Islands

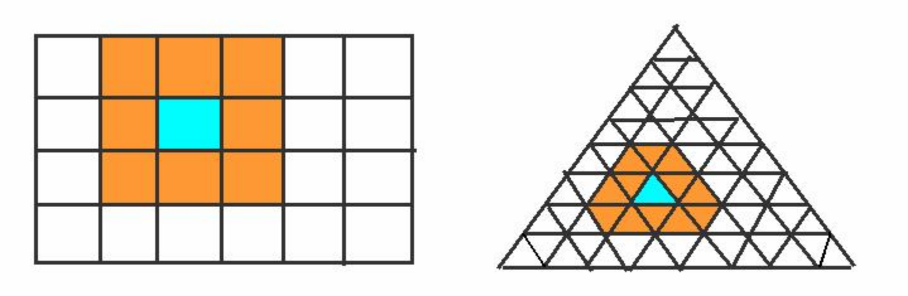
Eszter K. Horváth, Szeged

Coauthors: Péter Hajnal, Branimir Šešelja, Andreja Tepavčević

NSAC 2009

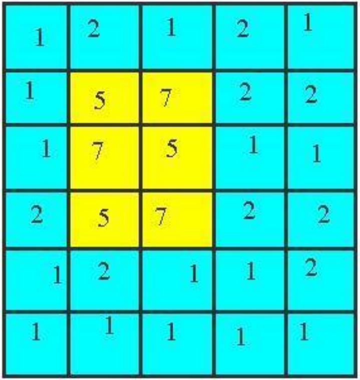
Definition/1

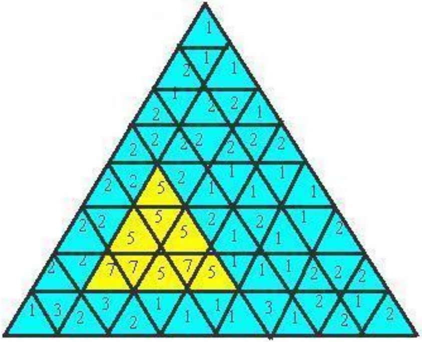
Grid, neighbourhood relation



Definition/2

We call a rectangle/triangle an *island*, if for the cell t, if we denote its height by a_t, then for each cell $\hat{t}$ neighbouring with a cell of the rectange/triangle T, the inequality $a_{\hat{t}} < min\{a_t : t \in T\}$ holds.





History/1

Coding theory

S. Földes and N. M. Singhi: On instantaneous codes, J. of Combinatorics, Information and System Sci., 31 (2006), 317-326.

History/1

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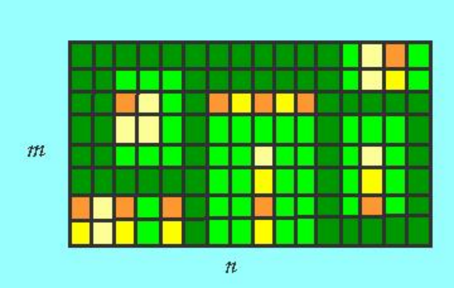
History/2

Rectangular islands

G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics 30 (2009), 208-215.

The maximum number of rectangular islands in a $m \times n$ rectangular board on square grid:

$$f(m,n) = [\frac{mn+m+n-1}{2}].$$



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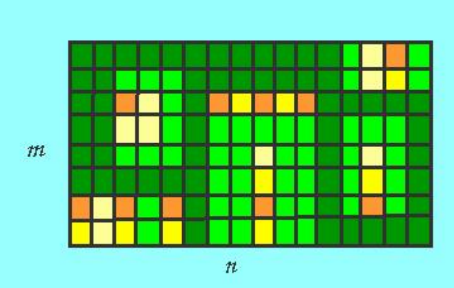

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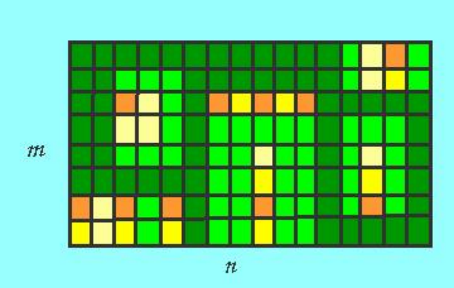

History/3

Rectangular islands in higher dimensions

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Triangular islands

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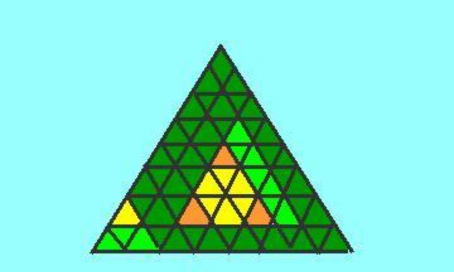
For the maximum number of triangular islands in an equilateral rectangle of side length n, $\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}$ holds.



