Islands and independence notions

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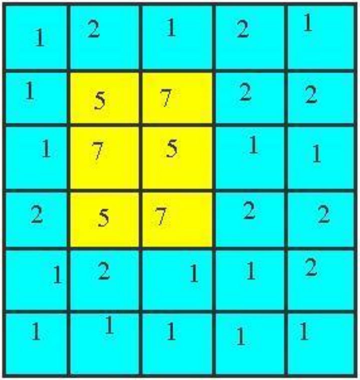
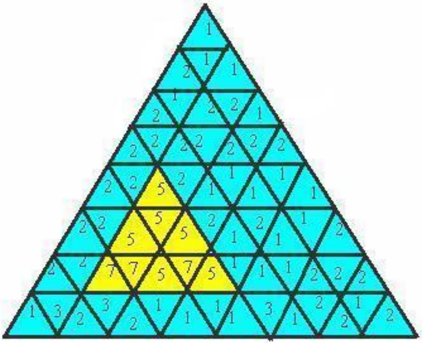
Islands?

Rectangular and triangular islands

We call a rectangle/triangle a *rectangular/triangular island*, if for the cell t, if we denote its height by a_t, then for each cell $\hat{t}$ neighbouring with a cell of the rectange/triangle T, the inequality $a_{\hat{t}} < min\{a_t : t \in T\}$ holds.



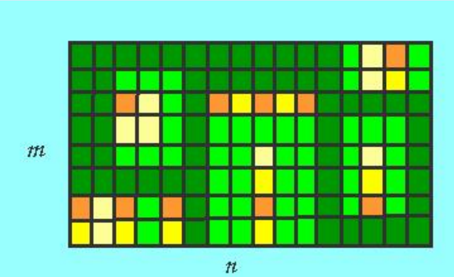


Rectangular islands

G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics 30 (2009), 208-215.

The maximum number of rectangular islands in a $m \times n$ rectangular board on square grid:

$$f(m,n) = \left[\frac{mn + m + n - 1}{2}\right].$$



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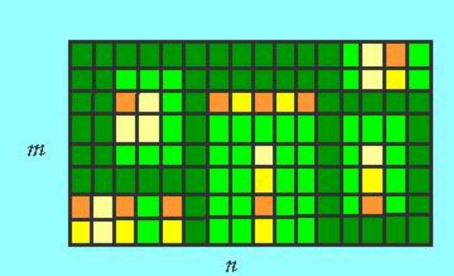

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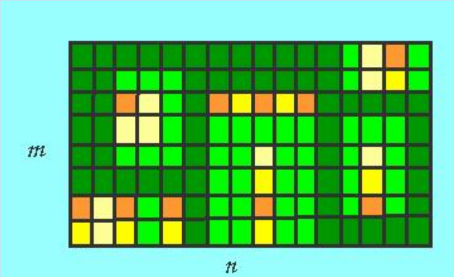
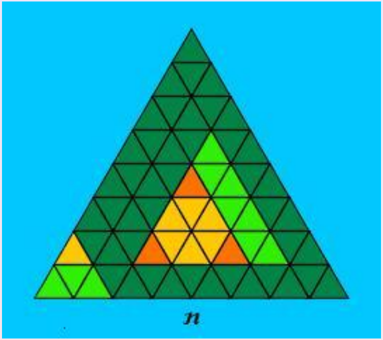


Triangular islands

For the maximum number of triangular islands in an equilateral triangle of side length n, $\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}$ holds.



