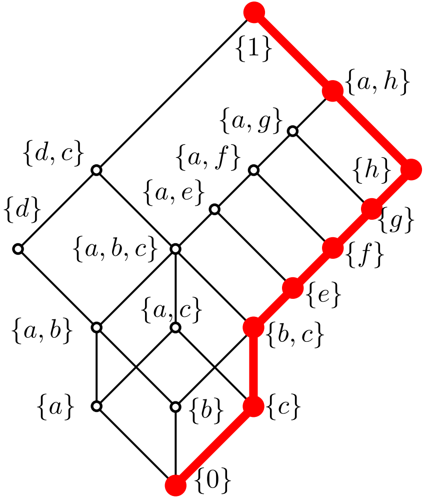

Proof

Any poset $(P, \leq)$ without least element becomes a poset with 0 by adding a new element 0 to P. In this way both the number of the elements in the CD-bases of $\mathbb{P}$ and the length of the maximal chains in $\mathcal{D}(P)$ are increased by one. Therefore, without loss of generality we may assume that $\mathbb{P}$ contains 0 and $|P| \geq 2$.

To prove the first part of Theorem 1.5, assume that B is a CD-base in $\mathbb{P}$. Then clearly $0 \in B$ and $|B| \geq 2$. Let $D_1 = \max(B)$. Take any $m_1 \in D_1$ and form $D_2 = \max(B \setminus \{m_1\})$. Then, in view of Lemma 1.7, $D_1, D_2 \in \mathcal{D}(P)$, $D_1 \succ D_2$, and D_1 is a maximal element in $\mathcal{D}(P)$. Further, suppose that we already have a sequence (D_i, m_i), $1 \leq i \leq k$ $(k \geq 2)$ such that $m_i \in D_i$, $D_1 \succ \ldots \succ D_k$ in $\mathcal{D}(P)$ and

$$D_k = \max(B \setminus \{m_1, \ldots, m_{k-1}\}).$$

We show that for all $i \in \{1, \ldots, k-1\}$ and $d \in D_k$ we have $d \not\geq m_i$. (5)

This is clear for $i = 1$ since $m_1 \in \max(B)$ and $d \in B$, $d \neq m_1$. If $2 \leq i \leq k-1$, then $m_i \in \max(B \setminus \{m_1, \ldots, m_{i-1}\})$, and since $d \in B \setminus \{m_1, \ldots, m_{i-1}\}$, $d \geq m_i$ would imply $m_i = d \in B \setminus \{m_1, \ldots, m_i, \ldots, m_{k-1}\}$, a contradiction.
Further, if $|B \setminus \{m_1, \ldots, m_{k-1}\}| \geq 2$, then form the next set $D_{k+1} := \max(B \setminus \{m_1, \ldots, m_{k-1}, m_k\})$ and let $m_{k+1} \in D_{k+1}$. Since D_{k+1} is an antichain in the CD-base B, it is a disjoint set, and clearly $D_{k+1} \neq D_k$.
In order to prove $D_k \succ D_{k+1}$, consider the subposet $(I(D_k), \leq)$. By Proposition 1.4, $B_k := B \cap I(D_k)$ is a CD-base in $(I(D_k), \leq)$. We claim that

$$B_k = B \setminus \{m_1, \ldots, m_{k-1}\}.$$

Indeed, $D_k = \max(B \setminus \{m_1, \ldots, m_{k-1}\})$ implies $B \setminus \{m_1, \ldots, m_{k-1}\} \subseteq B \cap I(D_k) = B_k$. On the other hand, (5) implies $\{m_1, \ldots, m_{k-1}\} \cap I(D_k) = \emptyset$, whence we get $B_k \subseteq B \setminus \{m_1, \ldots, m_{k-1}\}$, proving our claim. Hence $D_k = \max(B_k)$, and $D_{k+1} = \max\,(B \setminus \{m_1, \ldots, m_{k-1}, m_k\}) = \max(B_k \setminus \{m_k\})$.
Now, by applying Lemma 1.7, we obtain that $D_{k+1} \prec D_k$ holds in $\mathcal{D}(I(D_k))$. Finally, observe that any $S \in \mathcal{D}(P)$ with $S \leqslant D_k$ is also a disjoint set in $(I(D_k), \leq)$ according to (A). Moreover, since $D_{k+1} \prec D_k$ holds in $\mathcal{D}(I(D_k))$, $D_{k+1} \leqslant S \leqslant D_k$ implies either $S = D_k$ or $S = D_{k+1}$. This means that $D_{k+1} \prec D_k$ holds in $\mathcal{D}(P)$, too.
Thus we conclude by induction that the chain $D_1 \succ \ldots \succ D_k \succ \ldots$ can be continued as long as the condition $|B \setminus \{m_1, \ldots, m_{k-1}\}| \geq 2$ is still valid. Since P is finite, the process stops after finite - let say $n-1$ steps, when $|B \setminus \{m_1, \ldots, m_{n-1}\}| = 1$, and the last set is $D_n = B \setminus \{m_1, \ldots, m_{n-1}\}$. As $0 \in B$, and since $0 \notin \max(X)$ whenever $|X| \geq 2$, we get $\{0\} = B \setminus \{m_1, \ldots, m_{n-1}\} = D_n$. As D_1 is a maximal element and $D_n = \{0\}$ is the least element in $\mathcal{D}(P)$, $D_1 \succ \ldots \succ D_n$ is a maximal chain in $\mathcal{D}(P)$. Since $B = \{m_1, \ldots, m_{n-1}, 0\}$, we obtain $|B| = n$.