History/4

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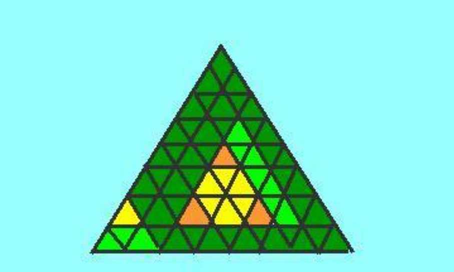

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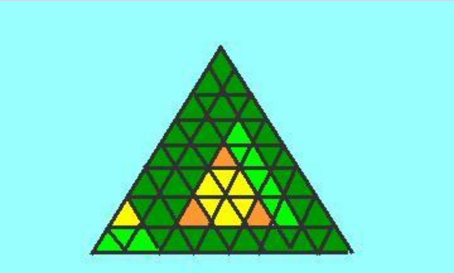
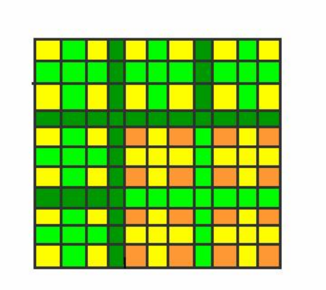


History/5

Square islands (also in higher dimensions)

E. K. Horváth, G. Horváth, Z. Németh, Cs. Szabó: The number of square islands on a rectangular sea, Acta Sci. Math., submitted. Available at http://www.math.u-szeged.hu/~horvath

$$\frac{1}{3}(rs-2r-2s) \leq f(r,s) \leq \frac{1}{3}(rs-1)$$



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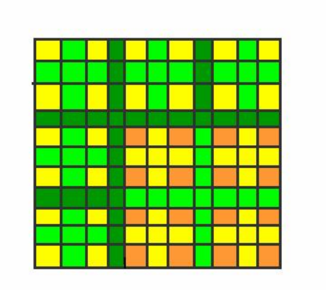

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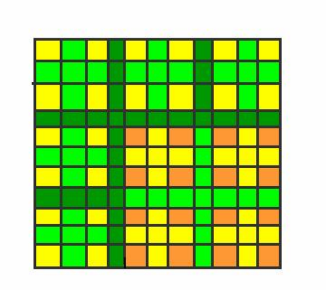

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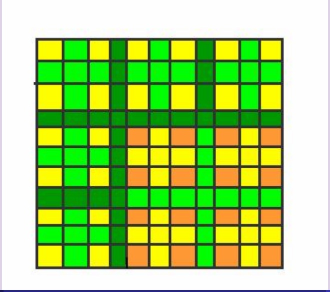

Some exact formulas

Peninsulas (semi islands) (J. Barát, P. Hajnal, E.K. Horváth):
$p(m,n) = f(m,n) = [(mn+m+n-1)/2]$.

Cylindric board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_1(m,n) = [\frac{(m+1)n}{2}]$.

Cylindric board, cylindric and rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_2(m,n) = [\frac{(m+1)n}{2}] + [\frac{(m-1)}{2}]$.

Torus board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
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History/6

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History/7

Further results on rectangular islands

Zs. Lengvárszky: The minimum cardinality of maximal systems of rectangular islands, European Journal of Combinatorics, **30** (2009), 216-219.

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Islands in Boolean algebras, i.e. in hypercubes /1

Joint work with Péter Hajnal

The board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $BA = \{0, 1\}^n$.

We consider two cells neighbouring if their Hamming distance is 1.

We denote the maximum number of islands in $BA = \{0, 1\}^n$ by $b(n)$.

Theorem 1

$b(n) = 1 + 2^{n-1}$.

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Proof:

$b(n) \geq 1 + 2^{n-1}$ because we can put one-cell islands to all vertices with an odd number of 1-s.

We show $b(n) \leq 1 + 2^{n-1}$ by induction on n. For $n = 0, 1$ the statement is easy to check.
For $n \geq 2$, we cut the hypercube into two half-hypercubes, of size 2^{n-1}. If one of them is an island, then the other cannot contain island.
If neither of them is an island, then by the induction hypothesis, in both half-hypercubes, the maximum cardinality of a system of islands is at most 2^{n-2}.

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Rectangular fuzzy relations/1

Joint work with Branimir Šešelja and Andreja Tepavčević

Let A and B nonempty sets and L a lattice. Then a *fuzzy relation* ρ is a mapping from $A \times B$ to L.

