Islands

Eszter K. Horváth, Szeged

Co-authors: Zoltán Németh, Gabriella Pluhár, János Barát, Péter Hajnal, Csaba Szabó, Gábor Horváth, Branimir Šešelja, Andreja Tepavčević, Attila Máder, Sándor Radeleczki

Szeged, 2011, May 20.

Project

Research is supported by the Hungary-Serbia IPA Cross-border Co-operation programme HU-SRB/0901/221/088 co-financed by the European Union.

Islands?

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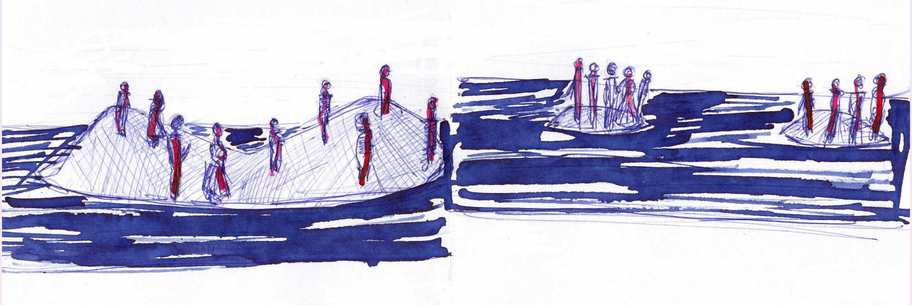
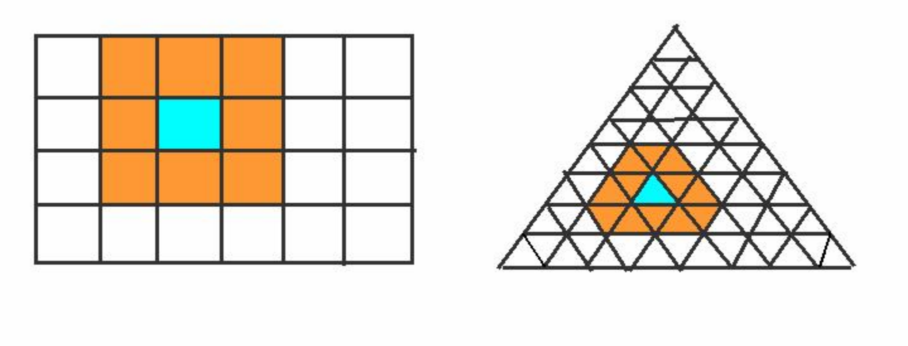
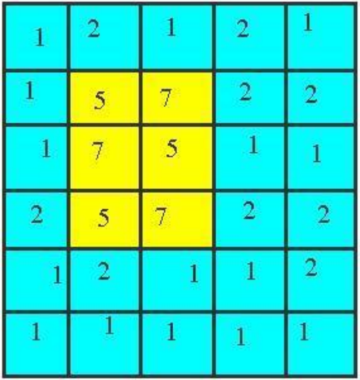


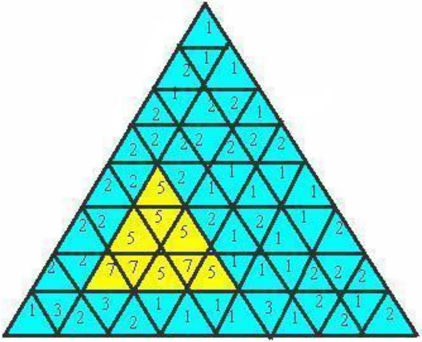
Grid, neighbourhood



Definition/2

We call a rectangle/triangle an *island*, if for the cell t, if we denote its height by a_t, then for each cell $\hat{t}$ neighbouring with a cell of the rectange/triangle T, the inequality $a_{\hat{t}} < min\{a_t : t \in T\}$ holds.





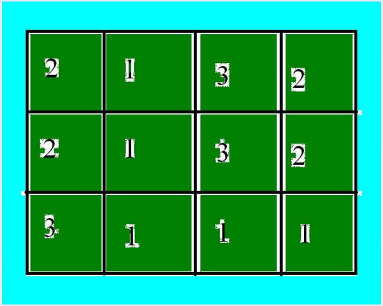
Count the islands! / 1

We put heights into the cells.
How many islands do we have?