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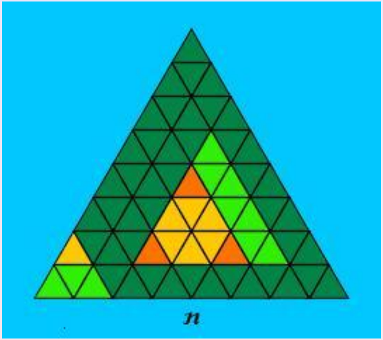

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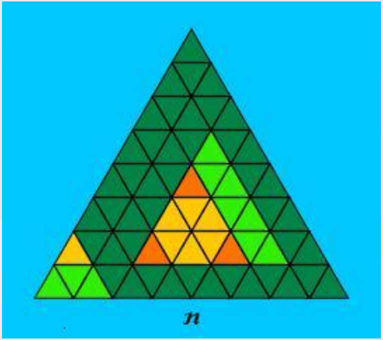

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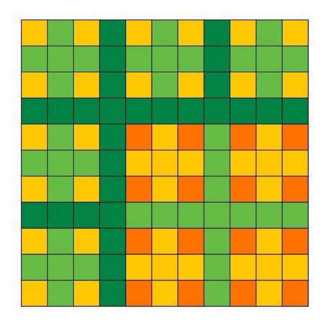
Square islands (also in higher dimensions)

$$\frac{1}{3}(rs - 2r - 2s) \leq f(r, s) \leq \frac{1}{3}(rs - 1)$$



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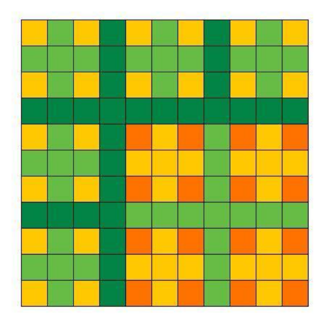

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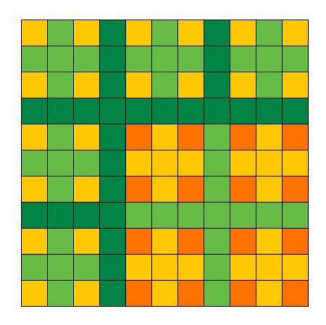

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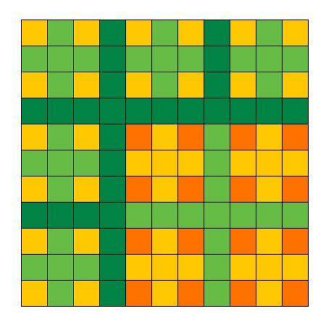

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CD-independent subsets in posets

Definitions

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set and $a, b \in P$. The elements a and b are called *disjoint* and we write $a \perp b$ if

- either $\mathbb{P}$ has least element $0 \in P$ and $\inf\{a, b\} = 0$,
- or if $\mathbb{P}$ is without 0, then a and b have no common lower bound.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$ or $x \perp y$ holds.

Maximal CD-independent sets (with respect to $\subseteq$) are called *CD-bases* in $\mathbb{P}$.

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CD-independent subsets in distributive lattices

Any two CD-bases of a finite distributive lattice have the same number of elements.

If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

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Definitions

$U \in \mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{P}(U)$
Let $h\colon U \to \mathbb{R}$ be a height function and let $S \in \mathcal{C}$ be a nonempty set.

We say that S is an *island* with respect to the triple $(\mathcal{C}, \mathcal{K}, h)$, if every $K \in \mathcal{K}$ with $S \prec K$ satisfies

$$\min h(K) < \min h(S).$$

We say that S is a *strict island* with respect to the triple $(\mathcal{C}, \mathcal{K}, h)$, if every $K \in \mathcal{K}$ with $S \prec K$ satisfies

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Example

Let $G = (U, E)$ be a connected simple graph with vertex set U and edge set E; let $\mathcal{K}$ consist of the connected subsets of U, and let $\mathcal{C} \subseteq \mathcal{K}$ such that $U \in \mathcal{C}$.

