CD-independent subsets

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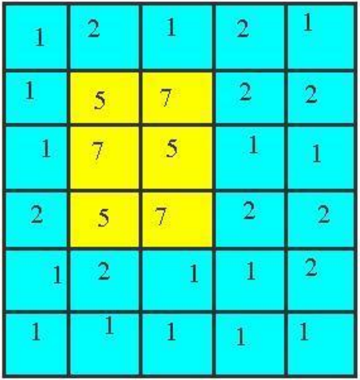
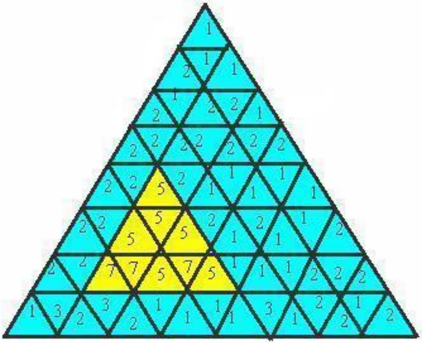
2010 July 5, Novi Sad

Island definition

We call a rectangle/triangle an *island*, if for the cell t, if we denote its height by a_t, then for each cell $\hat{t}$ neighbouring with a cell of the rectange/triangle T, the inequality $a_{\hat{t}} < min\{a_t : t \in T\}$ holds.





History/1

Coding theory

S. Földes and N. M. Singhi: On instantaneous codes, J. of Combinatorics, Information and System Sci., 31 (2006), 317-326.

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Rectangular islands

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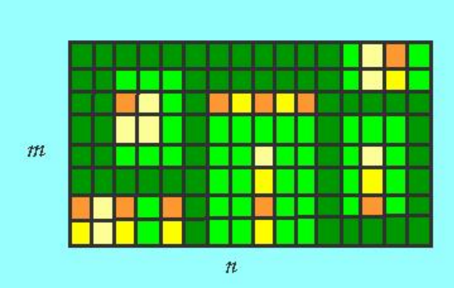
$$f(m,n) = [\frac{mn+m+n-1}{2}].$$



History/2

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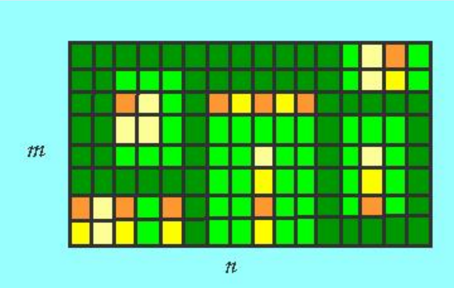

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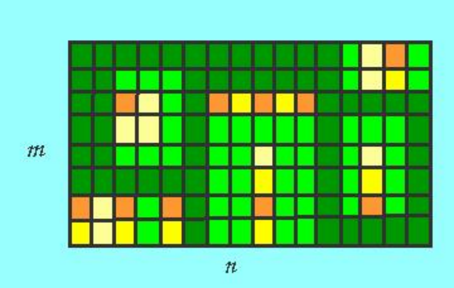

