Lattices and islands

Eszter K. Horváth, Szeged

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Luxembourg, 2011, June 16.

Islands?

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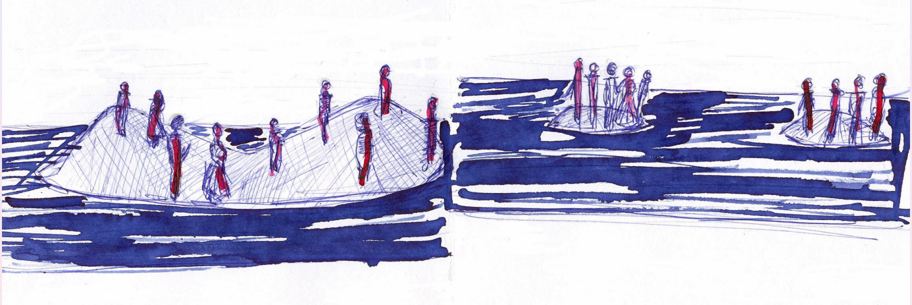
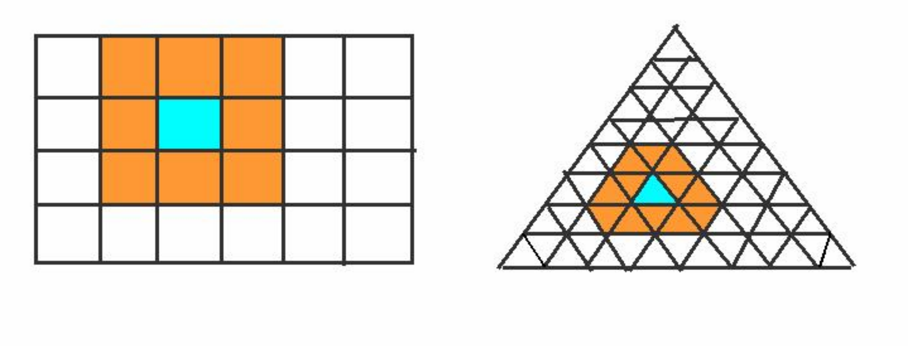
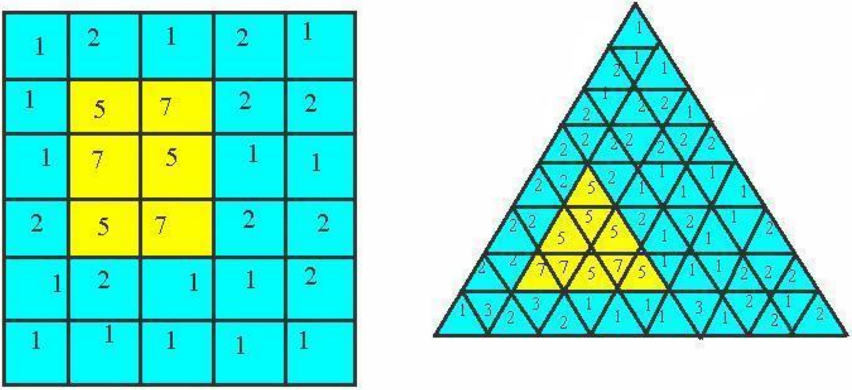


Definition/1

Grid, neighbourhood



We call a rectangle/triangle an *island*, if for the cell t, if we denote its height by a_t, then for each cell $\hat{t}$ neighbouring with a cell of the rectange/triangle T, the inequality $a_{\hat{t}} < min\{a_t : t \in T\}$ holds.



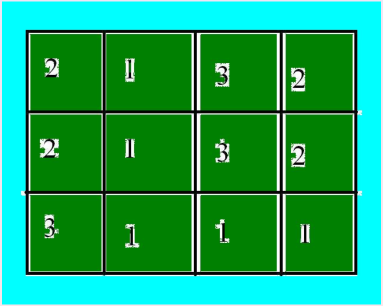
The number of islands / 1

We put heights into the cells.
How many islands do we have?

