Cuts of lattice-valued functions and applications to islands

Branimir Šešelja

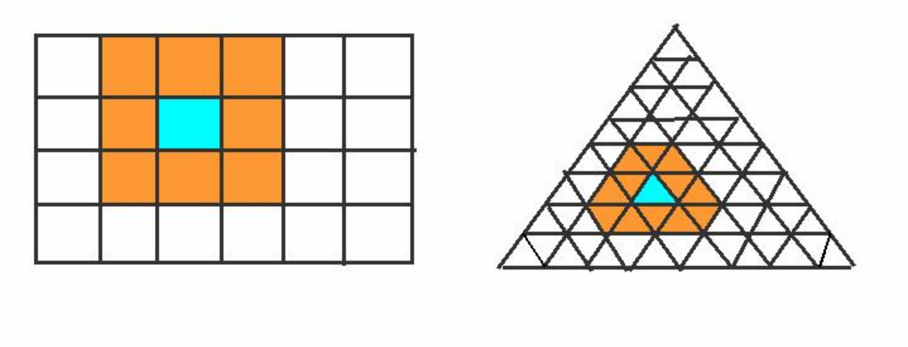
Andreja Tepavčević

Eszter K. Horváth

AAA 79

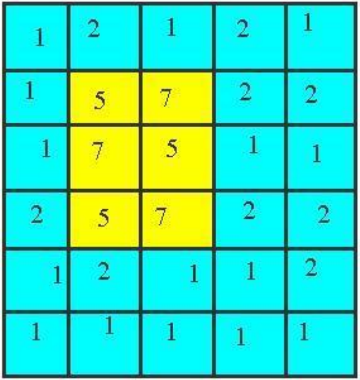
Definition/1

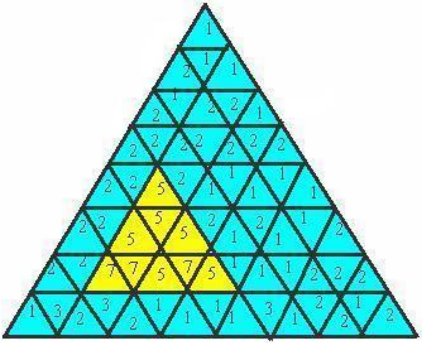
Grid, neighbourhood relation



Definition/2

We call a rectangle/triangle an *island*, if for the cell t, if we denote its height by a_t, then for each cell $\hat{t}$ neighbouring with a cell of the rectangle/triangle T, the inequality $a_{\hat{t}} < min\{a_t : t \in T\}$ holds.





History/1

Coding theory

S. Földes and N. M. Singhi: On instantaneous codes, J. of Combinatorics, Information and System Sci., 31 (2006), 317-326.

History/1

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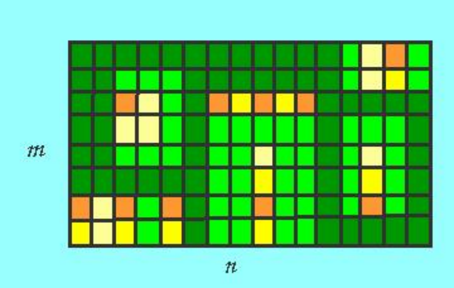
History/2

Rectangular islands

G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics 30 (2009), 208-215.

The maximum number of rectangular islands in a $m \times n$ rectangular board on square grid:

$$f(m,n) = [\frac{mn + m + n - 1}{2}].$$



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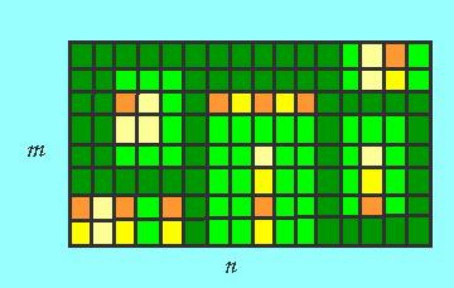