For every $p \in L$, cut relation is an ordinary relation ρ_p on $A \times B$ defined by

$$(x, y) \in \rho_p \ \text{ if and only if } \ \rho(x, y) \geq p.$$

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Rectangular fuzzy relations/2

We consider special lattice valued fuzzy relations:
The set $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$, $m, n \in \mathbb{N}$, is called *a* table of size $m \times n$. Such a table is the domain of a fuzzy relation. We consider

$$\Gamma_{\mathbb{N}} : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}.$$

Here the co-domain is the lattice $(\mathbb{N}, \leq)$, where $\mathbb{N}$ is the set of natural numbers under the usual ordering $\leq$ and suprema and infima are max and min, respectively. Moreover

$$\Gamma_{[0,1]} : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to [0, 1].$$

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Rectangular fuzzy relations/3

We say that two rectangles $\{\alpha, ..., \beta\} \times \{\gamma, ..., \delta\}$ and $\{\alpha_1, ..., \beta_1\} \times \{\gamma_1, ..., \delta_1\}$ are *distant* if they are disjoint and for every two cells, namely (a, b) from the first rectangle and (c, d) from the second, we have $(a-c)^2 + (b-d)^2 \geq 4$.

Fuzzy relation Γ is called *rectangular* if for every $p \in \mathbb{N}$, every nonempty p-cut of Γ is a union of distant rectangles.

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Rectangular fuzzy relations/4

5	5	3	5	5
4	4	2	4	4
2	2	1	2	2

$\Gamma_1 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\}$,
$\Gamma_2 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\} \setminus \{(3, 1)\}$,
$\Gamma_3 = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 3), (4, 2), (4, 3), (5, 2), (5, 3)\}$,
$\Gamma_4 = \{(1, 2), (1, 3), (2, 2), (2, 3), (4, 2), (4, 3), (5, 2), (5, 3)\}$ and
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Rectangular fuzzy relations/5
CHARACTERIZATION THEOREM / A

Theorem 2 / A

A fuzzy relation $\Gamma_{\mathbb{N}} : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ is rectangular if and only if for all (α, γ), $(\beta, \delta) \in \{1, 2, ..., m\} \times \{1, 2, ..., n\}$ either

- these are not neighboring cells and there is a cell (μ, ν) between (α, γ) and (β, δ) such that $\Gamma_{\mathbb{N}}(\mu, \nu) < \min\{\Gamma_{\mathbb{N}}(\alpha, \gamma), \Gamma_{\mathbb{N}}(\beta, \delta)\}$, or
- for all $(\mu, \nu) \in [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}] \times [\min\{\gamma, \delta\}, \max\{\gamma, \delta\}]$,

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Theorem 2 / B

A fuzzy relation $\Gamma_{[0,1]} : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to [0, 1]$ is rectangular if and only if for all (α, γ), $(\beta, \delta) \in \{1, 2, ..., m\} \times \{1, 2, ..., n\}$ either

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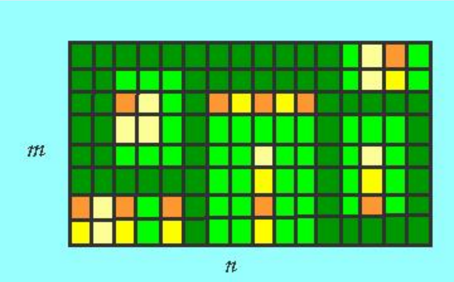
Rectangular fuzzy relations/7

Theorem 3

For every fuzzy relation $\Gamma_{\mathbb{N}} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, there is a rectangular fuzzy relation $\Phi_{\mathbb{N}} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, having the same islands.

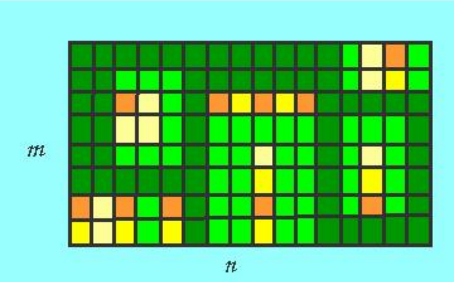
For every fuzzy relation $\Gamma_{[0,1]} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to [0, 1]$, there is a rectangular fuzzy relation $\Phi_{[0,1]} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to [0, 1]$, having the same islands.



Theorem 3

For every fuzzy relation $\Gamma_{\mathbb{N}} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \rightarrow \mathbb{N}$, there is a rectangular fuzzy relation $\Phi_{\mathbb{N}} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \rightarrow \mathbb{N}$, having the same islands.

For every fuzzy relation $\Gamma_{[0,1]} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \rightarrow [0, 1]$, there is a rectangular fuzzy relation $\Phi_{[0,1]} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \rightarrow [0, 1]$, having the same islands.