Example

Let $A_1, \ldots, A_n$ be nonempty sets, and let $\mathcal{I} \subseteq A_1 \times \cdots \times A_n$. Let us define

$$U = A_1 \times \cdots \times A_n,$$
$$\mathcal{K} = \{B_1 \times \cdots \times B_n \colon \emptyset \neq B_i \subseteq A_i,\ 1 \leq i \leq n\}$$
$$\mathcal{C} = \{C \in \mathcal{K} \colon C \subseteq \mathcal{I}\} \cup \{U\},$$

and let $h \colon U \longrightarrow \{0, 1\}$ be the height function given by

$$h(a_1, \ldots, a_n) := \begin{cases} 1, & \text{if } (a_1, \ldots, a_n) \in \mathcal{I}; \\ 0, & \text{if } (a_1, \ldots, a_n) \in U \setminus \mathcal{I}; \end{cases} \quad \text{for all } (a_1, \ldots, a_n) \in U.$$

It is easy to see that the islands corresponding to the triple $(\mathcal{C}, \mathcal{K}, h)$ are exactly U and the maximal elements of the poset $(\mathcal{C} \setminus \{U\}, \subseteq)$.

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Islands and admissible systems

Definition

Let $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$ be a family of sets such that $U \in \mathcal{H}$. We say that $\mathcal{H}$ is *admissible*, if for every nonempty antichain $\mathcal{A} \subseteq \mathcal{H}$

$$\exists H \in \mathcal{A}\ \forall K \in \mathcal{K} :\ H \subset K \implies K \nsubseteq \bigcup \mathcal{A}. \tag{1}$$

Proposition

Every system of islands is admissible.

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Proof

Let $h\colon U \to \mathbb{R}$ be a height function and let $\mathcal{S}$ be the system of islands corresponding to $(\mathcal{C}, \mathcal{K}, h)$. Clearly, we have $\emptyset \notin \mathcal{S}$ and $U \in \mathcal{S}$. Let us assume for contradiction that there exists an antichain $\mathcal{A} = \{S_i : i \in I\} \subseteq \mathcal{S}$ such that (1) does not hold. Then for every $i \in I$ there exists $K_i \in \mathcal{K}$ such that $S_i \subset K_i$ and $K_i \subseteq \bigcup_{i \in I} S_i$. Since S_i is an island, we have

$$\min h(S_i) > \min h(K_i) \geq \min h\Big(\bigcup_{i \in I} S_i\Big)$$

for all $i \in I$. Taking the minimum of these inequalities we arrive at the contradiction

$$\min \{\min h(S_i) \mid i \in I\} > \min h\Big(\bigcup_{i \in I} S_i\Big).$$

Canonical height functions

Let $\mathcal{H} \subseteq \mathcal{C}$ be an admissible family of sets.

We define subfamilies $\mathcal{H}^{(i)} \subseteq \mathcal{H}$ $(i = 0, 1, 2, \ldots)$ recursively as follows.
Let $\mathcal{H}^{(0)} = \{U\}$.

For $i > 0$, if $\mathcal{H} \neq \mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(i-1)}$, then let $\mathcal{H}^{(i)}$ consist of all those sets $H \in \mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(i-1)})$ that have the following property:

$$\forall K \in \mathcal{K} : \; H \subset K \implies K \not\subseteq \bigcup \left(\mathcal{H} \setminus (\mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(i-1)})\right). \qquad (2)$$

Since $\mathcal{H}$ is finite and admissible, after finitely many steps we obtain a partition $\mathcal{H} = \mathcal{H}^{(0)} \cup \cdots \cup \mathcal{H}^{(r)}$.

The *canonical height function corresponding to* $\mathcal{H}$ is the function $h_{\mathcal{H}} \colon U \to \mathbb{N}$ defined by

$$h_{\mathcal{H}}(x) := \max\left\{ i \in \{1, \ldots, r\} : x \in \bigcup \mathcal{H}^{(i)} \right\} \text{ for all } x \in U. \qquad (3)$$

