

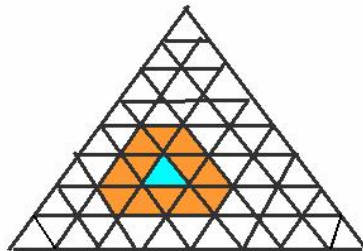
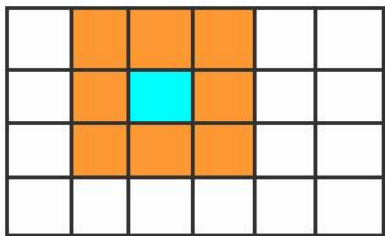
Szigetek és CD-independens halmazok

K. Horváth Eszter

Miskolc, 2010. április 7.

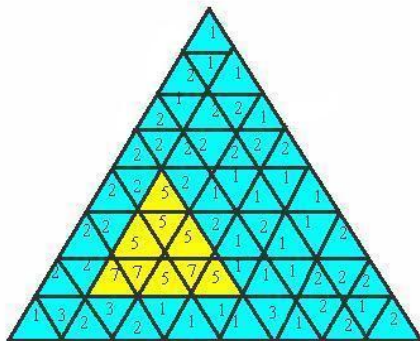
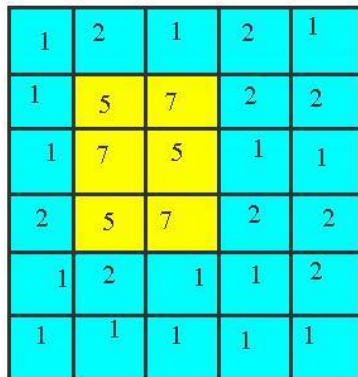
Definition/1

Grid, neighbourhood relation



Definition/2

We call a rectangle/triangle an *island*, if for the cell t , if we denote its height by a_t , then for each cell $\hat{t}$ neighbouring with a cell of the rectangle/triangle T , the inequality $a_{\hat{t}} < \min\{a_t : t \in T\}$ holds.



Coding theory

S. Földes and N. M. Singhi: On instantaneous codes, J. of Combinatorics, Information and System Sci., 31 (2006), 317-326.

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History/2

Rectangular islands

G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics 30 (2009), 208-215.

The maximum number of rectangular islands in a $m \times n$ rectangular board on square grid:

$$f(m, n) = \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor.$$



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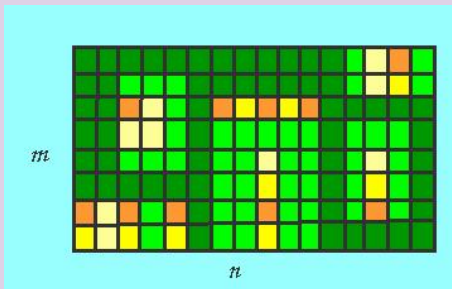


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Rectangular islands in higher dimensions

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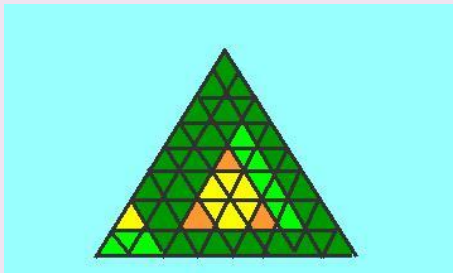
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Triangular islands

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For the maximum number of triangular islands in an equilateral triangle of side length n , $\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}$ holds.