2	1	3	2
2	1	3	2
3	1	1	1

Count the islands!
Water level: 0,5
Nomber of islands: 1

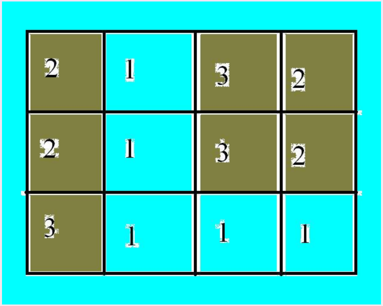
2	1	3	2
2	1	3	2
3	1	1	1



Count the islands! / 3

Water level: 1,5
Number of islands: 2

2	1	3	2
2	1	3	2
3	1	1	1

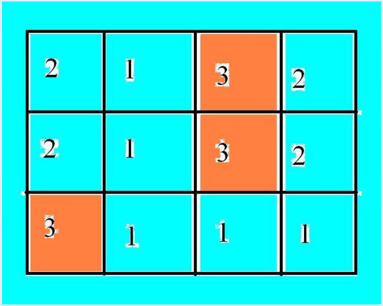


Count the islands! / 4

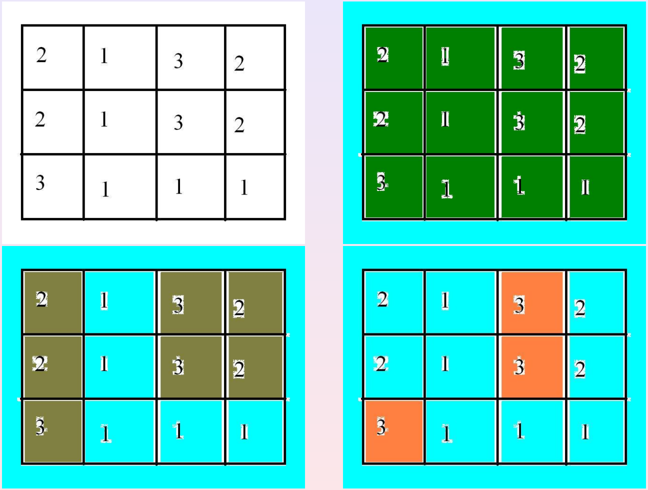
Water level: 2,5

Number of islands: 2

2	1	3	2
2	1	3	2
3	1	1	1

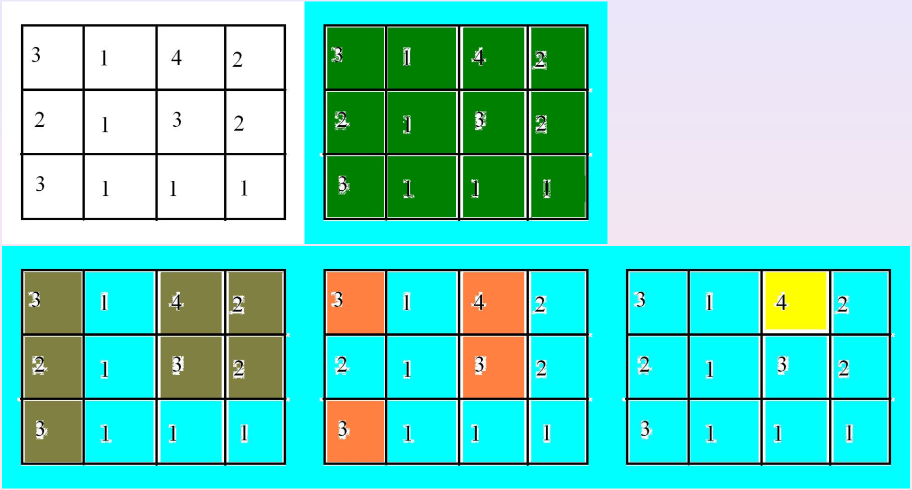


Altogether: $1 + 2 + 2 = 5$ islands.



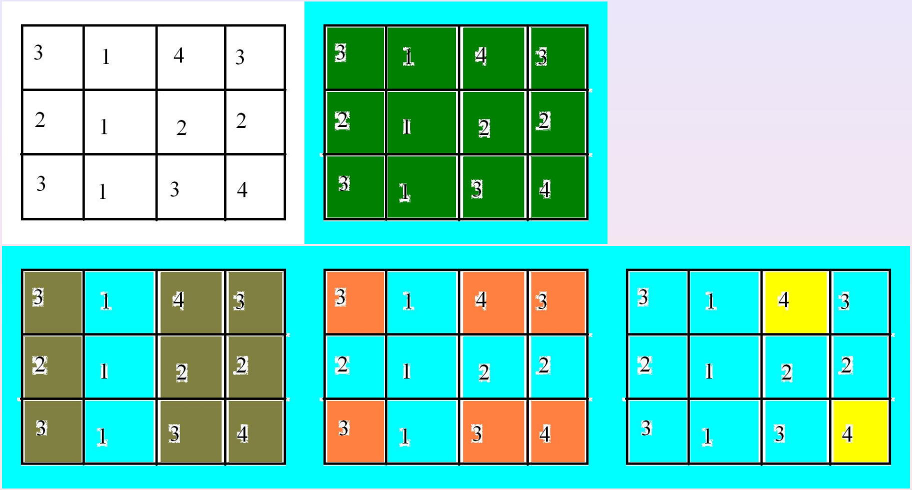
Could we make more islands onto this grid? (With other heights?)

Yes, we could make more islands, here we have $1 + 2 + 3 + 1 = 7$ islands.



Could we make more islands onto this grid? (With other heights?)

Yes, we could make more islands, here we have $1 + 2 + 4 + 2 = 9$ islands.



3	1	4	3
2	1	2	2
3	1	3	4

HOWEWER, WE CANNOT CREATE MORE !!!

The maximum number of islands on the $m \times n$ size grid (Gábor Czédli , Szeged, 2007. june 17.)

$$f(m,n) = \left[\frac{mn + m + n - 1}{2} \right].$$

Soon we prove the formula !

Coding theory

S. Földes and N. M. Singhi: On instantaneous codes, J. of Combinatorics, Information and System Sci., 31 (2006), 317-326.

History/1

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Rectangular islands

G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics 30 (2009), 208-215.

The maximum number of rectangular islands in a $m \times n$ rectangular board on square grid:

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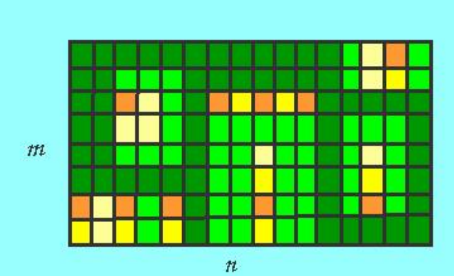

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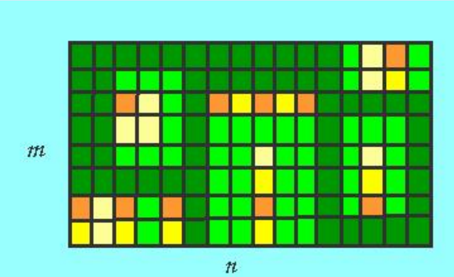