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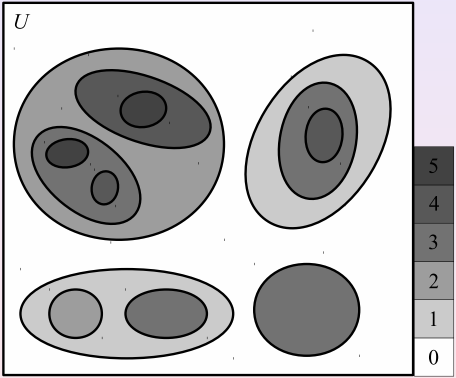

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If $\mathcal{H} \subseteq \mathcal{C}$ is an admissible family of sets and $h_{\mathcal{H}}$ is the corresponding canonical height function, then every member of $\mathcal{H}$ is an island with respect to $(\mathcal{C}, \mathcal{K}, h_{\mathcal{H}})$.

Theorem

A subfamily of $\mathcal{C}$ is a maximal system of islands if and only if it is a maximal admissible family.

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Subsets of island systems

The following two conditions are equivalent for any pair $(\mathcal{C}, \mathcal{K})$:

Any subset of a system of islands corresponding to $(\mathcal{C}, \mathcal{K})$ is also a system of islands.

The systems of islands corresponding to $(\mathcal{C}, \mathcal{K})$ are exactly the admissible families.

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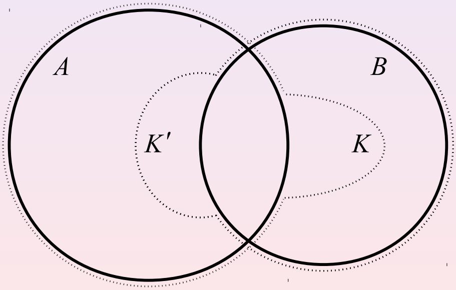
Island domains

Definition

A pair $(\mathcal{C}, \mathcal{K})$ is an *island domain* if

$$\forall A, B \in \mathcal{C} : \ (A \cap B \neq \emptyset \text{ and } B \not\subseteq A) \implies \exists K \in \mathcal{K} : A \subset K \subseteq A \cup B.$$



CDW-independence

Definition A family $\mathcal{H} \subseteq \mathcal{P}(U)$ is *weakly independent* if

$$H \subseteq \bigcup_{i \in I} H_i \implies \exists i \in I : H \subseteq H_i \tag{4}$$

holds for all $H \in \mathcal{H}, H_i \in \mathcal{H}\,(i \in I)$. If $\mathcal{H}$ is both CD-independent and weakly independent, then we say that $\mathcal{H}$ is *CDW-independent*.

Lemma

If $(\mathcal{C}, \mathcal{K})$ is an island domain, then every admissible subfamily of $\mathcal{C}$ is CDW-independent. [[But not conversely.]]

Island domains

Theorem

The following three conditions are equivalent for any pair $(\mathcal{C}, \mathcal{K})$:

(i) $(\mathcal{C}, \mathcal{K})$ is an island domain.

(ii) Every system of islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CD-independent.

(iii) Every system of islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CDW-independent.

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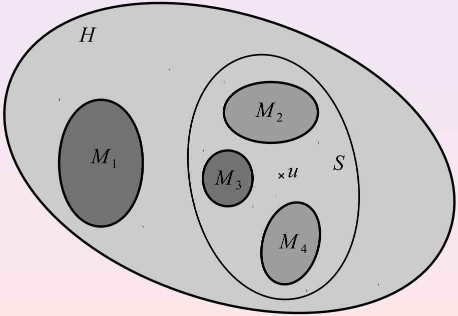
Island domains

Theorem

If $(\mathcal{C}, \mathcal{K})$ is an island domain, then a subfamily of $\mathcal{C}$ is a system of islands if and only if it is admissible.



Theorem

If $(\mathcal{C}, \mathcal{K})$ is an island domain and $\mathcal{S}$ is a system of islands corresponding to $(\mathcal{C}, \mathcal{K})$, then $|S| \leq |U|$.

Proof.