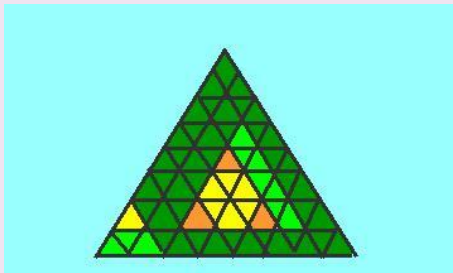


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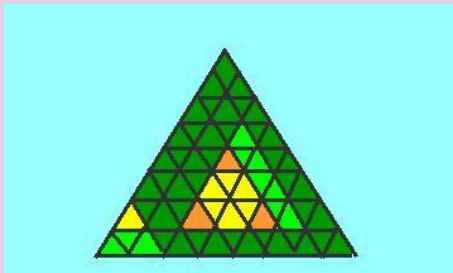


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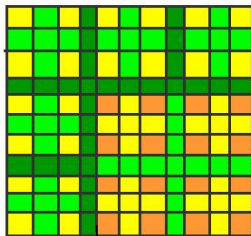
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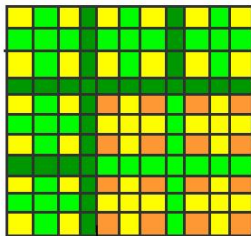
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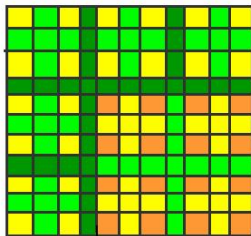
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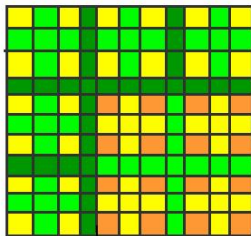
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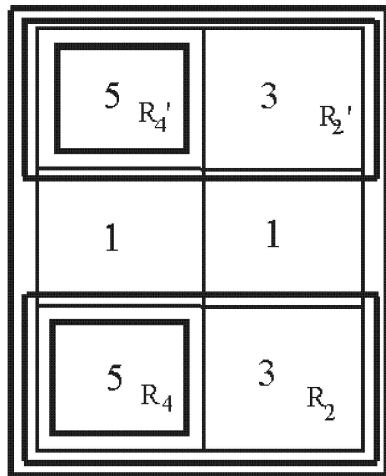
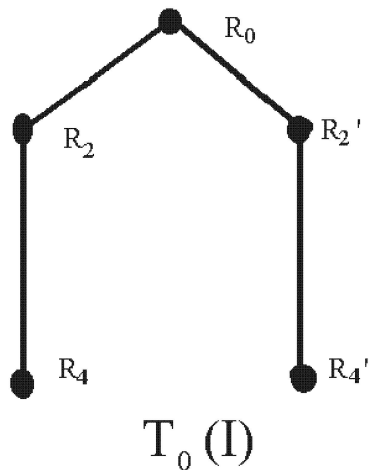
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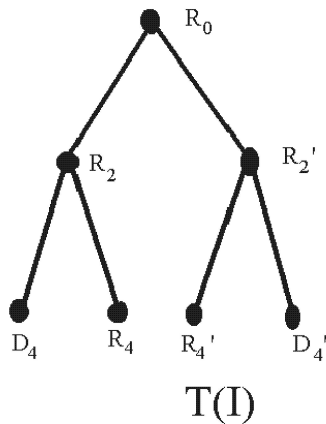
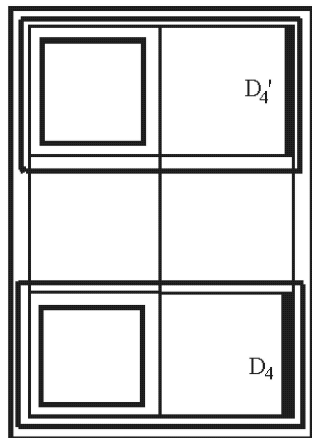
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TREE-GRAPH METHOD



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Lemma 2 (folklore)

- (i) Let T be a binary tree with ℓ leaves. Then the number of vertices of T depends only on ℓ , moreover $|V| = 2\ell - 1$.
- (ii) Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T . Then $|V| \leq 2\ell - 1$.

We have $4s + 2d \leq (n + 1)(m + 1)$.