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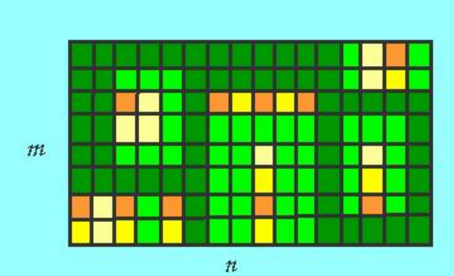

History/3

Rectangular islands in higher dimensions

G. Pluhár: The number of brick islands by means of distributive lattices, Acta Sci. Math., to appear.

History/3

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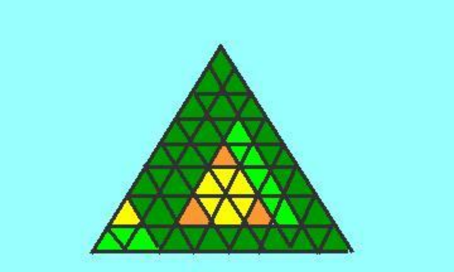
Triangular islands

E. K. Horváth, Z. Németh and G. Pluhár: The number of triangular islands on a triangular grid, Periodica Mathematica Hungarica, 58 (2009), 25–34.
Available at http://www.math.u-szeged.hu/~horvath

For the maximum number of triangular islands in an equilateral rectangle of side length n, $\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}$ holds.



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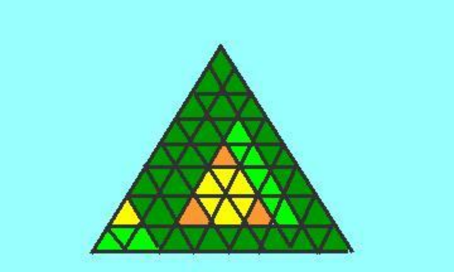

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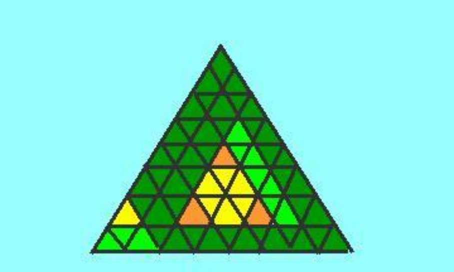
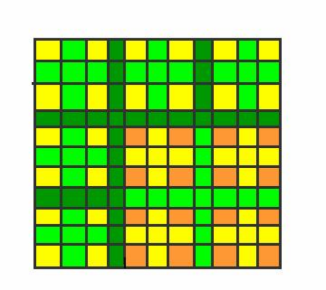


History/5

Square islands (also in higher dimensions)

E. K. Horváth, G. Horváth, Z. Németh, Cs. Szabó: The number of square islands on a rectangular sea, Acta Sci. Math., to appear. Available at http://www.math.u-szeged.hu/~horvath

$$\frac{1}{3}(rs - 2r - 2s) \leq f(r, s) \leq \frac{1}{3}(rs - 1)$$



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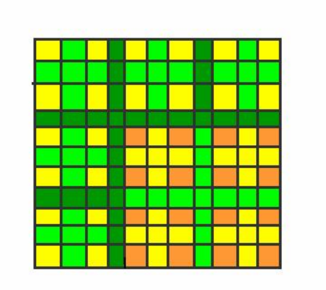

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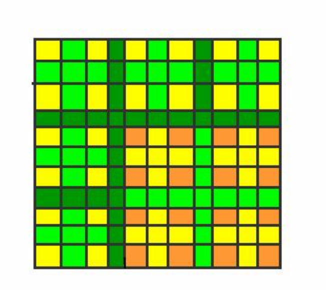

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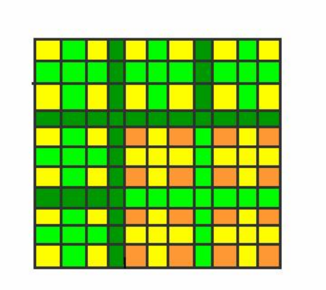


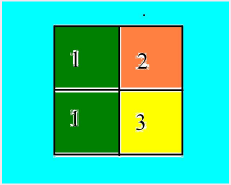
Proving $f(m,n) = \left[\frac{mn+m+n-1}{2}\right]$ THERE EXISTS:

By induction on the number of the cells: $f(m,n) \geq \left[\frac{mn+m+n-1}{2}\right]$.

If $m = 1$, then $\left[\frac{n+1+n-1}{2}\right] = n$, we put the numbers $1, 2, 3, \ldots, n$ in the cells and we will have exactly n islands.

If $n = 1$, then $\left[\frac{m+m+1-1}{2}\right] = m$.

If $m = n = 2$:



Az $f(m,n)=\left[\frac{mn+m+n-1}{2}\right]$ képlet bizonyítása, THERE EXISTS:

Let $m, n > 2$.

$$f(m,n) \geq f(m-2,n)+f(1,n)+1 \geq \left[\frac{(m-2)n+(m-2)+n-1}{2}\right]+\left[\frac{n+1+n-1}{2}\right]+1 =$$
$$= \left[\frac{(m-2)n+(m-2)+n-1+2n}{2}\right] + 1 = \left[\frac{mn+m+n-1}{2}\right].$$

Proving methods/1

LATTICE THEORETICAL METHOD

G. Czédli, A. P. Huhn and E. T. Schmidt: Weakly independent subsets in lattices, Algebra Universalis 20 (1985), 194-196.

Any two weak bases of a finite distributive lattice have the same number of elements.