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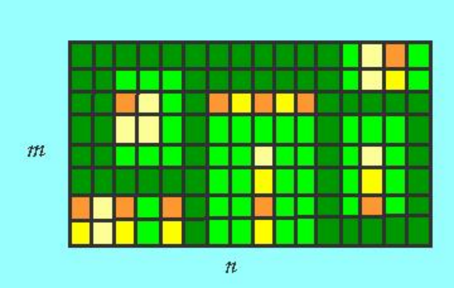

History/3

Rectangular islands in higher dimensions

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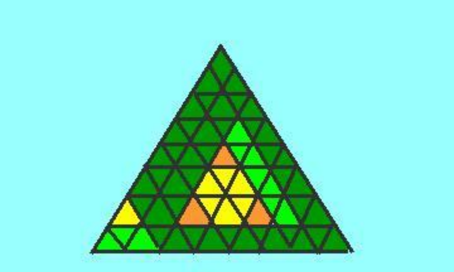
For the maximum number of triangular islands in an equilateral rectangle of side length n, $\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}$ holds.



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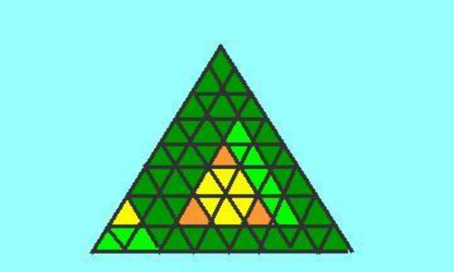

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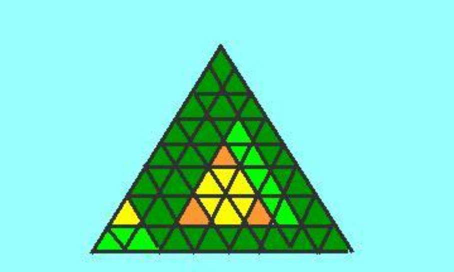

History/5

Square islands (also in higher dimensions)

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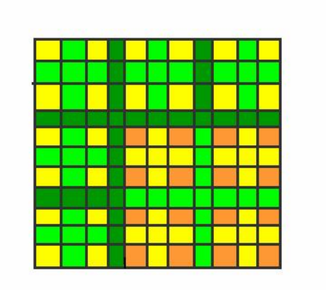

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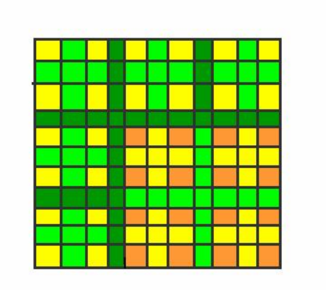

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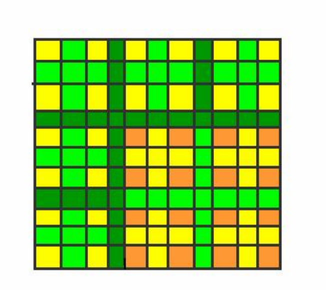

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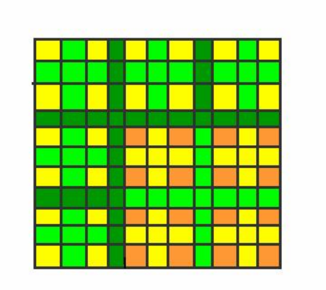

Proving methods/1

LATTICE THEORETICAL METHOD

G. Czédli, A. P. Huhn and E. T. Schmidt: Weakly independent subsets in lattices, Algebra Universalis 20 (1985), 194-196.

Any two weak bases of a finite distributive lattice have the same number of elements.

