Lattice-valued functions

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Lattice-valued functions

Let S be a nonempty set and L a complete lattice. Every mapping $\mu : S \to L$ is called a **lattice-valued** (L**-valued**) **function** on S.

Cut set (p-cut)

Let $p \in L$. A **cut set** of an L-valued function $\mu : S \to L$ (a p-cut) is a subset $\mu_p \subseteq S$ defined by:

$$x \in \mu_p \text{ if and only if } \mu(x) \geq p. \tag{1}$$

In other words, a p-cut of $\mu : S \to L$ is the inverse image of the principal filter $\uparrow p$, generated by $p \in L$:

$$\mu_p = \mu^{-1}(\uparrow p). \tag{2}$$

It is obvious that for every $p, q \in L$, $p \leq q$ implies $\mu_q \subseteq \mu_p$.

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Cuts and closure systems

If $\mu : S \to L$ is an L-valued function on S, then the collection μ_L of all cuts of μ is a closure system on S under the set-inclusion.

Let $\mathcal{F}$ be a closure system on a set S. Then there is a lattice L and an L-valued function $\mu : S \to L$, such that the collection μ_L of cuts of μ is $\mathcal{F}$.

A required lattice L is the collection $\mathcal{F}$ ordered by the *reversed-inclusion*, and that $\mu : S \to L$ can be defined as follows:

$$\mu(x) = \bigcap\{f \in \mathcal{F} \mid x \in f\}. \tag{3}$$

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The relation $\approx$ on L

Given an L-valued function $\mu : S \to L$, we define the relation $\approx$ on L: for $p, q \in L$

$$p \approx q \text{ if and only if } \mu_p = \mu_q. \quad (4)$$

The relation $\approx$ is an equivalence on L, and

$$p \approx q \text{ if and only if } \uparrow p \cap \mu(S) = \uparrow q \cap \mu(S), \quad (5)$$

where $\mu(S) = \{r \in L \mid r = \mu(x) \text{ for some } x \in S\}$.

We denote by $L/\approx$ the collection of equivalence classes under $\approx$.

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The collection of cuts

Let $(\mu_L, \leq)$ be the poset with $\mu_L = \{\mu_p \mid p \in L\}$ (the collection of cuts of μ) and the order $\leq$ being the inverse of the set-inclusion: for $\mu_p, \mu_q \in \mu_L$,

$$\mu_p \leq \mu_q \text{ if and only if } \mu_q \subseteq \mu_p.$$

$(\mu_L, \leq)$ is a complete lattice and for every collection $\{\mu_p \mid p \in L_1\}$, $L_1 \subseteq L$ of cuts of μ, we have

$$\bigcap\{\mu_p \mid p \in L_1\} = \mu_{\bigvee(p \mid p \in L_1)}. \tag{6}$$

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The quotient $L/\approx$

Each $\approx$-class contains its supremum:

$$\bigvee [p]_\approx \in [p]_\approx. \tag{7}$$

The mapping $p \mapsto \bigvee [p]_\approx$ is a closure operator on L.

The quotient $L/\approx$ can be ordered by the relation $\leq_{L/\approx}$ defined as follows:

$$[p]_\approx \leq_{L/\approx} [q]_\approx \text{ if and only if } \uparrow q \cap \mu(S) \subseteq \uparrow p \cap \mu(S).$$

The order $\leq_{L/\approx}$ of classes in $L/\approx$ corresponds to the order of suprema of classes in L (we denote the order in L by $\leq_L$):

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The poset $(L/{\approx}, \leq_{L/\approx})$

The poset $(L/{\approx}, \leq_{L/\approx})$ *is a complete lattice fulfilling:*

(*i*) $[p]_\approx \leq_{L/\approx} [q]_\approx$ *if and only if* $\bigvee[p]_\approx \leq_L \bigvee[q]_\approx$.