Proving methods/1

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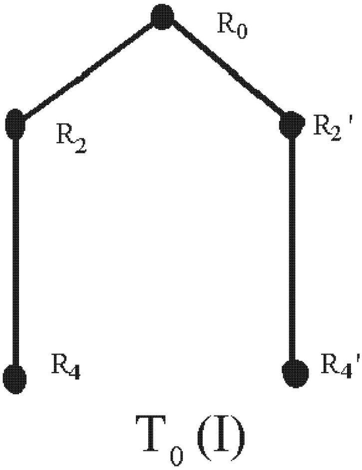
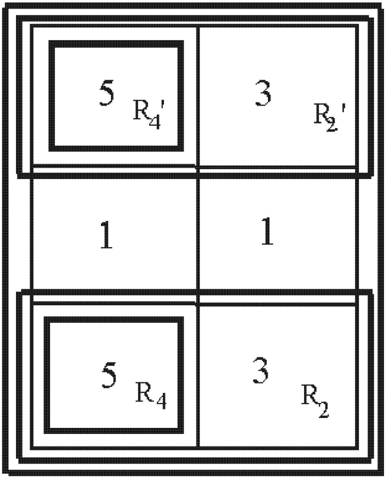
Proving methods/2

TREE-GRAPH METHOD





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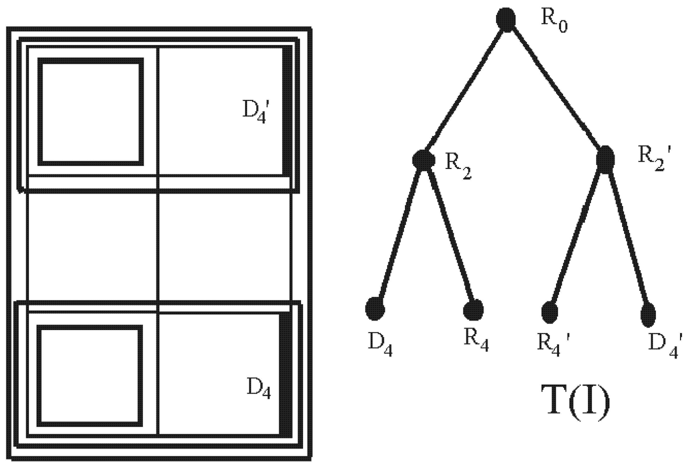

Proving methods/2

TREE-GRAPH METHOD

Lemma 2 (folklore)

(i) Let T be a binary tree with ℓ leaves. Then the number of vertices of T depends only on ℓ, moreover $|V| = 2\ell - 1$.

(ii) Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T. Then $|V| \leq 2\ell - 1$.

We have $4s + 2d \leq (n+1)(m+1)$.
The number of leaves of $T(\mathcal{I})$ is $\ell = s + d$. Hence by Lemma 2 the number of islands is

$$|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n+1)(m+1) - 1.$$

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.

Proving methods/3

ELEMENTARY METHOD

We define

$$\mu(R) = \mu(u, v) := (u + 1)(v + 1).$$

Now

$$f(m, n) = 1 + \sum_{R \in max\mathcal{I}} f(R) = 1 + \sum_{R \in max\mathcal{I}} ([\frac{(u+1)(v+1)}{2}] - 1)$$

$$= 1 + \sum_{R \in max\mathcal{I}} ([\frac{\mu(u, v)}{2}] - 1) \leq 1 - |max\mathcal{I}| + [\frac{\mu(\mathrm{C})}{2}].$$

If $|max\mathcal{I}| \geq 2$, then the proof is ready. Case $|max\mathcal{I}| = 1$ is an easy exercise.

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History/6

Some exact formulas

Cylindric board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_1(m,n) = [\frac{(m+1)n}{2}]$.

Cylindric board, cylindric and rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_2(m,n) = [\frac{(m+1)n}{2}] + [\frac{(m-1)}{2}]$.

Torus board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $m, n \geq 2$, then $t(m,n) = [\frac{mn}{2}]$.

Peninsulas (semi islands) (J. Barát, P. Hajnal, E.K. Horváth):
$p(m,n) = f(m,n) = [(mn + m + n - 1)/2]$.

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Further results on rectangular islands

Zs. Lengvárszky: The minimum cardinality of maximal systems of rectangular islands, European Journal of Combinatorics, **30** (2009), 216-219.

History/7

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History/8

The board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $BA = \{0,1\}^n$.

We consider two cells neighbouring if their Hamming distance is 1.

We denote the maximum number of islands in $BA = \{0,1\}^n$ by $b(n)$.

Island formula for Boolean algebras (P. Hajnal, E.K. Horváth)
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The board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $BA = \{0,1\}^n$.

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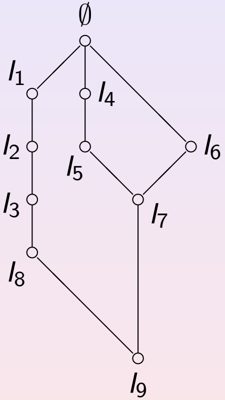
Constructing algorithm

Joint work with Branimir Šešelja and Andreja Tepavčević

CONSTRUCTING ALGORITHM

1. FOR $i = t$ TO 0
2. FOR $y = 1$ TO n
3. FOR $x = 1$ TO m
4. IF $h(x, y) = a_i$ THEN
5. j:= i
6. WHILE there is no island of h which is a subset of h_{a_j} that contains (x, y) DO j:=j-1
7. ENDWHILE
8. Let $h^*(x, y) := a_j$.
9. ENDIF
10. NEXT x
11. NEXT y
12. NEXT i
13. END.

The lattice of islands



$L = (\mathcal{I}_0(\Gamma), \supseteq)$

Height of the hills

We denote by $\Lambda_{max}(m, n)$ the maximum number of different nonempty p-cuts of a standard rectangular height function on the rectangular table of size $m \times n$.