Proving methods/1

LATTICE THEORETICAL METHOD

G. Czédli, A. P. Huhn and E. T. Schmidt: Weakly independent subsets in lattices, Algebra Universalis 20 (1985), 194-196.

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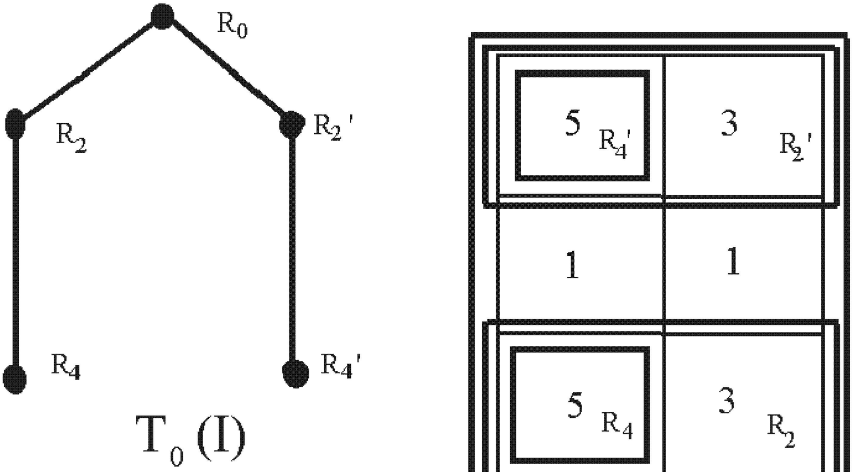
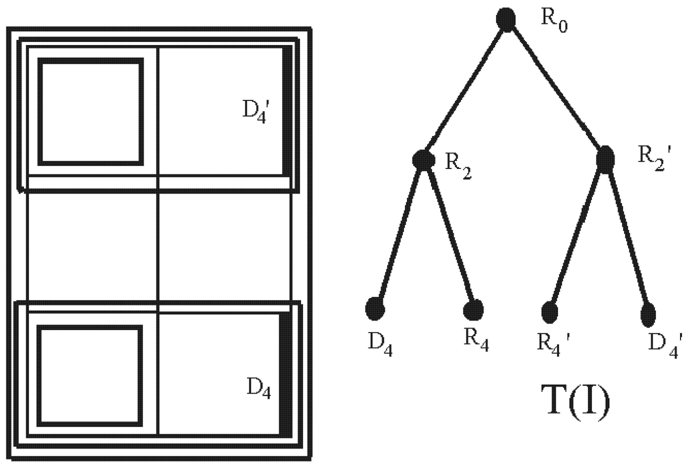
Proving methods/2

TREE-GRAPH METHOD (Barát, Hajnal, Horváth)



Proving methods/2

TREE-GRAPH METHOD



Proving methods/2

TREE-GRAPH METHOD

Lemma 2 (folklore)

(i) Let T be a binary tree with ℓ leaves. Then the number of vertices of T depends only on ℓ, moreover $|V| = 2\ell - 1$.

(ii) Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T. Then $|V| \leq 2\ell - 1$.

We have $4s + 2d \leq (n+1)(m+1)$.
The number of leaves of $T(\mathcal{I})$ is $\ell = s + d$. Hence by Lemma 2 the number of islands is

$$|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n+1)(m+1) - 1.$$

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LATTICE-VALUED REPRESENTATION

Theorem (Šešelja , Tepavčević, Horváth)
Let $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ **be a rectangular height function. Then there is a lattice** L **and an** L**-valued height functon** Φ**, such that the cuts of** Φ **are precisely all islands of** h**.**

Let $h : \{1,2,3,4,5\} \times \{1,2,3,4\} \to \mathbb{N}$ be a height function.

4	9	8	7	1	5
3	8	8	7	1	4
2	7	7	7	1	5
1	2	2	2	1	6
	1	2	3	4	5

Rectangular height functions

h is a rectangular height function. Its islands are:

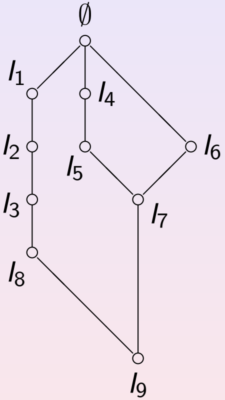
$I_1 = \{(1,4)\}$,
$I_2 = \{(1,3),(1,4),(2,3),(2,4)\}$,
$I_3 = \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$,
$I_4 = \{(5,1)\}$,
$I_5 = \{(5,1),(5,2)\}$,
$I_6 = \{(5,4)\}$,
$I_7 = \{(5,1),(5,2),(5,3),(5,4)\}$,
$I_8 = \{(1,2),(1,3),(1,4),(2,2),(2,3),$
$(2,4),(3,2),(3,3),(3,4),(1,1),(2,1),(3,1)\}$,
$I_9 = \{1,2,3,4,5\} \times \{1,2,3,4\}$.