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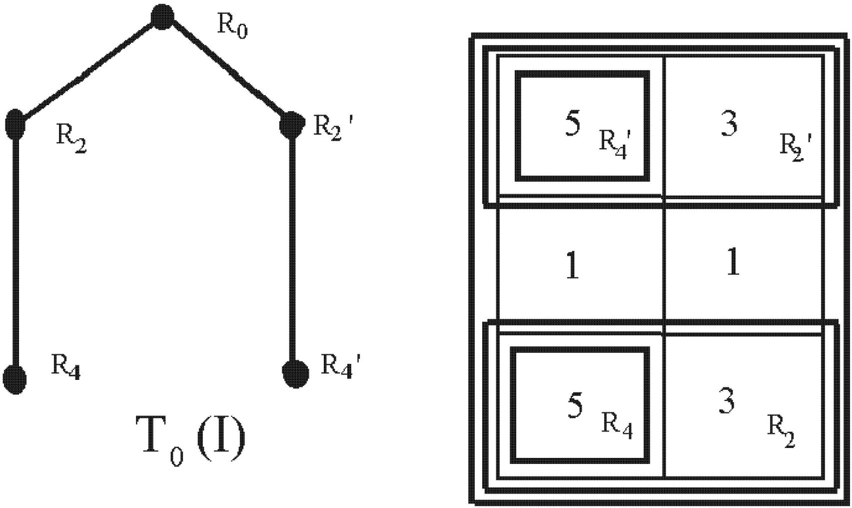
Proving methods/2

TREE-GRAPH METHOD



Proving methods/2

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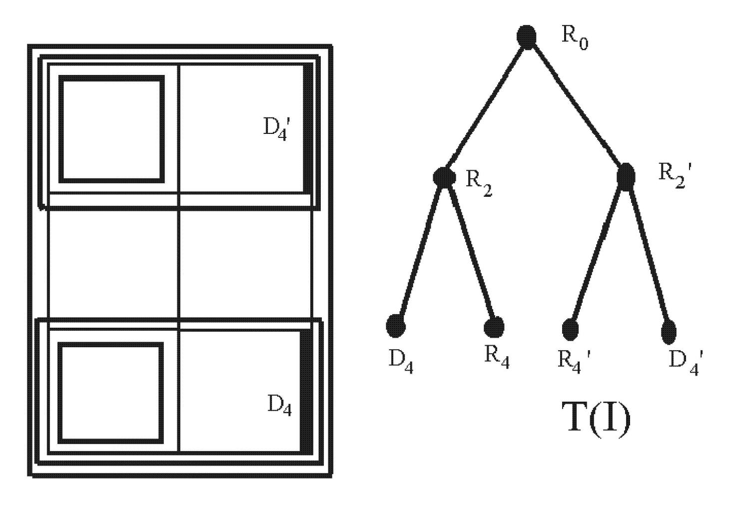

Proving methods/2

TREE-GRAPH METHOD

Lemma 2 (folklore)

(i) Let T be a binary tree with ℓ leaves. Then the number of vertices of T depends only on ℓ, moreover $|V| = 2\ell - 1$.

(ii) Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T. Then $|V| \leq 2\ell - 1$.

We have $4s + 2d \leq (n+1)(m+1)$.
The number of leaves of $T(\mathcal{I})$ is $\ell = s + d$. Hence by Lemma 2 the number of islands is

$$|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n+1)(m+1) - 1.$$

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Proving methods/3

ELEMENTARY METHOD

We define

$$\mu(R) = \mu(u, v) := (u + 1)(v + 1).$$

Now

$$f(m, n) = 1 + \sum_{R \in max\mathcal{I}} f(R) = 1 + \sum_{R \in max\mathcal{I}} ([\frac{(u+1)(v+1)}{2}] - 1)$$

$$= 1 + \sum_{R \in max\mathcal{I}} ([\frac{\mu(u, v)}{2}] - 1) \leq 1 - |max\mathcal{I}| + [\frac{\mu(C)}{2}].$$

If $|max\mathcal{I}| \geq 2$, then the proof is ready. Case $|max\mathcal{I}| = 1$ is an easy excersise.