Let $(\mathcal{C}, \mathcal{K})$ be an island domain and let $\mathcal{S} \subseteq \mathcal{C} \setminus \{\emptyset\}$ be a system of islands corresponding to $(\mathcal{C}, \mathcal{K})$. $\mathcal{S}$ is CDW-independent, and hence $\mathcal{S} \cup \{\emptyset\}$ is also CDW-independent. From the results of G. Czédli and E. T. Schmidt it follows that every maximal CDW-independent subset of $\mathcal{P}(U)$ has $|U| + 1$ elements. Thus we have $|\mathcal{S}| + 1 \leq |U| + 1$.

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Strict islands and proximity domains

$(\mathcal{C}, \mathcal{K})$

$\delta \subseteq \mathcal{C} \times \mathcal{C}$

$$A\delta B \Leftrightarrow \exists K \in \mathcal{K}: \quad A \preceq K \text{ and } K \cap B \neq \emptyset. \tag{5}$$

It is easy to verify that relation δ satisfies the following properties for all $A, B, C \in \mathcal{C}$:

$$A\delta B \Rightarrow B \neq \emptyset;$$
$$A \cap B \neq \emptyset \Rightarrow A\delta B;$$
$$A\delta(B \cup C) \Leftrightarrow (A\delta B \text{ or } A\delta C).$$

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Strict islands and proximity domains

We say that $A, B \in \mathcal{C}$ are *distant* if neither $A \delta B$ nor $B \delta A$ holds.

It is easy to see that in this case A and B are also incomparable (in fact, disjoint), whenever $A, B \neq \emptyset$.

A nonempty family $\mathcal{H} \subseteq \mathcal{C}$ will be called a *distant family*, if any two incomparable members of $\mathcal{H}$ are distant.

Lemma If $\mathcal{H} \subseteq \mathcal{C}$ is a distant family, then $\mathcal{H}$ is CDW-independent. Moreover, if $U \in \mathcal{H}$, then U is admissible.

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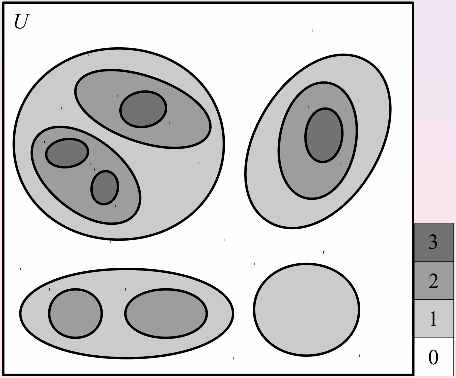
Standard height function

Let us consider a CD-independent family $\mathcal{H}$.

Clearly, for every $u \in U$, the set of members of $\mathcal{H}$ containing u is a finite chain.

The *standard height function* of $\mathcal{H}$ assigns to each element u the length of this chain, i.e., one less than the number of members of $\mathcal{H}$ that contain u.



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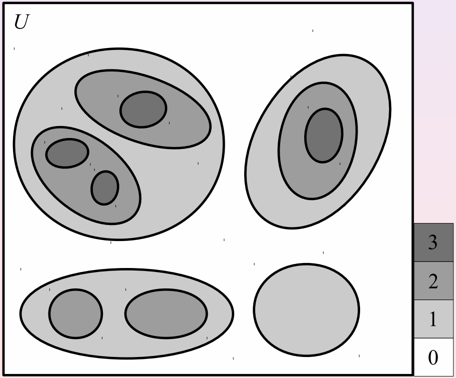

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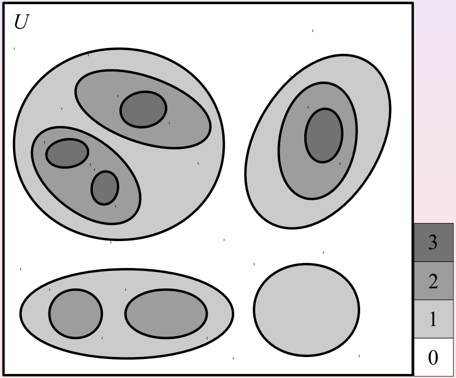