Rectangular height functions

Its cuts are:

$h_{10} = \emptyset$
$h_9 = I_1$ (one-element island)
$h_8 = I_2$ (four-element square island)
$h_7 = I_3$ (nine-element square island)
$h_6 = I_3 \cup I_4$ (this cut is a disjoint union of two islands)
$h_5 = I_3 \cup I_5 \cup I_6$ (union of three islands)
$h_4 = I_3 \cup I_7$ (union of two islands)
$h_2 = I_7 \cup I_8$ (union of two islands)
$h_1 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} = I_9$ (the whole domain)

Rectangular height functions



$L = (\mathcal{I}_0(\Gamma), \supseteq)$

CD-independent subsets in distributive lattices

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Any two CD-bases of a finite distributive lattice have the same number of elements.

If all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices.

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CD-independent subsets in posets

Definitions

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set and $a, b \in P$.
The elements a and b are called *disjoint*, and we write $a \perp b$, if

$\inf\{a, b\} = 0$, whenever $\mathbb{P}$ has least element $0 \in P$,
a and b have no common lowerbound, whenever $\mathbb{P}$ is without 0.

- Notice, that $a \perp b$ implies $x \perp y$ for all $x, y \in P$ with $x \leq a$ and $y \leq b$.

A nonempty set $X \subseteq P$ is called *CD-independent*, if for any $x, y \in X$ either $x \leq y$ or $y \leq x$ or $x \perp y$ holds.

Maximal CD-independent sets (with respect to $\subseteq$) are called *CD-bases* in $\mathbb{P}$.

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Disjoint systems

Definition

A nonempty set D of nonzero elements of P is called a *disjoint system* in $\mathbb{P}$, if $x \perp y$ holds for all $x, y \in D$, $x \neq y$.

Remarks

- Any disjoint system $D \subseteq P$ and any chain $C \subseteq P$ is a CD-independent set.
- D is a disjoint system, if and only if it is a CD-independent antichain in $\mathbb{P}$.
- *If X is a CD-independent set in $\mathbb{P}$, then any antichain $A \subseteq X$ is a disjoint system in $\mathbb{P}$.*

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Order ideals

Any antichain $A = \{a_i \mid i \in I\}$ of a poset $\mathbb{P}$ determines a unique order-ideal $I(A)$ of $\mathbb{P}$:

$$I(A) = \bigcup_{i \in I} (a_i] = \{x \in P \mid x \leq a_i, \text{ for some } i \in I\},$$

where $(a]$ stands for the principal ideal of an element $a \in P$.

Definition

If A_1, A_2 are antichains in $\mathbb{P}$, then we say that A_1 *is dominated by* A_2, and we denote it by $A_1 \leqslant A_2$, if

$$I(A_1) \subseteq I(A_2).$$

Remarks

- $\leqslant$ is a partial order
- $A_1 \leqslant A_2$ is satisfied if and only if

$$\text{for each } x \in A_1 \text{ there exists an } y \in A_2, \text{ with } x \leq y. \quad \text{(A)}$$

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- If D_1, D_2 are disjoint systems in P, then $D_1 \subseteq D_2$ implies $D_1 \leqslant D_2$.
- If $D_1 \leqslant D_2$, then for any $x \in D_1$ and $y \in D_2$ either $x \leq y$ or $x \perp y$ is satisfied.
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Tolerance relation

Definition

Let $\rho \subseteq P \times P$.

For any $x, y \in P$, $(x, y) \in \rho \Leftrightarrow$ either $x \leq y$ or $y \leq x$ or $x \perp y$.

Remarks

- ρ is a tolerance relation on P.
- The CD-bases of $\mathbb{P}$ are exactly the tolerance classes (tolerance blocks) of ρ.
- *Any poset $\mathbb{P} = (P, \leq)$ hast at least one CD-base, and the set P is covered by the CD-bases of $\mathbb{P}$.*

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Theorem

Let $(P, \leq)$ *be a finite poset and* B *a CD-base of it.*

There exists a maximal chain $D_1 \succ \ldots \succ D_n$ *in* $\mathcal{D}(P)$*, such that* $B = \bigcup_{i=1}^{n} D_i$ *and* $n = \mid B \mid$.

For any maximal chain $D_1 \prec \ldots \prec D_m$ *in* $\mathcal{D}(P)$ *the set* $D = \bigcup_{i=1}^{m} D_i$ *is a CD-base in* $(P, \leq)$ *with* $\mid D \mid = m$.