The number of leaves of $T(\mathcal{I})$ is $\ell = s + d$. Hence by Lemma 2 the number of islands is

$$|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n + 1)(m + 1) - 1.$$

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ELEMENTARY METHOD

We define

$$\mu(R) = \mu(u, v) := (u + 1)(v + 1).$$

Now

$$\begin{aligned} f(m, n) &= 1 + \sum_{R \in \max \mathcal{I}} f(R) = 1 + \sum_{R \in \max \mathcal{I}} \left(\left\lceil \frac{(u+1)(v+1)}{2} \right\rceil - 1 \right) \\ &= 1 + \sum_{R \in \max \mathcal{I}} \left(\left\lceil \frac{\mu(u, v)}{2} \right\rceil - 1 \right) \leq 1 - |\max \mathcal{I}| + \left\lceil \frac{\mu(C)}{2} \right\rceil. \end{aligned}$$

If $|\max \mathcal{I}| \geq 2$, then the proof is ready. Case $|\max \mathcal{I}| = 1$ is an easy exercise.

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Some exact formulas

Peninsulas (semi islands) (J. Barát, P. Hajnal, E.K. Horváth):

$$p(m, n) = f(m, n) = \lfloor (mn + m + n - 1)/2 \rfloor.$$

Cylindric board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):

If $n \geq 2$, then $h_1(m, n) = \lfloor \frac{(m+1)n}{2} \rfloor$.

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Further results on rectangular islands

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The board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $BA = \{0, 1\}^n$.

We consider two cells neighbouring if their Hamming distance is 1.

We denote the maximum number of islands in $BA = \{0, 1\}^n$ by $b(n)$.

Island formula for Boolean algebras (P. Hajnal, E.K. Horváth)

$$b(n) = 1 + 2^{n-1}.$$

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Rectangular height functions/1

Joint work with Branimir Šešelja and Andreja Tepavčević

A *height function* h is a mapping from $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ to $\mathbb{N}$, $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$.

The co-domain of the height function is the lattice $(\mathbb{N}, \leq)$, where $\mathbb{N}$ is the set of natural numbers under the usual ordering $\leq$ and suprema and infima are max and min, respectively.

For every $p \in \mathbb{N}$, the *cut of the height function*, i.e. the p -cut of h is an ordinary relation h_p on $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ defined by

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Rectangular height functions/2

We say that two rectangles $\{\alpha, \dots, \beta\} \times \{\gamma, \dots, \delta\}$ and $\{\alpha_1, \dots, \beta_1\} \times \{\gamma_1, \dots, \delta_1\}$ are *distant* if they are disjoint and for every two cells, namely (a, b) from the first rectangle and (c, d) from the second, we have $(a - c)^2 + (b - d)^2 \geq 4$.

The height function h is called *rectangular* if for every $p \in \mathbb{N}$, every nonempty p -cut of h is a union of distant rectangles.

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Rectangular height functions/3

5	5	3	5	5
4	4	2	4	4
2	2	1	2	2

$$\Gamma_1 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\},$$

$$\Gamma_2 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3\} \setminus \{(3, 1)\},$$

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Rectangular height functions/4

CHARACTERIZATION THEOREM

Theorem 1

A height function $h_{\mathbb{N}} : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ is rectangular if and only if for all $(\alpha, \gamma), (\beta, \delta) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ either

- these are not neighboring cells and there is a cell (μ, ν) between (α, γ) and (β, δ) such that $h_{\mathbb{N}}(\mu, \nu) < \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}$, or
- for all $(\mu, \nu) \in [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}] \times [\min\{\gamma, \delta\}, \max\{\gamma, \delta\}]$,

$$h_{\mathbb{N}}(\mu, \nu) \geq \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}.$$

Rectangular height functions/4

CHARACTERIZATION THEOREM

Theorem 1

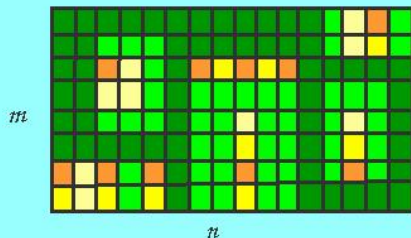
A height function $h_{\mathbb{N}} : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ is rectangular if and only if for all $(\alpha, \gamma), (\beta, \delta) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ either

- these are not neighboring cells and there is a cell (μ, ν) between (α, γ) and (β, δ) such that $h_{\mathbb{N}}(\mu, \nu) < \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}$, or
- for all $(\mu, \nu) \in [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}] \times [\min\{\gamma, \delta\}, \max\{\gamma, \delta\}]$,

$$h_{\mathbb{N}}(\mu, \nu) \geq \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}.$$

Theorem 2

For every height function $h : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N}$, there is a rectangular height function $h^* : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N}$, such that $\mathcal{I}_{rect}(h) = \mathcal{I}_{rect}(h^*)$.