(*ii*) *The mapping* $[p]_\approx \mapsto \bigvee[p]_\approx$ *is an injection of* $L/{\approx}$ *into* L.

The sub-poset $(\bigvee[p]_\approx, \leq_L)$ *of* L *is isomorphic to the lattice* $(L/{\approx}, \leq_{L/\approx})$ *under* $\bigvee[p]_\approx \mapsto [p]_\approx$.

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Canonical representation of lattice-valued functions

We take the lattice $(\mathcal{F}, \leq)$, where $\mathcal{F} = \mu_L \subseteq \mathcal{P}(S)$ is the collection of cuts of μ, and the order $\leq$ is the dual of the set inclusion.

Let $\widehat{\mu} : S \to \mathcal{F}$, where

$$\widehat{\mu}(x) := \bigcap \{\mu_p \in \mu_L \mid x \in \mu_p\}. \tag{8}$$

Properties:

$\widehat{\mu}$ has the same cuts as μ.

$\widehat{\mu}$ has one-element classes of the equivalence relation $\approx$.

Every $f \in \mathcal{F}$ is equal to the corresponding cut of $\widehat{\mu}$.

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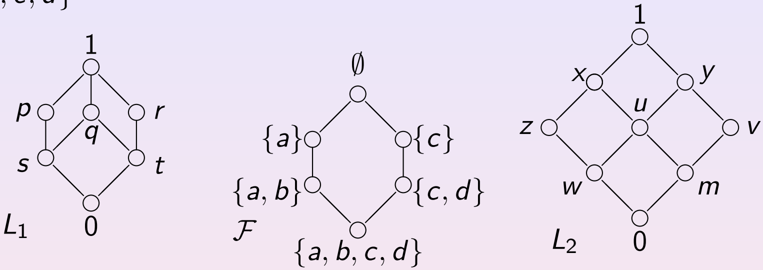
Example

$S = \{a, b, c, d\}$



$$\mu = \begin{pmatrix} a & b & c & d \\ p & s & r & t \end{pmatrix} \qquad \nu = \begin{pmatrix} a & b & c & d \\ z & w & m & v \end{pmatrix}$$

$$\widehat{\mu} = \widehat{\nu} = \begin{pmatrix} a & b & c & d \\ \{a\} & \{a, b\} & \{c\} & \{c, d\} \end{pmatrix}$$

Lattice-valued Boolean functions

A **Boolean function** is a mapping $f : \{0,1\}^n \to \{0,1\}$, $n \in \mathbb{N}$.

A **lattice-valued Boolean function** is a mapping

$$f : \{0,1\}^n \to L,$$

where L is a complete lattice.

We also deal with **lattice-valued** n**-variable functions** on a finite domain $\{0,1,\dots,k-1\}$:

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We use also p-**cuts** of lattice-valued functions as characteristic functions: for $f : \{0,1,\dots,k-1\}^n \to L$ and $p \in L$, we have

$$f_p : \{0,1,\dots,k-1\}^n \to \{0,1\},$$

such that $f_p(x_1,\dots,x_n) = 1$ if and only if $f(x_1,\dots,x_n) \geq p$.

Clearly, *a cut of a lattice-valued Boolean function is (as a characteristic function) a Boolean function.*

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Invariance group

As usual, by S_n we denote the symmetric group of all permutations over an n-element set. If f is an n-variable function on a finite domain X and $\sigma \in S_n$, then f is **invariant** under σ, symbolically $\sigma \vdash f$, if for all $(x_1, \ldots, x_n) \in X^n$

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

If f is invariant under all permutations in $G \leq S_n$ and not invariant under any permutation from $S_n \setminus G$, then G is called **the invariance group** of f, and it is denoted by $G(f)$.

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Representability

A group $G \leq S_n$ is said to be (k, m)-*representable* if there is a function $f : \{0, 1, \ldots, k-1\}^n \to \{1, \ldots, m\}$ whose invariance group is G.