2	1	3	2
2	1	3	2
3	1	1	1

The number of islands
Water level: 0,5
Nomber of islands: 1

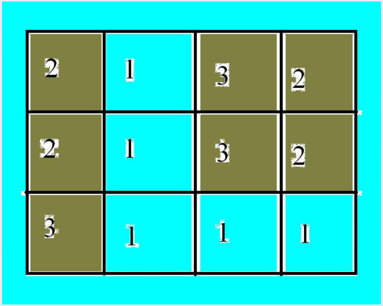
2	1	3	2
2	1	3	2
3	1	1	1



The number of islands / 3

Water level: 1,5
Number of islands: 2

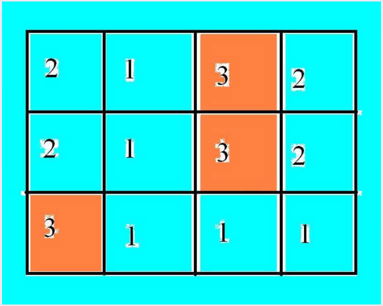
2	1	3	2
2	1	3	2
3	1	1	1



The number of islands / 4

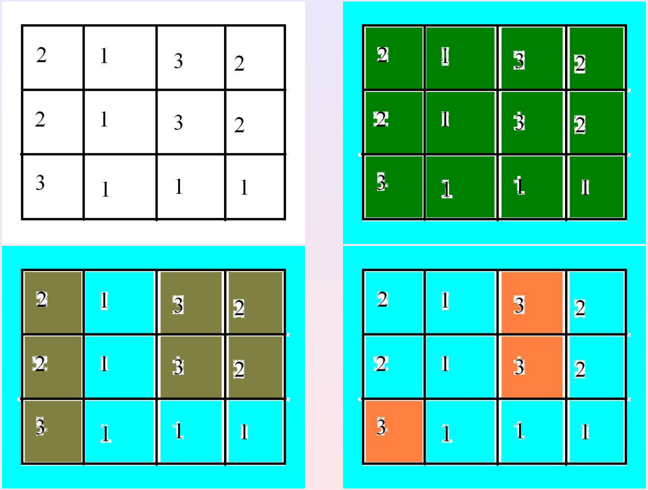
Water level: 2,5
Number of islands: 2

2	1	3	2
2	1	3	2
3	1	1	1



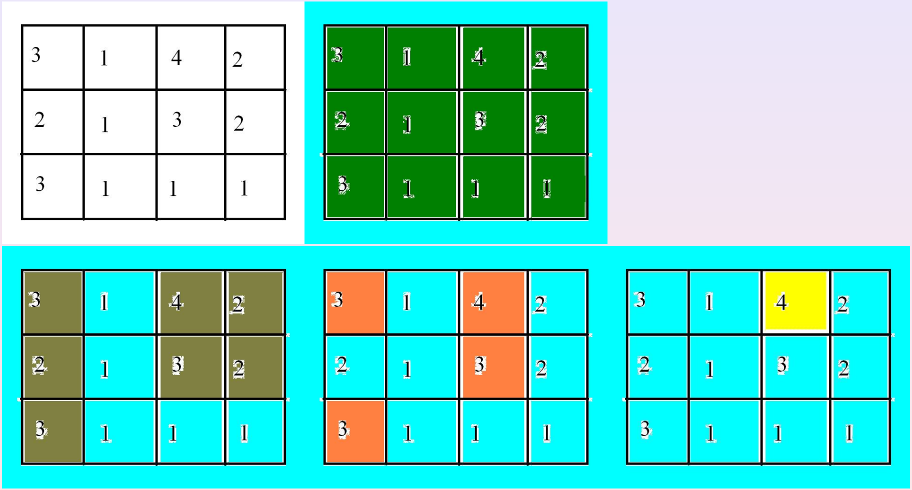
The number of islands / 5

Altogether: $1 + 2 + 2 = 5$ islands.



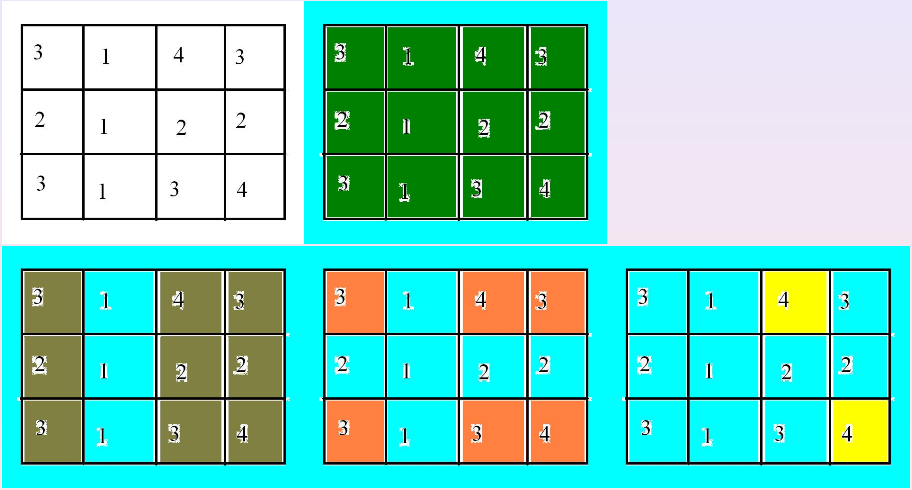
Could we make more islands onto this grid? (With other heights?)