Rectangular height functions/6

CONSTRUCTING ALGORITHM

1. FOR $i = t$ TO 0
2. FOR $y = 1$ TO n
3. FOR $x = 1$ TO m
4. IF $h(x, y) = a_i$ THEN
5. $j := i$
6. WHILE there is no island of h which is a subset of h_{a_j} that contains (x, y) DO $j := j - 1$
7. ENDWHILE
8. Let $h^*(x, y) := a_j$.
9. ENDIF
10. NEXT x
11. NEXT y
12. NEXT i
13. END.

Rectangular height functions/7

LATTICE-VALUED REPRESENTATION

Theorem 3

Let $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ be a rectangular height function. Then there is a lattice L and an L -valued mapping Φ , such that the cuts of Φ are precisely all islands of h .

Rectangular height functions/8

Let $h : \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} \rightarrow \mathbb{N}$ be a height function.

4	9	8	7	1	5
3	8	8	7	1	4
2	7	7	7	1	5
1	2	2	2	1	6
	1	2	3	4	5

Rectangular height functions/9

h is a rectangular height function. Its islands are:

$$I_1 = \{(1, 4)\},$$

$$I_2 = \{(1, 3), (1, 4), (2, 3), (2, 4)\},$$

$$I_3 = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\},$$

$$I_4 = \{(5, 1)\},$$

$$I_5 = \{(5, 1), (5, 2)\},$$

$$I_6 = \{(5, 4)\},$$

$$I_7 = \{(5, 1), (5, 2), (5, 3), (5, 4)\},$$

$$I_8 = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (1, 1), (2, 1), (3, 1)\},$$

$$I_9 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\}.$$

Rectangular height functions/10

Its cut relations are:

$$h_{10} = \emptyset$$

$$h_9 = I_1 \text{ (one-element island)}$$

$$h_8 = I_2 \text{ (four-element square island)}$$

$$h_7 = I_3 \text{ (nine-element square island)}$$

$$h_6 = I_3 \cup I_4 \text{ (this cut is a disjoint union of two islands)}$$

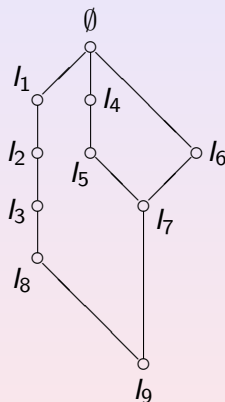
$$h_5 = I_3 \cup I_5 \cup I_6 \text{ (union of three islands)}$$

$$h_4 = I_3 \cup I_7 \text{ (union of two islands)}$$

$$h_2 = I_7 \cup I_8 \text{ (union of two islands)}$$

$$h_1 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} = I_9 \text{ (the whole domain)}$$

Rectangular height functions/11



$$L = (\mathcal{I}_0(\Gamma), \supseteq)$$

Theorem 4

For every rectangular height function

$$h^* : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N},$$

there is a rectangular height function

$$h^{**} : \{1, 2, \dots, n\} \times \{1, 2, \dots, m\} \rightarrow \mathbb{N},$$

such that $\mathcal{I}_{rect}(h^*) = \mathcal{I}_{rect}(h^{**})$ and in h^{**} every island appears exactly in one cut.

If a rectangular height function h^{**} has the property that each island appears exactly in one cut, then we call it *standard rectangular height function*.