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Thus we conclude by induction that the chain $D_1 \succ \ldots \succ D_k \succ \ldots$ can be continued as long as the condition $|B \setminus \{m_1, \ldots, m_{k-1}\}| \geq 2$ is still valid. Since P is finite, the process stops after finite - let say $n-1$ steps, when $|B \setminus \{m_1, \ldots, m_{n-1}\}| = 1$, and the last set is $D_n = B \setminus \{m_1, \ldots, m_{n-1}\}$. As $0 \in B$, and since $0 \notin \max(X)$ whenever $|X| \geq 2$, we get $\{0\} = B \setminus \{m_1, \ldots, m_{n-1}\} = D_n$. As D_1 is a maximal element and $D_n = \{0\}$ is the least element in $\mathcal{D}(P)$, $D_1 \succ \ldots \succ D_n$ is a maximal chain in $\mathcal{D}(P)$. Since $B = \{m_1, \ldots, m_{n-1}, 0\}$, we obtain $|B| = n$.

Bizonyítás

To prove the second part of Theorem 1.5, assume that the disjoint sets $D_1, \ldots, D_m$ form a maximal chain $\mathcal{C}$:

$$D_1 \prec \ldots \prec D_m$$

in $\mathcal{D}(P)$. Then $D_1 = \{0\}$. Let $D = \bigcup_{i=1}^{m} D_i$. First, we prove that the set D is CD-independent. Indeed, take any $x, y \in D$, i.e. $x \in D_i$ and $y \in D_j$ for some $1 \leq i \leq j \leq m$. Then $x \leq z$ for some $z \in D_j$ by (A). Assume that x and y are not comparable. Then $z \neq y$, and $z \perp y$ implies $x \perp y$ by (1). This means that D is CD-independent.
Now, assume that D is not a CD-base. Then there is an $x \in P \setminus D$ such that $D \cup \{x\}$ is CD-independent. Next, consider the set

$$\mathcal{E} = \{D_i \in \mathcal{C} \mid x \nleq d \text{ for all } d \in D_i\}.$$

Clearly, $D_1 = \{0\} \in \mathcal{E}$ since $x \nleq 0$. Let $D_i \in \mathcal{E}$. Then $d \perp x$ or $d < x$ holds for each $d \in D_i$ because $D \cup \{x\}$ is CD-independent. Thus $T_i := \{x\} \cup \{d \in D_i \mid d \nless x\}$ is a disjoint set, and $d < x$ or $d \in T_i$ holds for all $d \in D_i$. Hence

$$D_i < T_i, \qquad (6)$$

in view of (A) and $x \notin D_i$. Observe that $D_m \notin \mathcal{E}$ since $D_m < T_m$ is not possible because $\mathcal{C}$ is a maximal chain. Thus, there exists a $k \leq m-1$ such that $D_k \in \mathcal{E}$ but $D_{k+1} \notin \mathcal{E}$. This means that $x \nleq d$ for all $d \in D_k$, and $x \leq z$ holds for some $z \in D_{k+1}$. Then $T_k = \{x\} \cup \{d \in D_k \mid d \nless x\} \in \mathcal{D}(P)$ satisfies $D_k < T_k$ in virtue of (6). Since $T_k \setminus \{x\} \subseteq D_k < D_{k+1}$ and $x \leq z$, for each $t \in T_k$ there is a $v \in D_{k+1}$ with $t \leq v$. In view of (A) we get $D_k < T_k < D_{k+1}$ because $x \notin D_{k+1} \subseteq D$. Since this fact contradicts $D_k \prec D_{k+1}$, we conclude that D is a CD-base.
Further, in view of (4), it follows that any set $D_i \setminus D_{i-1}$, $2 \leq i \leq m$ contains exactly one element, let say, a_i. Observe also that

$$D = \bigcup_{i=1}^{m} D_i = D_1 \cup \left(\bigcup_{i=2}^{m} (D_i \setminus D_{i-1}) \right).$$