Theorem 5 $\Lambda_{max}(m, n) = m + n - 1.$

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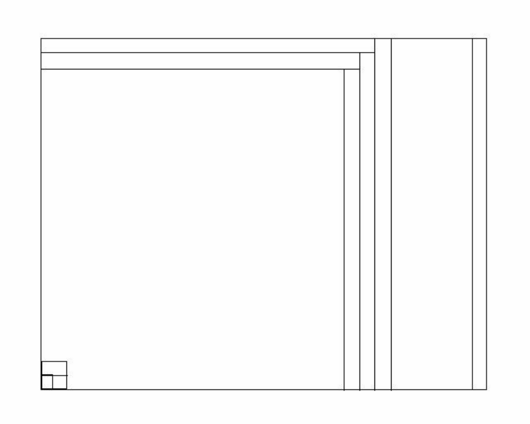

The maximum number of different nonempty p-cuts of a standard rectangular height function is equal to the minimum cardinality of maximal systems of islands.

Height of the hills

We denote by $\Lambda_h^{cz}(m, n)$ the number of different nonempty cuts of a standard rectangular height function h in the case h has maximally many islands, i.e., when the number of islands is

$$f(m, n) = \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor .$$

Theorem

Let $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ be a standard rectangular height function having maximally many islands $f(m, n)$. Then, $\Lambda_h^{cz}(m, n) \geq \lceil log_2(m + 1) \rceil + \lceil log_2(n + 1) \rceil - 1$.

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CD-independent subsets in posets

Definitions

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set and $a, b \in P$. The elements a and b are called *disjoint* and we write $a \perp b$ if

either $\mathbb{P}$ has least element $0 \in P$ and $\inf\{a, b\} = 0$,
or if $\mathbb{P}$ is without 0, then a and b have no common lowerbound.

- Notice, that $a \perp b$ implies $x \perp y$ for all $x, y \in P$ with $x \leq a$ and $y \leq b$.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$ or $x \perp y$ holds.

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Disjoint systems

Definition

A nonempty set D of nonzero elements of P is called a *disjoint system* in $\mathbb{P}$ if $x \perp y$ holds for all $x, y \in D$, $x \neq y$.

Remarks

- Any disjoint system $D \subseteq P$ and any chain $C \subseteq P$ is a CD-independent set.
- D is a disjoint system, if and only if it is a CD-independent antichain in $\mathbb{P}$.
- *If X is a CD-independent set in $\mathbb{P}$, then any antichain $A \subseteq X$ is a disjoint system in $\mathbb{P}$.*

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Order ideals

Any antichain $A = \{a_i \mid i \in I\}$ of a poset $\mathbb{P}$ determines a unique order-ideal $I(A)$ of $\mathbb{P}$:

$$I(A) = \bigcup_{i \in I} (a_i] = \{x \in P \mid x \leq a_i, \text{ for some } i \in I\},$$

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If A_1, A_2 are antichains in $\mathbb{P}$, then we say that A_1 *is dominated by* A_2, and we denote it by $A_1 \leqslant A_2$ if

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Tolerance relation

Definition

Let $\rho \subseteq P \times P$.

For any $x, y \in P$, $(x, y) \in \rho \Leftrightarrow$ either $x \leq y$ or $y \leq x$ or $x \perp y$.

Remarks

- ρ is a tolerance relation on P.
- The CD-bases of $\mathbb{P}$ are exactly the tolerance classes (tolerance blocks) of ρ.
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Theorem

Let B *be a CD-base of a finite poset* $(P, \leq)$, *and* let $|B| = n$.

Then there exists a maximal chain $\{D_i\}_{1 \leq i \leq n}$ *in* $\mathcal{D}(P)$ *such that* $B = \bigcup_{i=1}^{n} D_i$.

Moreover, for any maximal chain $\{D_i\}_{1 \leq i \leq m}$ *in* $\mathcal{D}(P)$ *the set* $D = \bigcup_{i=1}^{m} D_i$ *is a CD-base in* $(P, \leq)$ *with* $|D| = m$.

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Proof of the Theorem

Proposition

If B is a CD-base and $D \subseteq B$ is a disjoint system in the poset $(P, \leq)$, then $I(D) \cap B$ is also a CD-base in the subposet $(I(D), \leq)$.

Lemma

If $D_1 \prec D_2$ in $\mathcal{D}(P)$, then $D_2 = \{a\} \cup \{y \in D_1 \setminus \{0\} \mid y \perp a\}$ for some minimal element a of the set $S = \{s \in P \setminus (D_1 \cup \{0\}) \mid y \perp s \text{ or } y < s \text{ for all } y \in D_1\}$. Moreover, $D_1 \prec \{a\} \cup \{y \in D_1 \setminus \{0\} \mid y \perp a\}$ holds for any minimal element a of the set S.

Lemma

Assume that B is a CD-base with at least two elements in a finite poset $\mathbb{P} = (P, \leq)$, $M = \max(B)$, and $m \in M$. Then M and $N := \max(B \setminus \{m\})$ are disjoint sets. Moreover M is a maximal element in $\mathcal{D}(P)$, and $N \prec M$ holds in $\mathcal{D}(P)$.

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Corollary

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Let $\mathbb{P} = (P, \leq)$ *be a finite poset. Then the CD-bases of* $\mathbb{P}$ *have the same number of elements if and only if the poset* $\mathcal{D}(P)$ *is graded.*

Let $B \subseteq P$ *be a CD-base of* $\mathbb{P}$ *, and* $(B, \leq)$ *the poset under the restricted ordering. Then any maximal chain* $\mathcal{C} = \{D_i\}_{1 \leq i \leq m}$ *in* $\mathcal{D}(B)$ *is also a maximal chain in* $\mathcal{D}(P)$*.*

If D *is a disjoint set in* $\mathbb{P}$ *and the CD-bases of* $\mathbb{P}$ *have the same number of elements, then the CD-bases of the subposet* $(I(D), \leq)$ *also have the same number of elements.*

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$\mathcal{D}(P)$ is graded

The poset $\mathbb{P}$ is called *graded*, if all its maximal chains have the same cardinality.