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History/6

Some exact formulas

Peninsulas (semi islands) (J. Barát, P. Hajnal, E.K. Horváth):
$p(m,n) = f(m,n) = [(mn + m + n - 1)/2]$.

Cylindric board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_1(m,n) = [\frac{(m+1)n}{2}]$.

Cylindric board, cylindric and rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
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History/7

Further results on rectangular islands

Zs. Lengvárszky: The minimum cardinality of maximal systems of rectangular islands, European Journal of Combinatorics, **30** (2009), 216-219.

History/7

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History/8

The board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $BA = \{0,1\}^n$.

We consider two cells neighbouring if their Hamming distance is 1.

We denote the maximum number of islands in $BA = \{0,1\}^n$ by $b(n)$.

Island formula for Boolean algebras (P. Hajnal, E.K. Horváth)
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Rectangular height functions/1

Joint work with Branimir Šešelja and Andreja Tepavčević

A *height function* h is a mapping from $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$ to $\mathbb{N}$, $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$.

The co-domain of the height function is the lattice $(\mathbb{N}, \leq)$, where $\mathbb{N}$ is the set of natural numbers under the usual ordering $\leq$ and suprema and infima are max and min, respectively.

For every $p \in \mathbb{N}$, the *cut of the height function*, i.e. the *p-cut* of h is an ordinary relation h_p on $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$ defined by

$$(x, y) \in h_p \text{ if and only if } h(x, y) \geq p.$$

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Rectangular height functions/2

We say that two rectangles $\{\alpha, ..., \beta\} \times \{\gamma, ..., \delta\}$ and $\{\alpha_1, ..., \beta_1\} \times \{\gamma_1, ..., \delta_1\}$ are *distant* if they are disjoint and for every two cells, namely (a, b) from the first rectangle and (c, d) from the second, we have $(a - c)^2 + (b - d)^2 \geq 4$.

The height function h is called *rectangular* if for every $p \in \mathbb{N}$, every nonempty p-cut of h is a union of distant rectangles.

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Rectangular height functions/3

5	5	3	5	5
4	4	2	4	4
2	2	1	2	2

$\Gamma_1 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\}$,
$\Gamma_2 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\} \setminus \{(3, 1)\}$,
$\Gamma_3 = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 3), (4, 2), (4, 3), (5, 2), (5, 3)\}$,
$\Gamma_4 = \{(1, 2), (1, 3), (2, 2), (2, 3), (4, 2), (4, 3), (5, 2), (5, 3)\}$ and
$\Gamma_5 = \{(1, 3), (2, 3), (4, 3), (5, 3)\}$

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Rectangular height functions/4
CHARACTERIZATION THEOREM

Theorem 1

A height function $h_{\mathbb{N}} : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ is rectangular if and only if for all (α, γ), $(\beta, \delta) \in \{1, 2, ..., m\} \times \{1, 2, ..., n\}$ either

- these are not neighboring cells and there is a cell (μ, ν) between (α, γ) and (β, δ) such that $h_{\mathbb{N}}(\mu, \nu) < \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}$, or
- for all $(\mu, \nu) \in [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}] \times [\min\{\gamma, \delta\}, \max\{\gamma, \delta\}]$,

$$h_{\mathbb{N}}(\mu, \nu) \geq \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}.$$

Rectangular height functions/4
CHARACTERIZATION THEOREM

Theorem 1

A height function $h_{\mathbb{N}} : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ is rectangular if and only if for all (α, γ), $(\beta, \delta) \in \{1, 2, ..., m\} \times \{1, 2, ..., n\}$ either