Since $D_1 = \{0\}$ and $D_i \setminus D_{i-1} = \{a_i\}$, we get $D = \{0, a_2, \ldots, a_m\}$. We prove that all the elements $0, a_2, \ldots, a_m$ are different: Clearly, $0 \notin \{a_2, \ldots, a_m\}$. Take any $i, j \in \{2, \ldots, m\}$, $i < j$. Then $D_i \leqslant D_{j-1} \prec D_j$. As $a_i \in D_i$, there is a $b \in D_{j-1}$ with $0 < a_i \leq b$ by (A). As $a_j \in D_j \setminus D_{j-1}$, $b < a_j$ or $b \perp a_j$ holds by (2). Since both facts imply $a_i \neq a_j$, we conclude that D contains m different elements. □

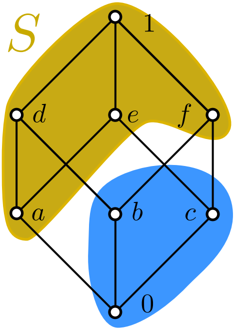
Lemma 1

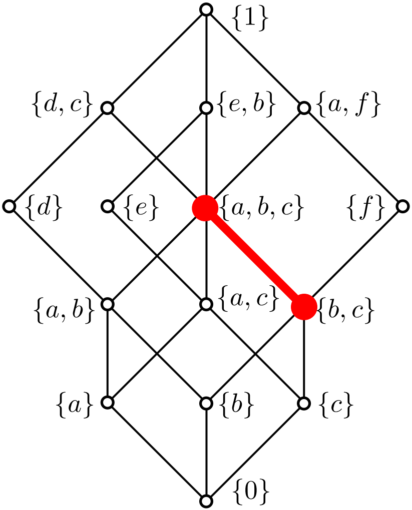
If $D_1 \prec D_2$ *in* $\mathcal{D}(P)$, *then* $D_2 = \{a\} \cup \left\{y \in D_1 \setminus \{0\} \mid y \perp a\right\}$ *for some minimal element* a *of the set*

$$S = \left\{s \in P \setminus (D_1 \cup \{0\}) \mid y \perp s \text{ or } y < s \text{ for all } y \in D_1\right\}.$$

Moreover, $D_1 \prec \{a\} \cup \left\{y \in D_1 \setminus \{0\} \mid y \perp a\right\}$ *holds for any minimal element* a *of the set* S.

Illustration for Lemma 1





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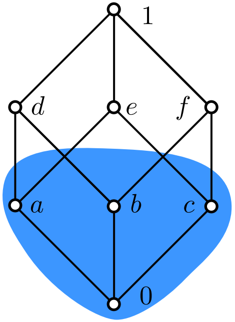

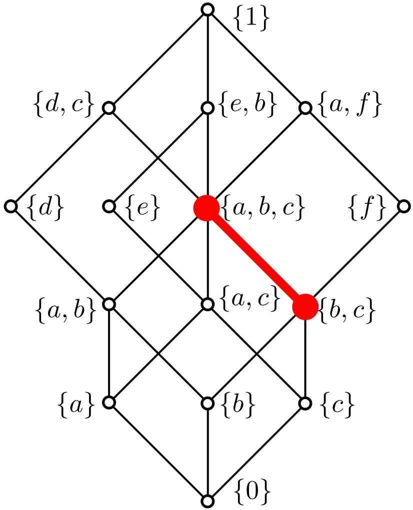

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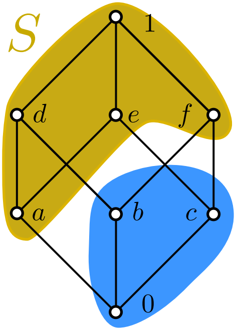

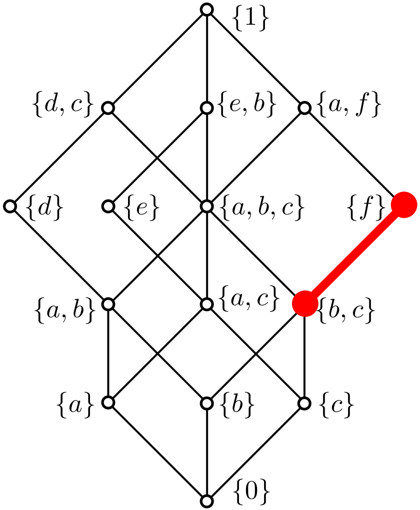

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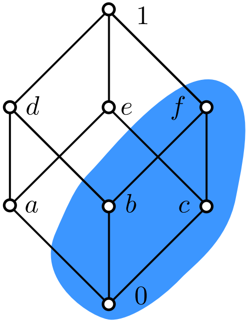
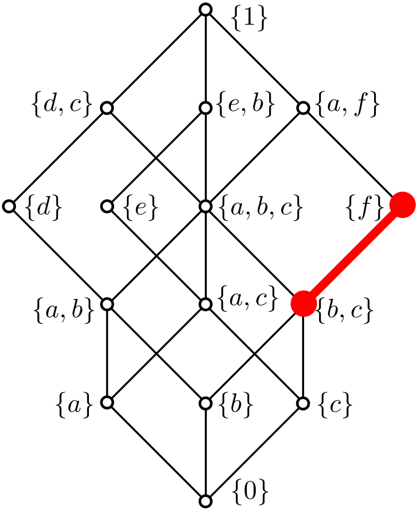