If G is the invariance group of a function $f : \{0, 1, \ldots, k-1\}^n \to \mathbb{N}$, then it is called (k, ∞)-*representable*.

$G \leq S_n$ is called *m-representable* if it is the invariance group of a function $f : \{0, 1\}^n \to \{1, \ldots, m\}$;

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Representability by lattice-valued functions

We say that a permutation group $G \leq S_n$ is (k, L)-**representable**, if there is a lattice-valued function $f : \{0, 1, \ldots, k-1\}^n \to L$, such that $\sigma \vdash f$ if and only if $\sigma \in G$.

In particular, a $(2, L)$-representable group is the invariance group of a lattice-valued Boolean function $f : \{0, 1\}^n \to L$.

The notion of $(2, L)$-representability is more general than $(2, 2)$-representability. An example is the Klein 4-group: $\{id, (12)(34), (13)(24), (14)(23)\}$, which is $(2, L)$ representable (for L being a three element chain), but not $(2, 2)$-representable.

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A Galois connection for invariance groups

Let $O_k^{(n)} = \{f \mid f : \mathbf{k}^n \to \mathbf{k}\}$ denote the set of all n-ary operations on $\mathbf{k}$, and for $F \subseteq O_k^{(n)}$ and $G \subseteq S_n$ let

$$F^{\vdash} := \{\sigma \in S_n \mid \forall f \in F : \sigma \vdash f\}, \qquad \overline{F}^{(k)} := (F^{\vdash})^{\vdash},$$
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The assignment $G \mapsto \overline{G}^{(k)}$ is a closure operator on S_n, and it is easy to see that $\overline{G}^{(k)}$ is a subgroup of S_n for every subset $G \subseteq S_n$ (even if G is not a group). For $G \leq S_n$, we call $\overline{G}^{(k)}$ the *Galois closure of G over* $\mathbf{k}$, and we say that G is *Galois closed over* $\mathbf{k}$ if $\overline{G}^{(k)} = G$.

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Galois closed groups

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For every $G \leq S_n$, we have

$$\overline{G}^{(k)} = \bigcap_{a \in \mathbf{k}^n} (S_n)_a \cdot G.$$

For arbitrary $k, n \geq 2$, characterize those subgroups of S_n that are Galois closed over $\mathbf{k}$.

Theorem (H., Makay, Pöschel, Waldhauser) Let $n > \max\left(2^d, d^2 + d\right)$ and $G \leq S_n$. Then G is not Galois closed over $\mathbf{k}$ if and only if $G = A_B \times L$ or $G <_{\mathrm{sd}} S_B \times L$, where $B \subseteq \mathbf{n}$ is such that $D := \mathbf{n} \setminus B$ has less than d elements, and L is an arbitrary permutation group on D.

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Cuts of composition of functions

Theorem Let L be a complete lattice, let $A \neq \emptyset$ be a set and let $\sigma : A \to A$, $\mu : A \to L$, $\psi : L \to L$. Then, for every $p \in L$,

$$(\sigma \circ \mu \circ \psi)_p = \sigma \circ \mu \circ \psi_p.$$

Corollary Let L be a complete lattice, let $A \neq \emptyset$ and let $\mu : A \to L$. Then the following holds.

(i) $\mu_p = \mu \circ (\mathcal{I}_L)_p$, where $\mathcal{I}_L$ is the identity mapping $\mathcal{I}_L : L \to L$.

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Invariance groups of lattice-valued functions via cuts

Proposition Let $f : \{0, \ldots, k-1\}^n \to L$ and $\sigma \in S_n$. Then

$$\sigma \vdash f \text{ if and only if for every } p \in L,\ \sigma \vdash f_p.$$

The invariance group of a lattice-valued function f depends only on the canonical representation of f.

If $f_1 : \{0, \ldots, k-1\}^n \to L_1$ and $f_2 : \{0, \ldots, k-1\}^n \to L_2$ are two n-variable lattice-valued functions on the same domain, then $\widehat{f_1} = \widehat{f_2}$ implies $G(f_1) = G(f_2)$.