Let $\mathbb{P} = (P, \leq)$ *be a finite poset with* 0. *Then the following conditions are equivalent:*

(i) *The CD-bases of* $\mathbb{P}$ *have the same number of elements,*

(ii) *$\mathcal{D}(P)$ is graded.*

A disjoint system D of a poset $(P, \leq)$ is called *complete*, if there is no $p \in P \setminus D$ such that $D \cup \{p\}$ is also a disjoint system.

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A disjoint system D of a poset $(P, \leq)$ is called *complete*, if there is no $p \in P \setminus D$ such that $D \cup \{p\}$ is also a disjoint system.

(iii) $\mathcal{DC}(P)$ *is graded.*

$\mathcal{D}(P)$ is graded

The poset $\mathbb{P}$ is called *graded*, if all its maximal chains have the same cardinality.

Let $\mathbb{P} = (P, \leq)$ *be a finite poset with* 0. *Then the following conditions are equivalent:*

(i) *The CD-bases of* $\mathbb{P}$ *have the same number of elements,*

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If $\mathbb{P}$ is a finite poset with 0

If all the principal ideals $(a]$ of $\mathbb{P}$ are weakly 0-modular, then $A(P) \cup C$ is a CD-base for every maximal chain C in $\mathbb{P}$.

If $\mathbb{P}$ has weakly modular principal ideals and $\mathcal{D}(P)$ is graded, then $\mathbb{P}$ is also graded, and any CD-base of $\mathbb{P}$ contains $|A(P)| + l(P)$ elements.

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Lemma

Let $\mathbb{P}$ *be a poset with* 0 *and* D_k, $k \in K$ $(K \neq \emptyset)$ *disjoint sets in* $\mathbb{P}$. *If the meet* $\bigwedge_{k \in K} a^{(k)}$ *of any system of elements* $a^{(k)} \in D_k$, $k \in K$ *exist in* $\mathbb{P}$, *then* $\bigwedge_{k \in K} D_k$ *also exists in* $\mathcal{D}(P)$.

A pair $a, b \in P$ with least upperbound $a \vee b$ in $\mathbb{P}$ is called a *distributive pair*, if $(c \wedge a) \vee (c \wedge b)$ exists in $\mathbb{P}$ for any $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$.
We say that $(P, \wedge)$ is *dp-distributive*, if any $a, b \in P$ with $a \wedge b = 0$ is a distributive pair.

Theorem

(i) *If* $\mathbb{P} = (P, \wedge)$ *is a semilattice with* 0*, then* $\mathcal{D}(P)$ *is a dp-distributive semilattice; if* $D_1 \cup D_2$ *is a CD-independent set for some* $D_1, D_2 \in \mathcal{D}(P)$*, then* D_1, D_2 *is a distributive pair in* $\mathcal{D}(P)$*.*

(ii) *If* $\mathbb{P}$ *is a complete lattice, then* $\mathcal{D}(P)$ *is a dp-distributive complete lattice.*

CD-bases in semilattices and lattices / 2

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Let $(P, \leq)$ be a poset and $A \subseteq P$. $(A, \leq)$ is called a *sublattice* of $(P, \leq)$, if $(A, \leq)$ is a lattice such that for any $a, b \in A$ the infimum and the supremum of $\{a, b\}$ is the same in the subposet $(A, \leq)$ and in $(P, \leq)$. If the relation $x \prec y$ in $(A, \leq)$ for some $x, y \in A$ implies $x \prec y$ in the poset $(P, \leq)$, then we say that $(A, \leq)$ is a *cover-preserving subposet* of $(P, \leq)$.

Theorem

Let $\mathbb{P} = (P, \leq)$ be a poset with 0 and B a CD-base of it. Then $(\mathcal{D}(B), \leqslant)$ is a distributive cover-preserving sublattice of the poset $(\mathcal{D}(P), \leqslant)$. If $\mathbb{P}$ is a $\wedge$-semilattice, then for any $D \in \mathcal{D}(P)$ and $D_1, D_2 \in \mathcal{D}(B)$ we have $(D_1 \vee D_2) \wedge D = (D_1 \wedge D) \vee (D_2 \wedge D)$ in $(\mathcal{D}(P), \leqslant)$.

Lemma

Let L be a finite weakly 0-distributive lattice and D a dual atom in $\mathcal{D}(L)$. Then either $D = \{d\}$, for some $d \in L$ with $d \prec 1$, or D consist of two different elements $d_1, d_2 \in L$ and $d_1 \vee d_2 = 1$.

Theorem

Let L be a finite, weakly 0-distributive lattice. Then the following are equivalent:

- (i) *L is graded, and $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.*
- (ii) *L is 0-modular, and the CD-bases of L have the same number of elements.*

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CD-bases in particular lattice classes

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High school competition exercise

Determine the maximum number of islands on n consecutive cells, if the possible heights on the grid are the following: $0, 1, 2, \ldots, h$; where $h \geq 1$.

The solution:

$I(n,h) = n - \left[\frac{n}{2^h}\right]$.

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Egy középiskolás ”versenyfeladat”, megoldással

Egydimenziós sziget

0	0	1	1	1	0	0	0

Szigetek száma: 1

0	1	2	2	1	0	1	2	1	2

Szigetek száma: 5

Egy középiskolás "versenyfeladat", megoldással

A feladat:

Legfeljebb hány sziget keletkezhet egy n hosszúságú négyzetrácson, ha a cellákba írt magasságok csak a következők lehetnek: $0, 1, 2, \ldots, h$; ahol $h \geq 1$. Feltételezzük, hogy a cellasor két végén 0 található (tehát a 0-dik és az $n+1$-edik cellában a magasság 0).