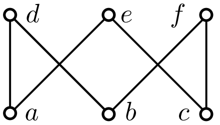


Lemma 2

Assume that B is a CD-base with at least two elements in a finite poset $\mathbb{P} = (P, \leq)$, $M = \max(B)$, and $m \in M$. Then M and $N := \max(B \setminus \{m\})$ are disjoint sets.

Moreover M is a maximal element in $\mathcal{D}(P)$, and $N \prec M$ holds in $\mathcal{D}(P)$.

Illustration for Lemma 2



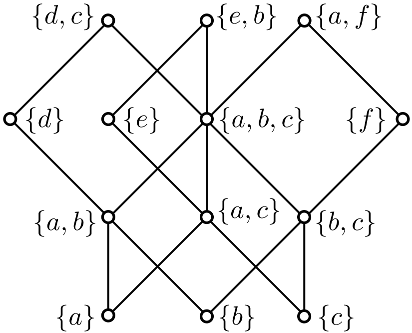
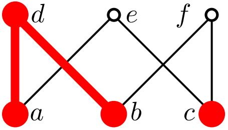
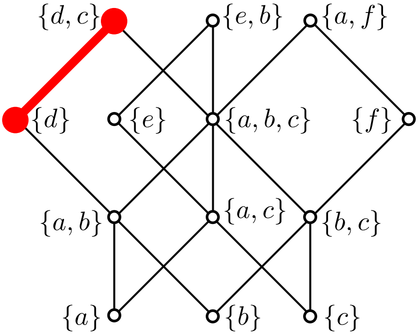
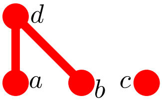


Illustration for Lemma 2







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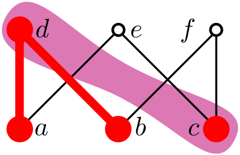

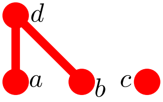

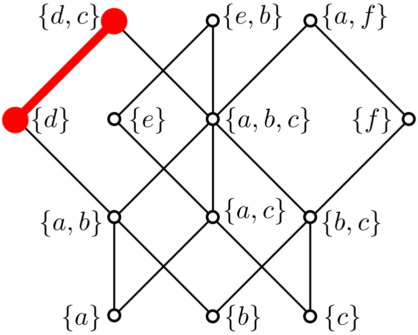

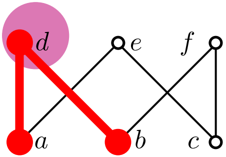
d
e
f
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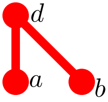
d
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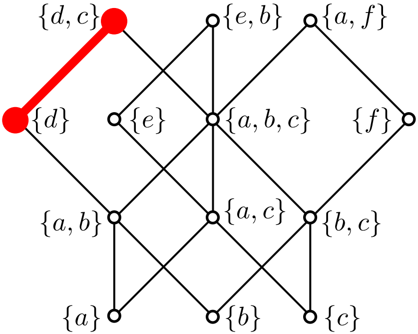
{d, c}
{e, b}
{a, f}
{d}
{e}
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{f}
{a, b}
{a, c}
{b, c}
{a}
{b}
{c}

Corollary

Let $\mathbb{P} = (P, \leq)$ be a finite poset.

The poset $\mathbb{P}$ is called *graded*, if all its maximal chains have the same cardinality.

The CD-bases of $\mathbb{P}$ have the same number of elements if and only if the poset $\mathcal{D}(P)$ is graded.

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Corollary

Let $B \subseteq P$ be a CD-base of $\mathbb{P}$, and $(B, \leq)$ the poset under the restricted ordering.

Then any maximal chain $\mathcal{C} = \{D_i\}_{1\leq i\leq m}$ in $\mathcal{D}(B)$ is also a maximal chain in $\mathcal{D}(P)$.