Yes, we could make more islands, here we have $1 + 2 + 3 + 1 = 7$ islands.



Could we make more islands onto this grid? (With other heights?)

Yes, we could make more islands, here we have $1 + 2 + 4 + 2 = 9$ islands.



HOWEWER, WE CANNOT CREATE MORE !!!

The maximum number of islands on the $m \times n$ size grid (Gábor Czédli , Szeged, 2007. june 17.)

$$f(m,n) = \left[\frac{mn+m+n-1}{2}\right].$$

Soon we prove the formula !

History/1

Coding theory

S. Földes and N. M. Singhi: On instantaneous codes, J. of Combinatorics, Information and System Sci., 31 (2006), 317-326.

History/1

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History/2

Rectangular islands

G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics 30 (2009), 208-215.

The maximum number of rectangular islands in a $m \times n$ rectangular board on square grid:

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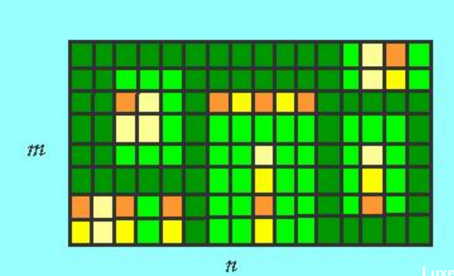

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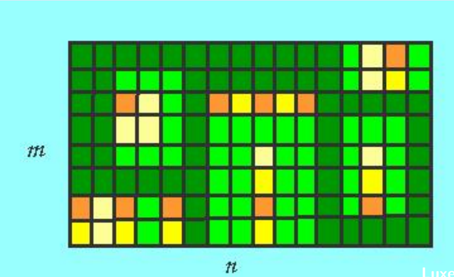

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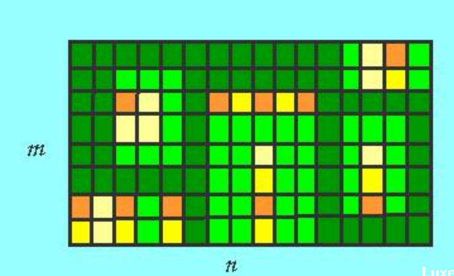

History/3

Rectangular islands in higher dimensions

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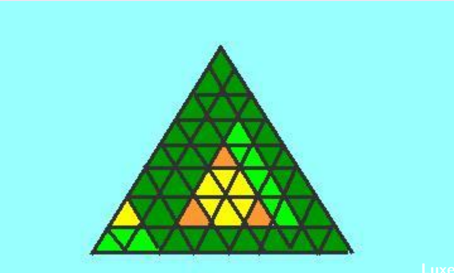
Triangular islands

E. K. Horváth, Z. Németh and G. Pluhár: The number of triangular islands on a triangular grid, Periodica Mathematica Hungarica, 58 (2009), 25–34.
Available at http://www.math.u-szeged.hu/~horvath

For the maximum number of triangular islands in an equilateral rectangle of side length n, $\frac{n^2+3n}{5} \leq f(n) \leq \frac{3n^2+9n+2}{14}$ holds.



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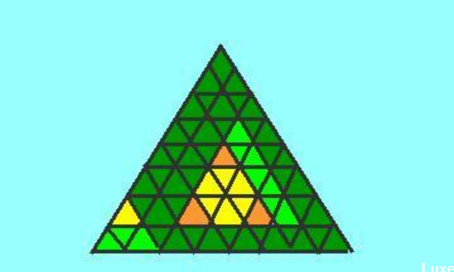

History/4

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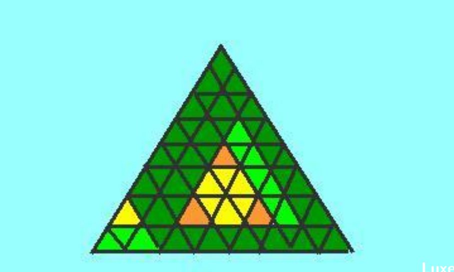
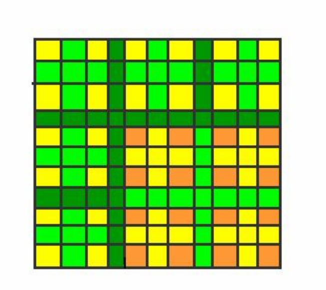