A megoldás:
$I(n, h) = n - \left[\frac{n}{2^h}\right]$.

Egy középiskolás "versenyfeladat", megoldással

Először h szerinti indukcióval bizonyítjuk, hogy van legalább $n - \left[\frac{n}{2^h}\right]$ számú sziget, méghozzá olyan módon, hogy minden második cella h magasságú, az elsővel kezdve.

$h = 1$:
$I(n,h) \geq n - \left[\frac{n}{2}\right]$.
1-essel kezdve felváltva írunk 1-est és 0-t, így éppen ennyi szigetet kapunk.

Ezután legyen $h > 1$.

Egy középiskolás "versenyfeladat", megoldással

Legyen először $n = 4k$. Az indukciós feltevés: $2k$ számú cellán az $0, 1, \dots, h-1$ magasságokkal keletkezhet legalább

$$2k - \left[\frac{2k}{2^{h-1}}\right]$$

sziget, és minden második cella magassága $h-1$, az elsővel kezdve. Ekkor a $h-1$-es magasságú cellák helyére három cellát "betoldunk" h, $h-1$, h magasságokkal. Ezen a módon $n = 4k$ cella keletkezik

$$2k - \left[\frac{2k}{2^{h-1}}\right] + 2k = 4k - \left[\frac{4k}{2^{h}}\right]$$

számú szigettel. Az ábrán $h = 3$; az eredetileg 2 magasságú cella helyére három cellát illesztettünk be 3, 2, 3 magasságokkal.

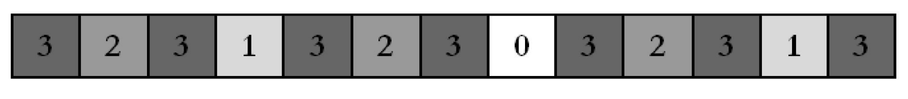
Egy középiskolás "versenyfeladat", megoldással

Legyen most $n = 4k + 1$. Ekkor a végére illesztett h magasságú cellával van

$$4k + 1 - \left[\frac{4k}{2^h}\right]$$

szigetünk, azonban

$$4k + 1 - \left[\frac{4k}{2^h}\right] \geq 4k + 1 - \left[\frac{4k+1}{2^h}\right].$$



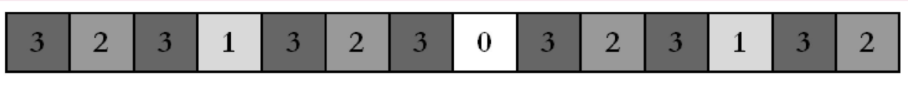
Egy középiskolás "versenyfeladat", megoldással

Legyen most $n = 4k + 2$. Ekkor a végére illesztett h és $h - 1$ magasságú cellákkal van

$$4k + 2 - \left[\frac{4k}{2^h}\right]$$

szigetünk, azonban

$$4k + 2 - \left[\frac{4k}{2^h}\right] \geq 4k + 2 - \left[\frac{4k+2}{2^h}\right].$$



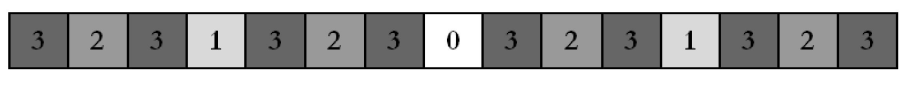
Egy középiskolás "versenyfeladat", megoldással

Legyen most $n = 4k + 3$. Ekkor a végére illesztett h, $h - 1$ és h magasságú cellákkal van

$$4k + 3 - \left[\frac{4k}{2^h}\right]$$

szigetünk, azonban

$$4k + 3 - \left[\frac{4k}{2^h}\right] \geq 4k + 3 - \left[\frac{4k+3}{2^h}\right].$$



Tehát beláttuk, hogy

$$I(n, h) \geq n - \left[\frac{n}{2^h}\right].$$

Egy középiskolás "versenyfeladat", megoldással

Most belátjuk, hogy nincs több sziget, vagyis

$$I(n,h) \leq n - \left[\frac{n}{2^h}\right],$$

n szerinti indukcióval.

Ha $n = 1$, akkor az állítás igaz.

Legyen $n > 1$. Az indukciós feltevés: bármely $n' < n$ esetén

$$I(n',h) = n' - \left[\frac{n'}{2^h}\right].$$

Először feltesszük, hogy van 0 magaságú cellánk. Egy ilyen 0-t tartalmazó cella k and l hosszúságú részre osztja a cellasort, ahol $k + l + 1 = n$, $k, l \geq 0$. Ha a szigetek száma most $|\mathcal{I}|$, akkor

$$|\mathcal{I}| \leq k - \left[\frac{k}{2^h}\right] + l - \left[\frac{l}{2^h}\right] = k + l + 1 - \left(\left[\frac{k}{2^h}\right] + \left[\frac{l}{2^h}\right] + 1\right).$$

Először belátjuk a következő egyenlőtlenséget:

$$\left[\frac{k}{2^h}\right] + \left[\frac{l}{2^h}\right] + 1 \geq \left[\frac{k+l+1}{2^h}\right].$$

Ehhez felhasználjuk az alábbiakat:

$$\left[\frac{k+l+1}{2^h}\right] \leq \frac{k}{2^h} + \frac{l}{2^h} + \frac{1}{2^h} \leq \left[\frac{k}{2^h}\right] + \frac{2^h-1}{2^h} + \left[\frac{l}{2^h}\right] + \frac{2^h-1}{2^h} + \frac{1}{2^h} =$$
$$= \left[\frac{k}{2^h}\right] + \left[\frac{l}{2^h}\right] + \frac{2^{h+1}-1}{2^h}.$$