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Illustration: P and $\mathcal{D}(P)$

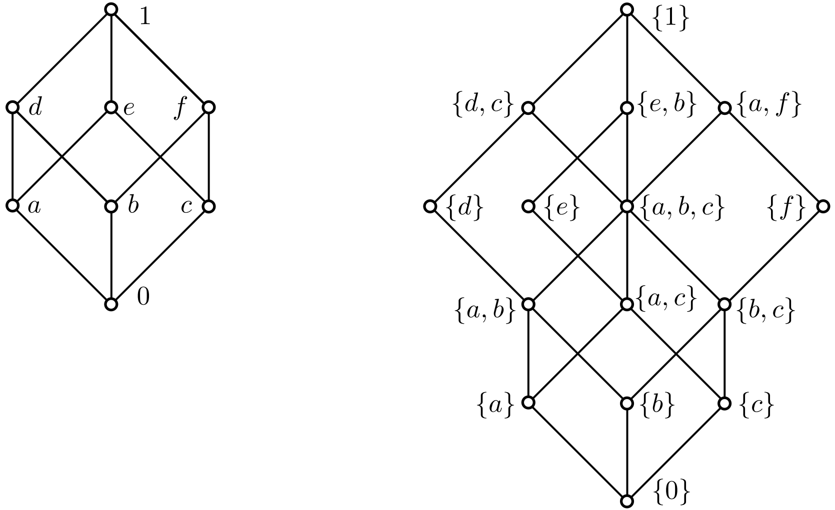
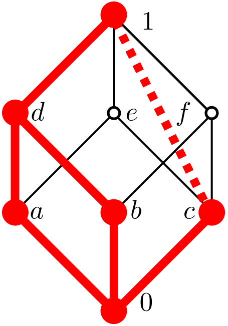
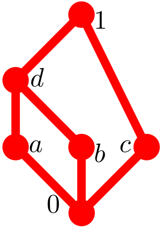


Illustration: P and $\mathcal{D}(P)$, B and $\mathcal{D}(B)$





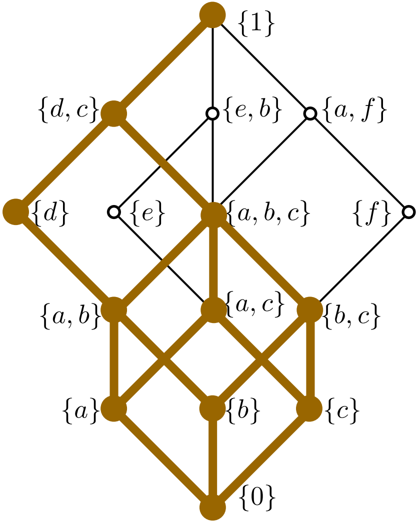
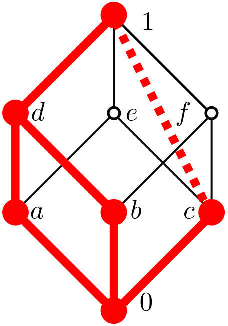
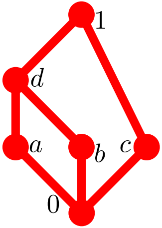


Illustration: P and $\mathcal{D}(P)$, B and $\mathcal{D}(B)$; a maximal chain





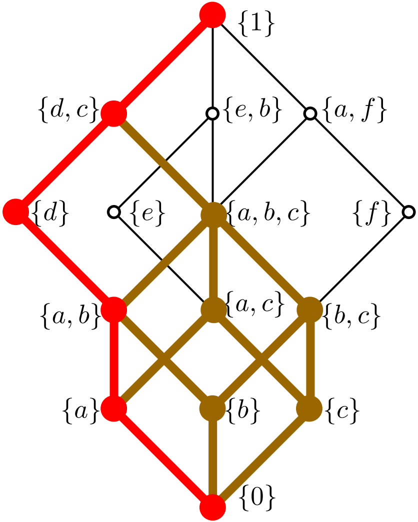
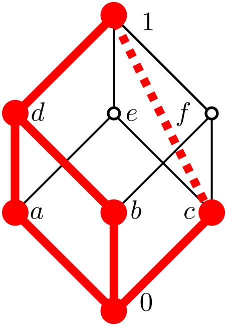
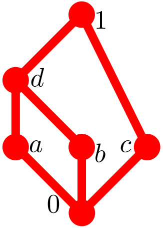
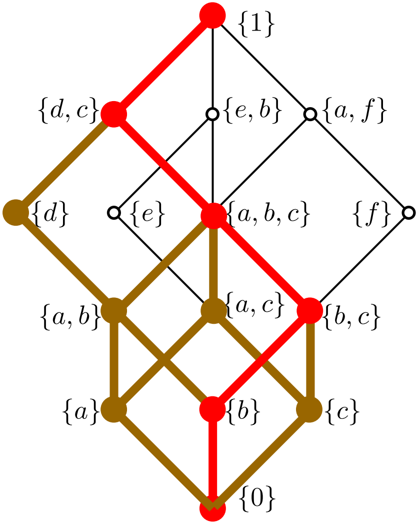