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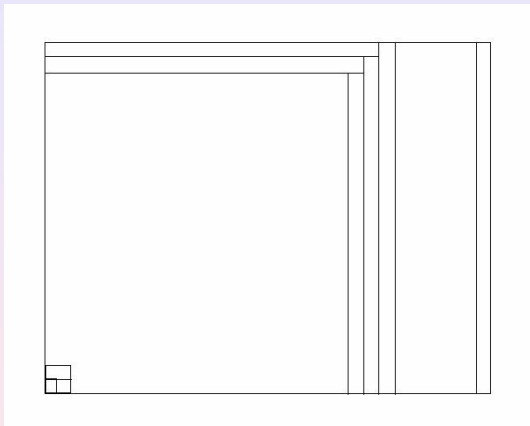
We denote by $\Lambda_{\max}(m, n)$ the maximum number of different nonempty p -cuts of a standard rectangular height function on the rectangular table of size $m \times n$.

Theorem 5 $\Lambda_{\max}(m, n) = m + n - 1$.

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Rectangular height functions/14



The maximum number of different nonempty p -cuts of a standard rectangular height function is equal to the minimum cardinality of maximal systems of islands.

Lemma 1

If $m \geq 3$ and $n \geq 3$ and a height function

$h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ has maximally many islands, then it has exactly two maximal islands.

Lemma 2

If $m \geq 3$ or $n \geq 3$, then for any odd number $t = 2k + 1$ with $1 \leq t \leq \max\{m - 2, n - 2\}$, there is a standard rectangular height function $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ having the maximum number of islands $f(m, n)$, such that one of the side-lengths of one of the maximal islands is equal to t .

(Remark: The statement is not true for even side-lengths, one can construct counterexample easily!)

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We denote by $\Lambda_h^{cz}(m, n)$ the number of different nonempty cuts of a standard rectangular height function h in the case h has maximally many islands, i.e., when the number of islands is

$$f(m, n) = \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor.$$

Theorem 6

Let $h : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \rightarrow \mathbb{N}$ be a standard rectangular height function having maximally many islands $f(m, n)$. Then,
 $\Lambda_h^{cz}(m, n) \geq \lceil \log_2(m+1) \rceil + \lceil \log_2(n+1) \rceil - 1.$

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Islands and formal concepts

Neither every island is a formal concept, nor every formal concept is an island.

Let (C, D) be a formal concept of the context $\mathbb{K} := (A, B, I)$. Then, there are linear orderings $\leq_1$ and $\leq_2$ on A and B , respectively, such that $C \times D$ is an island of $I \subseteq (A, \leq_1) \times (B, \leq_2)$ if and only if there is an element $a \in A$ such that $a \notin C$ and $b \in B$ such that $b \notin D$, with $(a, b) \notin I$, $(a, y) \notin I$ for every $y \in D$ and $(x, b) \notin I$ for every $x \in C$.

If $A_1 \times B_1$ is an island of a relation $I \subseteq (A, \leq_1) \times (B, \leq_2)$, then (A_1, B_1) is a concept if and only if there is no $a \in A \setminus A_1$ such that $(a, b) \in I$ for all $b \in B_1$ and there is no $b \in B \setminus B_1$ such that $(a, b) \in I$ for all $a \in A_1$.

Let I be a relation $I \subseteq (A, \leq_1) \times (B, \leq_2)$. Then, every island in a relation I is a concept if and only if every $x \in A$ belongs to not more than one island and every $y \in B$ belongs to not more than one island.

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CD-independence in posets

Join work with Sándor Radeleczki

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set and $a, b \in P$. The elements a and b are called *disjoint*, and we write $a \perp b$, if

$\inf\{a, b\} = 0$, whenever $\mathbb{P}$ has a least element $0 \in P$,
 a and b have no common lowerbound, whenever $\mathbb{P}$ is without 0 .

Notice, that $a \perp b$ implies $x \perp y$ for all $x, y \in P$ with $x \leq a$ and $y \leq b$. A nonempty set $X \subseteq P$ is called *CD-independent*, if for any $x, y \in X$ either $x \leq y$ or $y \leq x$ or $x \perp y$ holds.