Rectangular fuzzy relations/8
CONSTRUCTING ALGORITHM

Let $\Gamma_{[0,1]} : \{1,2,...,m\} \times \{1,2,...,n\} \to [0,1]$ be a fuzzy relation. Let $\{a_1, a_2, ..., a_h\}$ be the set of different values of $\Gamma_{[0,1]}$, such that $0 \leq a_0 < a_1 < ... < a_h \leq 1$.

1. Let $i := h$
2. Let $(x, y) = (1, 1)$
3. If $\Gamma(x, y) \neq a_i$, then go to 6
4. Let $\Phi(x, y) := \Gamma(x, y)$.
5. Take a_k to be $\Phi(x, y)$. If there is an island of $\Gamma(x, y)$ that contains (x, y) which is a subset of Γ_{a_k} then go to 6.
Otherwise $\Phi(x, y) = a_{k-1}$.
6. If $x < m$, then $x := x + 1$, go to 3. Otherwise, go to 7.
7. If $y < n$, then $y := y + 1$ and $x := 1$, go to 3. Otherwise, if $x < m$ go to 6 and if $x = m$ go to 8.
8. If $i \neq 0$, then $i := i - 1$ and go to 2. Otherwise go to 9.
9. End.

Rectangular fuzzy relations/9
LATTICE-VALUED REPRESENTATION

Theorem 4

Let $\Gamma_{\mathbb{N}} : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \rightarrow \mathbb{N}$ be a rectangular fuzzy relation. Then there is a lattice L and an L-valued relation Φ, such that the cuts of Φ are precisely all islands of $\Gamma_{\mathbb{N}}$.

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Let $\Gamma : \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} \rightarrow [0, 1]$ be a fuzzy relation.

4	0.9	0.8	0.7	0.1	0.5
3	0.8	0.8	0.7	0.1	0.4
2	0.7	0.7	0.7	0.1	0.5
1	0.2	0.2	0.2	0.1	0.6
	1	2	3	4	5

Rectangular fuzzy relations/11

Γ is a rectangular fuzzy relation. Its islands are:

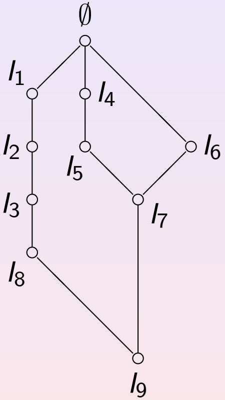
$I_1 = \{(1,4)\}$,
$I_2 = \{(1,3),(1,4),(2,3),(2,4)\}$,
$I_3 = \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$,
$I_4 = \{(5,1)\}$,
$I_5 = \{(5,1),(5,2)\}$,
$I_6 = \{(5,4)\}$,
$I_7 = \{(5,1),(5,2),(5,3),(5,4)\}$,
$I_8 = \{(1,2),(1,3),(1,4),(2,2),(2,3),$
$(2,4),(3,2),(3,3),(3,4),(1,1),(2,1),(3,1)\}$,
$I_9 = \{1,2,3,4,5\} \times \{1,2,3,4\}$.

Rectangular fuzzy relations/12

Its cut relations are:

$\Gamma_1 = \emptyset$
$\Gamma_{0.9} = I_1$ (one-element island)
$\Gamma_{0.8} = I_2$ (four-element square island)
$\Gamma_{0.7} = I_3$ (nine-element square island)
$\Gamma_{0.6} = I_3 \cup I_4$ (this cut is a disjoint union of two islands)
$\Gamma_{0.5} = I_3 \cup I_5 \cup I_6$ (union of three islands)
$\Gamma_{0.4} = I_3 \cup I_7$ (union of two islands)
$\Gamma_{0.2} = I_7 \cup I_8$ (union of two islands)
$\Gamma_{0.1} = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} = I_9$ (the whole domain)

Rectangular fuzzy relations/13



$$L = (\mathcal{I}_0(\Gamma), \supseteq)$$

Theorem 5
For every rectangular fuzzy relation $\Phi_{\mathbb{N}} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, there is a rectangular fuzzy relation $\Psi_{\mathbb{N}} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, having the same islands and in $\Psi_{\mathbb{N}}$ every island appears exactly in one cut.

If a fuzzy rectangular relation $\Psi_{\mathbb{N}}$ has the property that each island appears exactly in one cut, then we call it *standard fuzzy rectangular relation*. We denote by $\Lambda(m, n)$ the maximum number of different p-cuts of a standard fuzzy rectangular relation on the rectangular table of size $m \times n$.

Theorem 6
$\Lambda(m, n) = m + n - 1.$

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Rectangular fuzzy relations/14

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