Vegyük az egészrészét a következő (imént kapott) egyenlőtlenség mindkét oldalának:

$$\left[\frac{k+l+1}{2^h}\right] \leq \left[\frac{k}{2^h}\right] + \left[\frac{l}{2^h}\right] + \frac{2^{h+1}-1}{2^h},$$

kapjuk:

$$\left[\frac{k+l+1}{2^h}\right] \leq \left[\frac{k}{2^h}\right] + \left[\frac{l}{2^h}\right] + 1.$$

Ez utóbbiból pedig adódik a következő:

$$|I| \leq k+l+1-\left(\left[\frac{k}{2^h}\right] + \left[\frac{l}{2^h}\right] + 1\right) \leq k+l+1-\left[\frac{k+l+1}{2^h}\right] = n-\left[\frac{n}{2^h}\right].$$

Egy középiskolás "versenyfeladat", megoldással

Ha nem használjuk a 0 magasságot, akkor először m-et írjunk a határoló cellákba (a 0-dik és az $n+1$-edik cellába), ahol m a celláinkban szereplő számok minimuma. Az előbb igazoltak miatt legfeljebb

$$n - \left[\frac{n}{2^{h-m}}\right]$$

szigetünk van. Csökkentjük a határcellák magasságát $m-1$-re, ekkor a teljes cellasor szigetté válik, vagyis most legfeljebb

$$n - \left[\frac{n}{2^{h-m}}\right] + 1$$

szigetünk van. Mivel $\left[\frac{n}{2^{h-m}}\right] \geq \left[\frac{n}{2^{h-1}}\right]$, kapjuk, hogy

$$n - \left[\frac{n}{2^{h-m}}\right] + 1 \leq n - \left[\frac{n}{2^{h-1}}\right] + 1.$$

Egy középiskolás "versenyfeladat", megoldással

Azonban

$$n - \left[\frac{n}{2^{h-1}}\right] + 1 = n - \left[\frac{2n - 2^h}{2^h}\right].$$

Ha $n \geq 2^h$, akkor $2n - 2^h \geq n$, tehát legfeljebb

$$n - \left[\frac{2n - 2^h}{2^h}\right] \leq n - \left[\frac{n}{2^h}\right]$$

szigetünk van.

Ha $n < 2^h$, akkor $\left[\frac{n}{2^h}\right] = 0$, tehát elég belátni, hogy a szigetek száma az n hosszúságú cellasoron nem lehet több, mint n (h-tól függetlenül). Ezt n szerinti indukcióval látjuk be. Ha $n = 1$, akkor legfeljebb egyetlen szigetünk van. Legyen $n > 1$. Az indukciós feltevés: ha $n' < n$, akkor az n' hosszúságú cellasoron a szigetek száma nem haladhatja meg n'-t. Egy minimális magasságú cella k és $n - k - 1$ hosszúságú részre osztja az n hosszúságú cellasort, ahol $k \geq 0$. Azonban a teljes cellasor lehet sziget, vagyis az indukciós feltevés alkalmazása után adódik, hogy a szigetek száma legfeljebb $k + n - k - 1 + 1 = n$.

Téglalapszigetek hengeren (bizonyítás)

Törlünk egy cellaoszlopot, $m \times (n-1)$ méretű téglalapot kapunk. Ezért

$$c_1(m,n) \geq f(m,n-1)+1 = [(mn+n)/2] \,.$$

Legyen $\mathcal{I}^*$ maximális sok szigetet tartalmazó szigetrendszer. Ekkor

$$c_1(m,n) = 1 + \sum_{R \in max \mathcal{I}^*} f(R) = 1 + \sum_{R \in max \mathcal{I}^*} \left(\left[\frac{(u+1)(v+1)}{2} \right] - 1 \right) =$$

$$= 1 - |max(\mathcal{I}^*)| + \sum_{R \in max \mathcal{I}^*} \left[\frac{(u+1)(v+1)}{2} \right] \leq$$

$$\leq 1 - 1 + \left[\frac{(m+1)n}{2} \right] = \left[\frac{(m+1)n}{2} \right].$$

Nyilván $-|max(\mathcal{I}^*)| \leq -1$ ha $|max(\mathcal{I}^*)| \geq 1$; valamint ábra felrajzolásával kiderül, hogy

$$\sum_{R \in max \mathcal{I}^*} \left[\frac{(u+1)(v+1)}{2} \right] \leq \left[\frac{(m+1)n}{2} \right].$$

Téglalapszigetek tóruszon (bizonyítás)

Méret: $m \times n$.

Egy oszlopot és egy sort kihagyva $(m-1) \times (n-1)$ téglalap adódik. Ezért:

$$t(m,n) \geq f(m-1,n-1)+1 = \left[\frac{mn}{2}\right].$$

Ismét $\mathcal{I}^*$ egy maximálisan sok szigetet tartalmazó szigetrendszer. Ekkor

$$t(m,n) = 1 + \sum_{R \in max \mathcal{I}^*} f(R) = 1 + \sum_{R \in max \mathcal{I}^*} \left(\left[\frac{(u+1)(v+1)}{2} \right] - 1 \right) =$$

$$= 1 - |max(\mathcal{I}^*)| + \sum_{R \in \max \mathcal{I}^*} \left[\frac{(u+1)(v+1)}{2} \right] \leq 1 - 1 + \left[\frac{mn}{2}\right] = \left[\frac{mn}{2}\right].$$

Ismét felhasználtuk, hogy $-|max(\mathcal{I}^*)| \leq -1$ ha $|max(\mathcal{I}^*)| \geq 1$, továbbá azt is, hogy

$$\sum_{R \in \max \mathcal{I}^*} \left[\frac{(u+1)(v+1)}{2} \right] \leq + \left[\frac{mn}{2}\right].$$