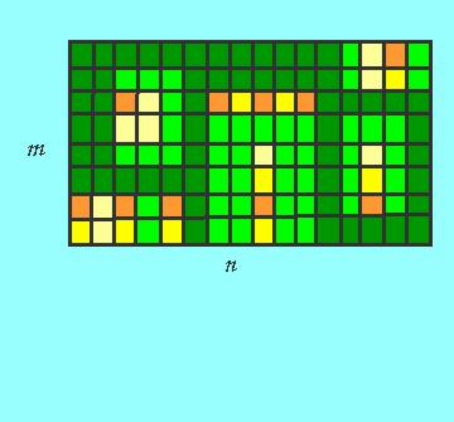
- these are not neighboring cells and there is a cell (μ, ν) between (α, γ) and (β, δ) such that $h_{\mathbb{N}}(\mu, \nu) < \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}$, or
- for all $(\mu, \nu) \in [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}] \times [\min\{\gamma, \delta\}, \max\{\gamma, \delta\}]$,

$$h_{\mathbb{N}}(\mu, \nu) \geq \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}.$$

Rectangular height functions/5

Theorem 2

For every height function $h : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, there is a rectangular height function $h^* : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, such that $\mathcal{I}_{rect}(h) = \mathcal{I}_{rect}(h^*)$.



Rectangular height functions/6
CONSTRUCTING ALGORITHM

1. FOR $i = t$ TO 0
2. FOR $y = 1$ TO n
3. FOR $x = 1$ TO m
4. IF $h(x, y) = a_i$ THEN
5. j:= i
6. WHILE there is no island of h which is a subset of h_{a_j} that contains (x, y) DO j:=j-1
7. ENDWHILE
8. Let $h^*(x, y) := a_j$.
9. ENDIF
10. NEXT x
11. NEXT y
12. NEXT i
13. END.

Rectangular height functions/7
LATTICE-VALUED REPRESENTATION

Theorem 3

Let $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ be a rectangular height function. Then there is a lattice L and an L-valued mapping Φ, such that the cuts of Φ are precisely all islands of h.

Rectangular height functions/8

Let $h : \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} \to \mathbb{N}$ be a height function.

4	9	8	7	1	5
3	8	8	7	1	4
2	7	7	7	1	5
1	2	2	2	1	6
	1	2	3	4	5

Rectangular height functions/9

h is a rectangular height function. Its islands are:

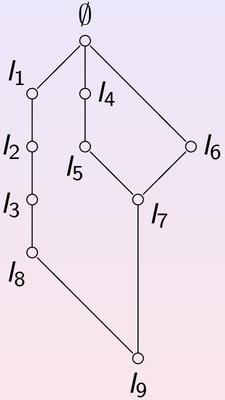
$I_1 = \{(1,4)\}$,
$I_2 = \{(1,3),(1,4),(2,3),(2,4)\}$,
$I_3 = \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$,
$I_4 = \{(5,1)\}$,
$I_5 = \{(5,1),(5,2)\}$,
$I_6 = \{(5,4)\}$,
$I_7 = \{(5,1),(5,2),(5,3),(5,4)\}$,
$I_8 = \{(1,2),(1,3),(1,4),(2,2),(2,3),$
$(2,4),(3,2),(3,3),(3,4),(1,1),(2,1),(3,1)\}$,
$I_9 = \{1,2,3,4,5\} \times \{1,2,3,4\}$.

Rectangular height functions/10

Its cut relations are:

$h_{10} = \emptyset$
$h_9 = I_1$ (one-element island)
$h_8 = I_2$ (four-element square island)
$h_7 = I_3$ (nine-element square island)
$h_6 = I_3 \cup I_4$ (this cut is a disjoint union of two islands)
$h_5 = I_3 \cup I_5 \cup I_6$ (union of three islands)
$h_4 = I_3 \cup I_7$ (union of two islands)
$h_2 = I_7 \cup I_8$ (union of two islands)
$h_1 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} = I_9$ (the whole domain)

Rectangular height functions/11



$L = (\mathcal{I}_0(\Gamma), \supseteq)$

Rectangular height functions/12

Theorem 4

For every rectangular height function

$$h^* : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N},$$

there is a rectangular height function

$$h^{**} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N},$$

such that $\mathcal{I}_{rect}(h^*) = \mathcal{I}_{rect}(h^{**})$ and in h^{**} every island appears exactly in one cut.