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Representation theorem

For every $n \in \mathbb{N}$, there is a lattice L and a lattice valued Boolean function $F : \{0,1\}^n \to L$ satisfying the following: If $G \leq S_n$ and $G = G(f)$ for a Boolean function f, then $G = G(F_p)$, for a cut F_p.

Representation theorem on the k-element set

Every subgroups of S_n is an invariance group of a function $\{0, \ldots, k-1\}^n \to \{0, \ldots, k-1\}$ if and only if $k \geq n$.

If $k \geq n$, then for every subgroup G of S_n there exists a function $f : \{0, \ldots, k-1\}^n \to \{0, 1\}$ such that the invariance group of f is exactly G.

For $k, n \in \mathbb{N}$ and $k \geq n$, there is a lattice L and a lattice valued function $F : \{0, \ldots, k-1\}^n \to L$ such that the following holds: If $G \leq S_n$, then $G = G(F_p)$ for a cut F_p of of F.

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Linear combination

A *lattice-valued Boolean function* is a map $\mu\colon \{0,1\}^n \to L$ where L is a bounded lattice and $n \in \langle 1, 2, 3, \dots \rangle$.

We say that μ can be given by a *linear combination* (in L) if there are $w_1, \dots, w_n \in L$ such that, for all $x = \{x_1, \dots, x_n\} \in \{0,1\}^n$,

$$\mu(x) = \bigvee_{i=1}^{n} w_i x_i, \quad \text{that is,} \quad \mu(x) = \bigvee_{i=1}^{n} (w_i \wedge x_i). \qquad (9)$$

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Cuts and closure systems

For $p \in L$, the set

$$\mu_p := \{x \in \{0,1\}^n : \mu(x) \geq p\} \tag{10}$$

is called a *cut* of μ.

A *closure system* $\mathcal{F}$ over B_n is a $\cap$-subsemilattice of the powerset lattice $P(B_n) = \langle P(B_n); \cup, \cap \rangle$ such that $B_n \in \mathcal{F}$. By finiteness, $\mathcal{F}$ is necessarily a complete $\cap$-semilattice.

A closure system $\mathcal{F}$ determines a *closure operator* in the standard way. We only need the closures of singleton sets, that is,

$$\text{for } x \in B_n, \text{ we have } \overline{x} := \bigcap\{f \in \mathcal{F} : x \in f\}. \tag{11}$$

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$\{\vee, 0\}$-homomorphism

If $\mu\colon B_n \to L$ such that $\mu(0) = 0$ and, for all $x, y \in B_n$, $\mu(x \vee y) = \mu(x) \vee \mu(y)$, then μ is called a $\{\vee, 0\}$-*homomorphism*.

A lattice-valued function $B_n \to L$ can be given by a linear combination in L iff it is a $\{\vee, 0\}$-homomorphism.

$\mu(x \vee y) = \bigvee_i w_i(x_i \vee y_i) = \bigvee_i (w_i x_i \vee w_i y_i) = \bigvee_i w_i x_i \vee \bigvee_i w_i y_i = \mu(x) \vee \mu(y)$.

Let $e^{(i)} = \langle 0, \dots, 0, 1, 0, \dots, 0 \rangle \in B_n$ where 1 stands in the i-th place. Define $w_i := \mu(e^{(i)})$. Observe that $\mu(e^{(i)} \cdot 1) = w_i = w_i \cdot 1$ and $\mu(e^{(i)} \cdot 0) = 0 = w_i \cdot 0$, that is, $\mu(e^{(i)} \cdot x_i) = w_i \cdot x_i$. Therefore, for $x \in B_n$, we obtain $\mu(x) = \mu(\bigvee_i e^{(i)} x_i) = \bigvee_i \mu(e^{(i)} x_i) = \bigvee_i w_i \cdot x_i$.