Illustration: P and $\mathcal{D}(P)$, B and $\mathcal{D}(B)$; other



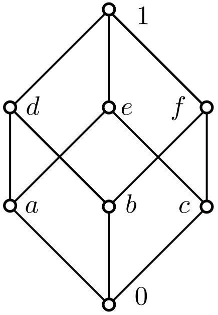


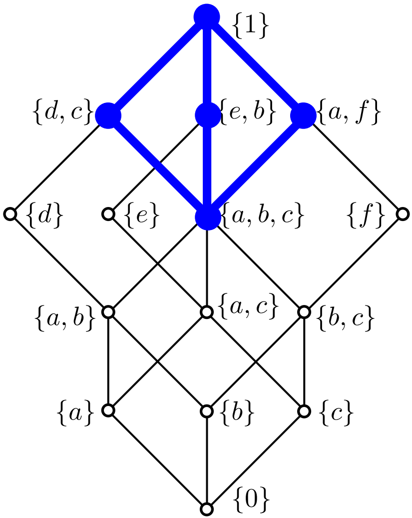


$\mathcal{DC}(P)$

A set of pairwise disjoint elements D of a poset $(P, \leq)$ is called *complete,* if there is no $p \in P \setminus D$ such that $D \cup \{p\}$ is also a set of pairwise disjoint elements.

P, $\mathcal{D}(P)$ and $\mathcal{DC}(P)$





Equivalent conditions

Let $\mathbb{P} = (P, \leq)$ be a finite poset with 0*. Then the following conditions are equivalent:*

(i) *The CD-bases of $\mathbb{P}$ have the same number of elements,*

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Weakly 0-modular lattices

A poset with least element 0 and greatest element 1 is called *bounded*.

A lattice $\mathbb{L} = (L, \leq)$ with 0 is called 0-*modular* if for all $a, b, c \in L$

$$a \leq b \text{ and } b \wedge c = 0 \text{ imply } b \wedge (a \vee c) = a \qquad (\mathrm{M}_0,)$$

We know that that the tolerance lattices of algebras belonging to congruence distributive varieties are 0-*modular* (but not necessarily modular).

If (M_0) is satisfied under the assumptions that a is an atom and $c \prec b \vee c$, then L is called *weakly* 0-*modular*.

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Weakly 0-modular lattices

L is called *lower-semimodular* if for all $a, b, c \in L$, $b \prec c$ implies $a \wedge b \preceq a \wedge c$.

It belongs to the folklore that join-semidistributivity and lower semimodularity characterize the closure lattices of finite convex geometries.

It is easy to see that any lower-semimodular lattice and any 0-modular lattice is weakly 0-modular.

We say that a poset $\mathbb{P}$ with 0 is *weakly 0-modular* if the above weak form of (M_0) holds whenever $\sup\{a, c\}$ and $\sup\{b, c\}$ exist in $\mathbb{P}$.

If $\mathbb{P}$ is a finite bounded poset

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If all the principal ideals $\downarrow a$ of $\mathbb{P}$ are weakly 0-modular, then $A(\mathbb{P}) \cup C$ is a CD-base for every maximal chain C in $\mathbb{P}$.

If each principal ideal of $\mathbb{P}$ is weakly 0-modular and $\mathcal{D}(\mathbb{P})$ is graded, then $\mathbb{P}$ is also graded, and any CD-base of $\mathbb{P}$ contains $|A(\mathbb{P})| + l(\mathbb{P})$ elements.

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Lemma

Let $\mathbb{P}$ be a poset with 0. Assume that $K \neq \varnothing$ is an index set and, for each $k \in K$, D_k is a set of pairwise disjoint elements in $\mathbb{P}$. If for every choice function $f \in \prod_{k \in K} D_k$ the meet $\bigwedge_{k \in K} f(k)$ exists in $\mathbb{P}$, then $\bigwedge_{k \in K} D_k$ exists in $\mathcal{D}(\mathbb{P})$.

In particular, for $K = \{1, 2\}$ and $D_1 = \{a_i \mid i \in I\}$, $D_2 = \{b_j \mid j \in J\} \in \mathcal{D}(\mathbb{P})$ such that all the $a_i \wedge b_j$ exists, we have

$$D_1 \wedge D_2 = \begin{cases} M\} & \text{if } M \neq \emptyset; \\ \{0\} & \text{otherwise,} \end{cases}$$

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Theorem 2. (E. K. H., S. Radeleczki)

(i) If $\mathbb{P} = (P, \wedge)$ is a semilattice with 0, then $\mathcal{D}(\mathbb{P})$ is a dp-distributive semilattice. Furthermore, for all $D_1, D_2 \in \mathcal{D}(\mathbb{P})$, if $D_1 \cup D_2$ is a CD-independent set, then D_1, D_2 is a distributive pair in $\mathcal{D}(\mathbb{P})$.