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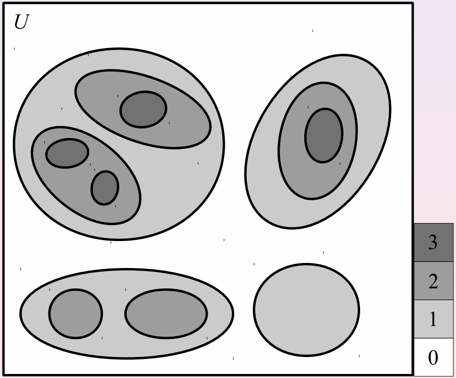

Theorem

Let $(\mathcal{C}, \mathcal{K})$ be an island domain and let $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$ with $U \in \mathcal{H}$. If $\mathcal{H}$ is a distant family, then $\mathcal{H}$ is a system of strict islands; moreover, $\mathcal{H}$ is the system of strict islands corresponding to its standard height function.

Strict islands and proximity domains

The pair $(\mathcal{C}, \mathcal{K})$ is called a *proximity domain*, if it is an island domain and the relation δ is symmetric for nonempty sets, that is

$$\forall A, B \in \mathcal{C} \setminus \{\emptyset\} : \; A\delta B \Leftrightarrow B\delta A. \tag{6}$$

If a relation δ defined on $\mathcal{P}(U)$ satisfies the mentioned three properties and δ is symmetric for nonempty sets, then (U, δ) is called a *proximity space*.

The notion goes back to Frigyes Riesz (1908), however this axiomatization is due to Vadim A. Efremovich.

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Strict islands and proximity domains

Proposition

If $(\mathcal{C}, \mathcal{K})$ is a proximity domain, then any system of strict islands corresponding to $(\mathcal{C}, \mathcal{K})$ is a distant system.

Corollary

If $(\mathcal{C}, \mathcal{K})$ is a proximity domain, and $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$ with $U \in \mathcal{H}$, then $\mathcal{H}$ is a system of strict islands if and only if $\mathcal{H}$ is a distant family. Moreover, in this case $\mathcal{H}$ is the system of strict islands corresponding to its standard height function.

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Strict islands and proximity domains

Finally, let us consider the following condition on $(\mathcal{C}, \mathcal{K})$, which is stronger than that of being an island domain:

$$\forall K_1, K_2 \in \mathcal{K}: \ K_1 \cap K_2 \neq \emptyset \implies K_1 \cup K_2 \in \mathcal{K}. \tag{7}$$

Theorem

Suppose that $(\mathcal{C}, \mathcal{K})$ satisfies condition (7), and assume that for all $C \in \mathcal{C}$, $K \in \mathcal{K}$ with $C \prec K$ we have $|K \setminus C| = 1$. Then $(\mathcal{C}, \mathcal{K})$ is a proximity domain, and islands and strict islands corresponding to $(\mathcal{C}, \mathcal{K})$ coincide. Therefore, if $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$ and $U \in \mathcal{H}$, then $\mathcal{H}$ is a system of (strict) islands if and only if $\mathcal{H}$ is a distant family. Moreover, in this case $\mathcal{H}$ is the system of (strict) islands corresponding to its standard height function.

Strict islands and proximity domains

Corollary

Let G be a graph with vertex set U; let $(\mathcal{C}, \mathcal{K})$ be an island domain corresponding to $(\mathcal{C}, \mathcal{K})$, and let $\mathcal{H} \subseteq \mathcal{C} \setminus \{\emptyset\}$ with $U \in \mathcal{H}$. Then $\mathcal{H}$ is a system of (strict) islands if and only if $\mathcal{H}$ is distant; moreover, in this case $\mathcal{H}$ is the system of (strict) islands corresponding to its standard height function.