If a rectangular height function h^{**} has the property that each island appears exactly in one cut, then we call it *standard rectangular height function.*

Rectangular height functions/12

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Rectangular height functions/13

We denote by $\Lambda_{max}(m, n)$ the maximum number of different nonempty p-cuts of a standard rectangular height function on the rectangular table of size $m \times n$.

Theorem 5 $\Lambda_{max}(m, n) = m + n - 1.$

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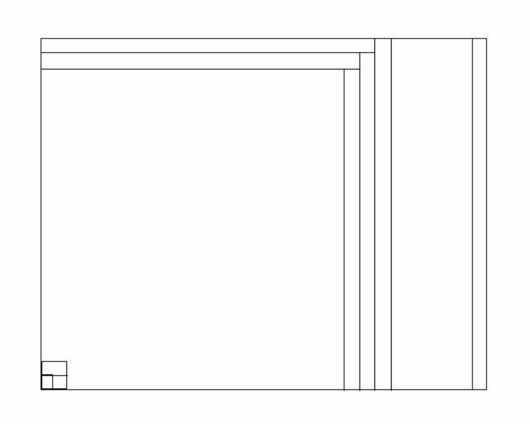
Rectangular height functions/14



The maximum number of different nonempty p-cuts of a standard rectangular height function is equal to the minimum cardinality of maximal systems of islands.

Lemma 1

If $m \geq 3$ and $n \geq 3$ and a height function $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ has maximally many islands, then it has exactly two maximal islands.

Lemma 2

If $m \geq 3$ or $n \geq 3$, then for any odd number $t = 2k + 1$ with $1 \leq t \leq max\{m - 2, n - 2\}$, there is a standard rectangular height function $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ having the maximum number of islands f(m,n), such that one of the side-lengths of one of the maximal islands is equal to t.
(Remark: The statement is not true for even side-lengths, one can construct counterexample easily!)

Rectangular height functions/15

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Rectangular height functions/16

We denote by $\Lambda_h^{cz}(m,n)$ the number of different nonempty cuts of a standard rectangular height function h in the case h has maximally many islands, i.e., when the number of islands is

$$f(m,n) = \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor .$$

Theorem 6

Let $h : \{1,2,...,m\} \times \{1,2,...,n\} \to \mathbb{N}$ be a standard rectangular height function having maximally many islands $f(m,n)$. Then, $\Lambda_h^{cz}(m,n) \geq \lceil log_2(m+1) \rceil + \lceil log_2(n+1) \rceil - 1$.

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Islands and formal concepts

Neither every island is a formal concept, nor every formal concept is an island.

Let (C, D) be a formal concept of the context $\mathbb{K} := (A, B, I)$. Then, there are linear orderings $\leq_1$ and $\leq_2$ on A and B, respectively, such that $C \times D$ is an island of $I \subseteq (A, \leq_1) \times (B, \leq_2)$ if and only if there is an element $a \in A$ such that $a \notin C$ and $b \in B$ such that $b \notin D$, with $(a, b) \notin I$, $(a, y) \notin I$ for every $y \in D$ and $(x, b) \notin I$ for every $x \in C$.

If $A_1 \times B_1$ is an island of a relation $I \subseteq (A, \leq_1) \times (B, \leq_2)$, then (A_1, B_1) is a concept if and only if there is no $a \in A \setminus A_1$ such that $(a, b) \in I$ for all $b \in B_1$ and there is no $b \in B \setminus B_1$ such that $(a, b) \in I$ for all $a \in A_1$.

Let I be a relation $I \subseteq (A, \leq_1) \times (B, \leq_2)$. Then, every island in a relation I is a concept if and only if every $x \in A$ belongs to not more than one island and every $y \in B$ belongs to not more then one island.

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