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CD-bases in semilattices and lattices / 2

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CD-bases in particular lattice classes

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We say that $\mathbb{L}$ is *weakly* 0*-distributive* if this implication holds under the condition $a \wedge b = 0$.

Remark

If D is a set of pairwise disjoint elements in a weakly 0-distributive lattice and $|D| \geq 2$, then it is easy to see that replacing two different elements $d_1, d_2 \in D$ by their join $d_1 \vee d_2$, we obtain again a set of pairwise disjoint elements.

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Lemma

Let $\mathbb{L}$ be a finite weakly 0-distributive lattice and D a dual atom of the poset $\mathcal{D}(\mathbb{L})$.

Then either $D = \{d\}$ for some $d \in L$ with $d \prec 1$, or D consist of two different elements $d_1, d_2 \in L$ with $d_1 \vee d_2 = 1$.

Let $\mathbb{L}$ be a graded lattice, and $a \in L$. Then the *height* of a is the length of the interval $[0, a]$, denoted by $l(a)$.

A graded lattice $\mathbb{L}$ is 0-modular, whenever $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.

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Let L be a finite, weakly 0-distributive lattice. Then the following are equivalent:

- (i) *L is graded, and $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.*
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- (ii) *L is 0-modular, and the CD-bases of L have the same number of elements.*

Modular pair

We say that two elements $a, b \in L$ form a *modular pair* in the lattice L and we write $(a, b)M$ if for all $x \in L$, $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$.

Also, a and b form a *dual-modular pair* if for all $x \in L$, $x \geq b$ implies $x \wedge (a \vee b) = (x \wedge a) \vee b$. This is denoted by $(a, b)M^*$.

Clearly, if a and b form a distributive pair, then $(a, b)M^*$ is satisfied.

By means of modular pairs, the 0-modularity condition can be reformulated as follows: For all $a, b \in L$,

Lemma (M. Stern) In a graded lattice of finite length, $(a, b)M$ implies $l(a) + l(b) \leq l(a \wedge b) + l(a \vee b)$.

With the help of the previous Lemma of M. Stern above, using an N_5 sublattice containing 0 as well as the dual lattice, we obtain

Proposition If $\mathbb{L}$ is a lattice with 0 such that $(a, b)M^*$ holds for all $a, b \in L$ with $a \wedge b = 0$, then L is 0-modular. If in addition $\mathbb{L}$ is a graded lattice of finite length, then $l(a \vee b) = l(a) + l(b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.

Corollary (i) Let $\mathbb{L}$ be a finite, weakly 0-distributive lattice such that for each $a, b \in L$ with $a \wedge b = 0$, condition $(a, b)M^*$ holds. Then the CD-bases of $\mathbb{L}$ have the same number of elements if and only if $\mathbb{L}$ is graded.
(ii) If $\mathbb{L}$ is a finite pseudocomplemented modular lattice, then the CD-bases of $\mathbb{L}$ have the same number of elements.

dp-distributive lattices

As any dp-distributive lattice $\mathbb{L}$ is weakly 0-distributive, and $(a, b)M^*$ holds for all $a, b \in L$ with $a \wedge b = 0$ since (a, b) is a distributive pair, we obtain

Corollary

(i) Any dp-distributive lattice is 0-modular. If $\mathbb{L}$ is a dp-distributive graded lattice with finite length, then $l(a \vee b) = l(a) + l(b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.
(ii) The CD-bases in a finite dp-distributive lattice $\mathbb{L}$ have the same number of elements if and only if $\mathbb{L}$ is graded.

Interval system

An *interval system* $(V, \mathcal{I})$ is an algebraic closure system satisfying the axioms:

(I_0) $\{x\} \in \mathcal{I}$ for all $x \in V$, and $\emptyset \in \mathcal{I}$;

(I_1) $A, B \in \mathcal{I}$ and $A \cap B \neq \emptyset$ imply $A \cup B \in \mathcal{I}$;

(I_2) For any $A, B \in \mathcal{I}$ the relations $A \cap B \neq \emptyset$, $A \nsubseteq B$ and $B \nsubseteq A$ imply $A \setminus B \in \mathcal{I}$ (and $B \setminus A \in \mathcal{I}$).

The *modules* (*X-sets, or autonomous sets*) of an undirected graph $G = (V, E)$, the *intervals* of an *n*-ary relation $R \subseteq V^n$ on the set V for $n \geq 2$ – in particular, the usual intervals of a linearly ordered set $(V, \leq)$ – form interval systems.

Let us consider now the condition:

$(\mathcal{II})$ If $a \wedge b \neq 0$, then $(x \leq a \vee b$ and $x \wedge a = 0) \Rightarrow x \leq b$ for all $a, b, x \in L$.