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If $\varnothing \neq X \subseteq B_n$ such that $(\forall x \in X)(\forall y \in B_n)(x \leq y \quad \text{then} \quad y \in X)$, then X is an *up-set* of B_n.

The lattice-valued function $\mu \colon B_n \to L$ is isotone iff all the cuts of μ are up-sets.

Up-sets

If $\varnothing \neq X \subseteq B_n$ such that $(\forall x \in X)(\forall y \in B_n)(x \leq y \quad \text{then} \quad y \in X)$, then X is an *up-set* of B_n.

The lattice-valued function $\mu\colon B_n \to L$ is isotone iff all the cuts of μ are up-sets.

Closure systems of up-sets, linear combinations

Let $\mathcal{F}$ be a set consisting of some up-sets of B_n. Then, the following three conditions are equivalent.

(i) $\mathcal{F}$ is a closure system over B_n, and for all $x, y \in B_n$, $\overline{x} \subseteq \overline{y}$ implies $\overline{x \vee y} = \overline{x}$.

(ii) $\mathcal{F}$ is a closure system over B_n, and for all $x, y \in B_n$, $\overline{x \vee y} = \overline{x} \cap \overline{y}$.

(iii) There exist a bounded lattice L and a lattice-valued function $\mu \colon B_n \to L$ given by a linear combination such that $\mathcal{F}$ is the family of cuts of μ.

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Threshold functions

A classical **threshold function** is a Boolean function $f : \{0,1\}^n \to \{0,1\}$ such that there exist real numbers $w_1, \ldots, w_n, t$, fulfilling

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \sum_{i=1}^{n} w_i \cdot x_i \geq t, \tag{12}$$

where w_i is called the **weight** of x_i, for $i = 1, 2, \ldots, n$ and t is a constant called the **threshold value**.

Properties of threshold functions

Threshold functions are not closed under superposition, so they do not constitute a clone.

It is easy to see that threshold functions with positive weights and a threshold value are isotone.

However, an isotone Boolean function is not necessarily threshold, e.g. $f = x \cdot y \vee w \cdot z$ is isotone, but not a threshold function. To see this, it is enough to consider its invariance group, which is the following: $D8 = \{(), (1324), (12)(34), (1423), (12), (34), (13)(24), (14)(23)\}$, however the invariance group of any threshold function is a direct product of symmetric groups.

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Lattice-induced threshold function

For $x \in \{0,1\}$, and $w \in L$, we define a mapping $L \times \{0,1\}$ into L denoted by "$\cdot$", as follows:

$$w \cdot x := \begin{cases} w, & \text{if} \quad x = 1 \\ 0, & \text{if} \quad x = 0. \end{cases} \tag{13}$$

A function $f : \{0,1\}^n \to \{0,1\}$ is a **lattice induced threshold function**, if there is a complete lattice L and $w_1, \ldots, w_n, t \in L$, such that

$$f(x_1, \ldots, x_n) = 1 \text{ if and only if } \bigvee_{i=1}^{n} (w_i \cdot x_i) \geq t. \tag{14}$$

Theorem

Every lattice-induced threshold function is isotone.

Every isotone Boolean function is a lattice induced threshold function. The corresponding lattice in each case can be the free distributive lattice with n generators.

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Main representative of isotone functions

Let $B = (\{0,1\}^n, \leq)$, $n \in \mathbb{N}$, let L_D a free distributive lattice with n generators $w_1, \ldots, w_n$ and $\overline{\beta} : B \to L_D$, an L_D-valued function on B defined in the following way: for $x = (x_1, \ldots, x_n) \in B$

$$\overline{\beta}(x) = \bigvee_{i=1}^{n} (w_i \cdot x_i), \tag{15}$$

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Every up-set of a finite Boolean lattice $B = (\{0,1\}^n, \leq)$, $n \in \mathbb{N}$, is a cut of $\overline{\beta}$.

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