Maximal CD-independent sets (with respect to $\subseteq$) are called *CD-bases* in $\mathbb{P}$.

If $\mathbb{P}$ contains a least element 0 (a greatest element 1) and B is a CD-base, then obviously, $0 \in B$ ($1 \in B$).

Disjoint system

A nonempty set D of nonzero elements of P is called a *disjoint system* in $\mathbb{P}$, if $x \perp y$ holds for all $x, y \in D$, $x \neq y$.

If $\mathbb{P}$ is with 0-element, then $\{0\}$ is considered to be a disjoint system, too. Clearly, any disjoint system $D \subseteq P$ and any chain $C \subseteq P$ is a CD-independent set, and observe also, that D is a disjoint system, if and only if it is a CD-independent antichain in $\mathbb{P}$.

Let $\mathcal{D}(P)$ stand for the set of all disjoint systems of $\mathbb{P}$. Since the disjoint systems of $\mathbb{P}$ are also antichains, restricting $\leq$ to $\mathcal{D}(P)$, we obtain a poset $(\mathcal{D}(P), \leq)$. Clearly, if $\mathbb{P}$ has a least element 0, then $\{0\}$ itself is the least element of $(\mathcal{D}(P), \leq)$.

Proposition 1. *Any poset $\mathbb{P} = (P, \leq)$ has at least one CD-base, and the set P is covered by the CD-bases of $\mathbb{P}$.*

We recall that any antichain $A = \{a_i \mid i \in I\}$ of a poset $\mathbb{P}$ determines a unique order-ideal $I(A)$ of $\mathbb{P}$, namely

$$I(A) = \bigcup_{i \in I} (a_i] = \{x \in P \mid x \leq a_i, \text{ for some } i \in I\},$$

where $(a]$ stands for the principal ideal of an element $a \in P$.

Proposition 2. *If B is a CD-base and $D \subseteq B$ is a disjoint system in the poset $(P, \leq)$, then $I(D) \cap B$ is also a CD-base in the subposet $(I(D), \leq)$.*

Theorem 7. Let $(P, \leq)$ be a finite poset and B a CD-base of it. Then the following assertions hold:

(i) There exists a maximal chain $D_1 \succ \dots \succ D_n$ in $\mathcal{D}(P)$, such that

$$B = \bigcup_{i=1}^n D_i \text{ and } n = |B|.$$

(ii) For any maximal chain $D_1 \prec \dots \prec D_m$ in $\mathcal{D}(P)$ the set $D = \bigcup_{i=1}^m D_i$ is a CD-base in $(P, \leq)$ with $|D| = m$.

$\mathbb{P}$ is called *graded*, if all its maximal chains have the same cardinality.

Proposition 3. *Let $\mathbb{P} = (P, \leq)$ be a finite poset. Then the CD-bases of $\mathbb{P}$ have the same number of elements if and only if the poset $\mathcal{D}(P)$ is graded.*

Corollary 2. *Let $(P, \leq)$ be a finite poset and $(B, \leq)$ its subposet corresponding to a CD-base $B \subseteq P$. Then any maximal chain $\mathcal{C} : D_1 \prec \dots \prec D_n$ in $\mathcal{D}(B)$ is also a maximal chain in $\mathcal{D}(P)$.*

The complete disjoint systems of $\mathbb{P}$ form a principal filter $[A(P))$ in $\mathcal{D}(P)$. Their subposet $([A(P)), \leq)$ will be denoted by $\mathcal{DC}(P)$.

Proposition 4. *Let $\mathbb{P} = (P, \leq)$ be a finite poset with 0. Then the following conditions are equivalent.*

- (i) *The CD-bases of $\mathbb{P}$ have the same number of elements,*
- (ii) *$\mathcal{D}(P)$ is graded.*
- (iii) *$\mathcal{DC}(P)$ is graded.*

Let $\mathbb{P} = (P, \wedge)$ be a semilattice with 0. Now, for any $a, b \in P$ the relation $a \perp b$ means that $a \wedge b = 0$. Hence, a set $\{a_i \mid i \in I\}$ of nonzero elements is a disjoint system if and only if $a_i \wedge a_j = 0$, for all $i, j \in I$, $i \neq j$. A pair of elements $a, b \in P$ with a least upperbound $a \vee b$ in $\mathbb{P}$ is called a *distributive pair*, if $(c \wedge a) \vee (c \wedge b)$ there exists in $\mathbb{P}$ for any $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$.