Lattices with 0 satisfying condition $(\mathcal{II})$ and with the property that $\uparrow a$ is a modular lattice for all $a \in L$, $a \neq 0$, can be considered as a generalization of the lattice $(\mathcal{I}, \subseteq)$ of an interval system $(V, \mathcal{I})$. To study their CD-bases, first we proved:

Lemma Let $\mathbb{L}$ be an atomic lattice satisfying condition $(\mathcal{II})$. Assume $D \in \mathcal{D}(\mathbb{L})$ and define $S_D = \{s \in L \setminus (D \cup \{0\}) \mid d \wedge s = 0$ or $d < s$, for all $d \in D\}$. Then for all $b, c \in S_D$ with $b \wedge c \neq 0$ and all $d \in D$, $d \wedge (b \vee c) \neq 0$ if and only if $0 < d < b$ or $0 < d < c$ holds.

Generalizing interval systems

Let us consider now the condition:

Remark Let $\mathbb{L}$ be a finite lattice and $D = \{d_j \mid j \in J\} \in \mathcal{DC}(\mathbb{L})$. If $D \prec D'$ for some $D' \in D(\mathbb{L})$; then, there is a minimal element $a \in S_D$ such that $D' = \{a\} \cup \{d_j \in D \setminus \{0\} \mid d_j \wedge a = 0\}$. In this case there exists a set $K \subseteq J$ such that

$$K = \{j \in J \mid d_j < a\} \neq \emptyset \text{ and } D' = \{a\} \cup \{d_j \mid j \in J \setminus K\}. \quad (14)$$

Birkhoff's condition

It is well-known that a finite lattice $\mathbb{L}$ is semimodular if and only if it satisfies *Birkhoff's condition*, namely, for all $a, b \in L$

(Bi) $$a \wedge b \prec a, b \text{ implies } a, b \prec a \vee b.$$

We also say that a pair $a, b \in L$ satisfies Birkhoff's condition if the above implication (Bi) is valid for a, b. It is known that any distributive pair $a, b \in L$ satisfies Birkhoff's condition.

Theorem 5. (K. H. E., Radeleczki S.) Let $\mathbb{L}$ be a finite lattice satisfying condition $(\mathcal{II})$ such that any principal filter $\uparrow a$ with $a \in L \setminus \{0\}$ is a modular lattice. Then $\mathcal{DC}(\mathbb{L})$ is a semimodular lattice.

Corollary (i) If $\mathbb{L}$ is a finite distributive lattice, then $\mathcal{DC}(\mathbb{L})$ is a semimodular lattice.

(ii) If $\mathbb{L}$ is a finite lattice that satisfies the conditions in Theorem 3, then its CD-bases have the same number of elements.

By applying this to interval systems we obtain:

Corollary

If $(V, \mathcal{I})$ is a finite interval system, then the CD-bases of the lattice $(\mathcal{I}, \subseteq)$ contain the same number of elements.

CD-bases

Corollary (i) If $\mathbb{L}$ is a finite distributive lattice, then $\mathcal{DC}(\mathbb{L})$ is a semimodular lattice.

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By applying this to interval systems we obtain:

Corollary

If $(V, \mathcal{I})$ is a finite interval system, then the CD-*bases of the lattice $(\mathcal{I}, \subseteq)$ contain the same number of elements.*

Island domain

$U \in \mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{P}(U)$

Let $h\colon U \to \mathbb{R}$ be a height function and let $S \in \mathcal{C}$ be a nonempty set.

We say that S is an *pre-island* with respect to the triple $(\mathcal{C}, \mathcal{K}, h)$, if every $K \in \mathcal{K}$ with $S \prec K$ satisfies

$$\min h(K) < \min h(S).$$

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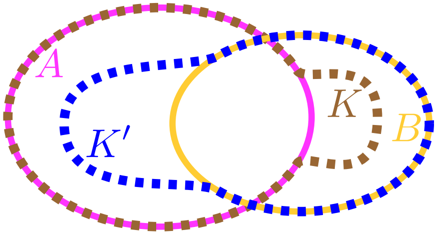
Connective island domains

Definition

A pair $(\mathcal{C}, \mathcal{K})$ is an *connective island domain* if

$$\forall A, B \in \mathcal{C}: \quad (A \cap B \neq \emptyset \text{ and } B \not\subseteq A) \implies \exists K \in \mathcal{K} : A \subset K \subseteq A \cup B.$$



Connective island domains

Theorem 5. (S. Foldes, E. K. H., S. Radeleczki, T. Waldhauser)

The following three conditions are equivalent for any pair $(\mathcal{C}, \mathcal{K})$:

(i) $(\mathcal{C}, \mathcal{K})$ is a connective island domain.

(ii) Every system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CD-independent.

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Thank you for your attention!