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Proof of the Theorem

Proposition

If B is a CD-base and $D \subseteq B$ is a disjoint system in the poset $(P, \leq)$, then $I(D) \cap B$ is also a CD-base in the subposet $(I(D), \leq)$.

Lemma

$D_1 \prec D_2$ holds in $\mathcal{D}(P)$ if and only if $D_2 = \{a\} \cup \{y \in D_1 \mid a \perp y\}$, where a is a minimal element of the set

$$S = \{x \in P \setminus D_1 \mid x \perp y \text{ or } x > y, \text{ for all } y \in D_1\}.$$

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Let B be a CD-base with at least two elements in a finite poset $\mathbb{P} = (P, \leq)$, $M = \max(B)$, and for arbitrary $m \in M$ let $N = \max(B \setminus \{m\})$. Then M and N are disjoint systems, M is a maximal element in $\mathcal{D}(P)$, and $N \prec M$ holds in $\mathcal{D}(P)$.

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Corollary

Let $\mathbb{P} = (P, \leq)$ *be a finite poset.*

(i) *If* $B \subseteq P$ *is a CD-base and* $(B, \leq)$ *is the subposet corresponding to it, then any maximal chain* $\mathcal{C} : D_1 \prec ... \prec D_n$ *in* $\mathcal{D}(B)$ is *also a maximal chain in* $\mathcal{D}(P)$.

(ii) *If* D *is a disjoint system in* $\mathbb{P}$, *and the CD-bases of* $\mathbb{P}$ *have the same number of elements, then the CD-bases of the subposet* $I(D)$ *also have the same number of elements.*

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$\mathcal{D}(P)$ is graded

The poset $\mathbb{P}$ is called *graded*, if all its maximal chains have the same cardinality.

Let $\mathbb{P} = (P, \leq)$ *be a finite poset with* 0*. Then the following conditions are equivalent:*

(i) *The CD-bases of* $\mathbb{P}$ *have the same number of elements,*

(ii) $\mathcal{D}(P)$ *is graded.*

A disjoint system D of a poset $(P, \leq)$ is called *complete,* if there is no $p \in P \setminus D$ such that $D \cup \{p\}$ is also a disjoint system.

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(a) *If all the CD-bases of $\mathbb{P}$ consist of n elements, then* $n \geq |A(P)| + l(P)$.

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CD-bases in semilattices and lattices

A pair $a, b \in P$ with least upperbound $a \vee b$ in $\mathbb{P}$ is called a *distributive pair*, if $(c \wedge a) \vee (c \wedge b)$ exists in $\mathbb{P}$ for any $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$.
We say that $(P, \wedge)$ is *dp-distributive*, if any $a, b \in P$ with $a \wedge b = 0$ is a distributive pair.

Theorem

If $\mathbb{P} = (P, \wedge)$ is a semilattice with 0, then $\mathcal{D}(P)$ is a semilattice with 0; if $D_1 \cup D_2$ is a CD-independent set for some $D_1, D_2 \in \mathcal{D}(P)$, then D_1, D_2 is a distributive pair in $\mathcal{D}(P)$. If $\mathbb{P}$ is a complete lattice, then $\mathcal{D}(P)$ is a complete lattice, too.

Proposition

Let $\mathbb{P} = (P, \leq)$ be a poset with 0 and B a CD-base of it. Then $(\mathcal{D}(B), \leqslant)$ is a distributive cover-preserving sublattice of the poset $(\mathcal{D}(P), \leqslant)$. If $\mathbb{P}$ is a $\wedge$-semilattice, then for any $D \in \mathcal{D}(P)$ and $D_1, D_2 \in \mathcal{D}(B)$ we have $(D_1 \vee D_2) \wedge D = (D_1 \wedge D) \vee (D_2 \wedge D)$.

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CD-bases in particular lattice classes

Lemma

Let L be a finite weakly 0-distributive lattice and D a dual atom in $\mathcal{D}(L)$. Then either $D = \{d\}$, for some $d \in L$ with $d \prec 1$, or D consist of two elements $d_1, d_2 \in L$ and $d_1 \vee d_2 = 1$.

Theorem

Let L be a finite 0-modular and weakly 0-distributive lattice. Then the following are equivalent:

- (i) *L is graded and $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.*
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