Theorem 8. *If $\mathbb{P} = (P, \wedge)$ is a semilattice with 0, then $\mathcal{D}(P)$ is a semilattice with 0; if $D_1 \cup D_2$ is a CD-independent set for some $D_1, D_2 \in \mathcal{D}(P)$, then D_1, D_2 is a distributive pair in $\mathcal{D}(P)$. If $\mathbb{P}$ is a complete lattice, then $\mathcal{D}(P)$ is a complete lattice, too.*

We say that $(P, \wedge)$ is *dp-distributive* (*distributive with respect to disjoint pairs*), if any $a, b \in P$ with $a \wedge b = 0$ is a distributive pair.

Corollary 2. *If $\mathbb{P} = (P, \wedge)$ is a semilattice with 0, then $\mathcal{D}(P)$ is a dp-distributive semilattice.*

Let $(P, \leq)$ be a poset and $A \subseteq P$. $(A, \leq)$ is called a *sublattice* of $(P, \leq)$, if $(A, \leq)$ is a lattice such that for any $a, b \in A$ the infimum and the supremum of $\{a, b\}$ is the same in the subposet $(A, \leq)$ and in $(P, \leq)$. If the relation $x \prec y$ in $(A, \leq)$, for some $x, y \in A$ implies $x \prec y$ in the poset $(P, \leq)$, then we say that $(A, \leq)$ is an *cover-preserving subposet* of $(P, \leq)$.

Proposition 5. *Let $\mathbb{P} = (P, \leq)$ be a poset with 0 and B a CD-base of it. Then $(\mathcal{D}(B), \leq)$ is a distributive cover-preserving sublattice of the poset $(\mathcal{D}(P), \leq)$. If $\mathbb{P}$ is a $\wedge$ -semilattice, then for any $D \in \mathcal{D}(P)$ and $D_1, D_2 \in \mathcal{D}(B)$ we have $(D_1 \vee D_2) \wedge D = (D_1 \wedge D) \vee (D_2 \wedge D)$.*

A lattice L with 0 is called *pseudocomplemented* if for each $x \in L$ there exists an element $x^* \in L$ such that for any $y \in L$, $y \wedge x = 0 \Leftrightarrow y \leq x^*$. It is known that an algebraic lattice L is pseudocomplemented if and only if it is *0-distributive*, that is, for any $a, b, x \in L$, $x \wedge a = 0$ and $x \wedge b = 0$ imply $x \wedge (a \vee b) = 0$. We say that L is *weakly 0-distributive* if this implication holds under the assumption $a \wedge b = 0$.

We say that two elements $a, b \in L$ form a *modular pair* in the lattice L , and we write $(a, b)M$, if for any $x \in L$, $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$. a, b is called a *dual-modular pair* if for any $x \in L$, $x \geq b$ implies $x \wedge (a \vee b) = (x \wedge a) \vee b$. This is denoted by $(a, b)M^*$. Clearly, if a, b is a distributive pair, then $(a, b)M^*$ is satisfied. We say that a lattice L satisfies condition M_0^* , if for all $a, b \in L$ with $a \wedge b = 0$, $(a, b)M^*$ holds.

Theorem 9. *Let L be a finite weakly 0-distributive lattice that satisfies condition M_0^* . Then the CD-bases of L have the same number of elements if and only if L is graded.*

If $a \wedge b \neq 0$, then $(x \leq a \vee b \text{ and } x \wedge a = 0) \implies x \leq b$, for any $a, b, x \in L$. ($\mathcal{I}$)

Theorem 10. *Let L be a finite, weakly modular lattice satisfying condition ($\mathcal{I}$). Then the CD-bases of L have the same number of elements.*