History/5

Square islands (also in higher dimensions)

E. K. Horváth, G. Horváth, Z. Németh, Cs. Szabó: The number of square islands on a rectangular sea, Acta Sci. Math., to appear. Available at http://www.math.u-szeged.hu/~horvath

$$\frac{1}{3}(rs - 2r - 2s) \leq f(r,s) \leq \frac{1}{3}(rs - 1)$$



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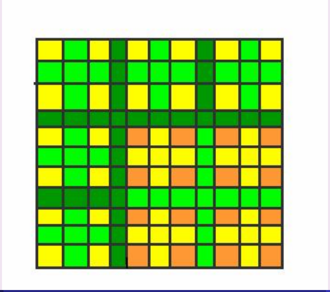

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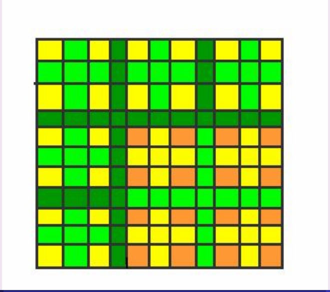

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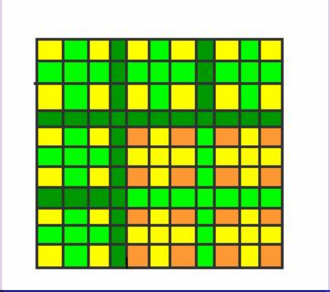


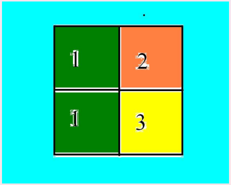
Proving $f(m,n) = \left[\frac{mn+m+n-1}{2}\right]$ THERE EXISTS:

By induction on the number of the cells: $f(m,n) \geq \left[\frac{mn+m+n-1}{2}\right]$.

If $m = 1$, then $\left[\frac{n+1+n-1}{2}\right] = n$, we put the numbers $1, 2, 3, \ldots, n$ in the cells and we will have exactly n islands.

If $n = 1$, then $\left[\frac{m+m+1-1}{2}\right] = m$.

If $m = n = 2$:



Az $f(m,n) = \left[\frac{mn+m+n-1}{2}\right]$ képlet bizonyítása, THERE EXISTS:

Let $m, n > 2$.

$$f(m,n) \geq f(m-2,n) + f(1,n) + 1 \geq \left[\frac{(m-2)n+(m-2)+n-1}{2}\right] + \left[\frac{n+1+n-1}{2}\right] + 1 =$$
$$= \left[\frac{(m-2)n+(m-2)+n-1+2n}{2}\right] + 1 = \left[\frac{mn+m+n-1}{2}\right].$$

Proving methods/1

LATTICE THEORETICAL METHOD

G. Czédli, A. P. Huhn and E. T. Schmidt: Weakly independent subsets in lattices, Algebra Universalis 20 (1985), 194-196.

Any two weak bases of a finite distributive lattice have the same number of elements.

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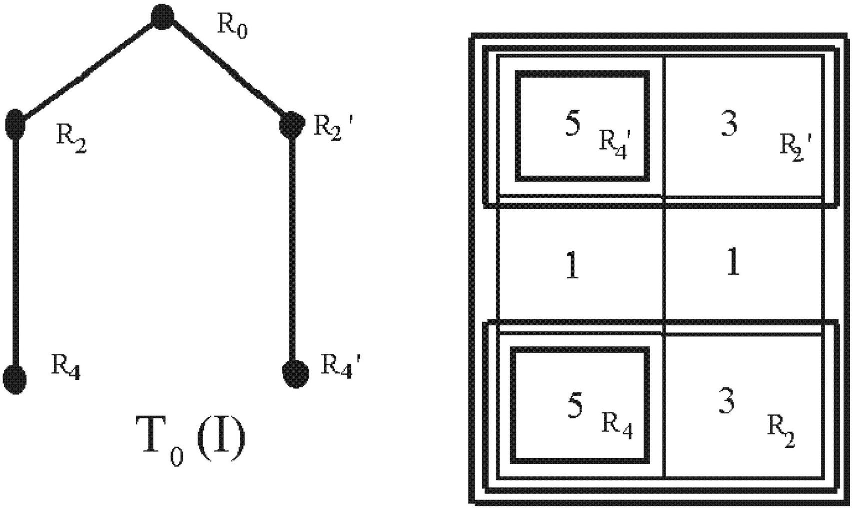
Proving methods/2

TREE-GRAPH METHOD



Proving methods/2

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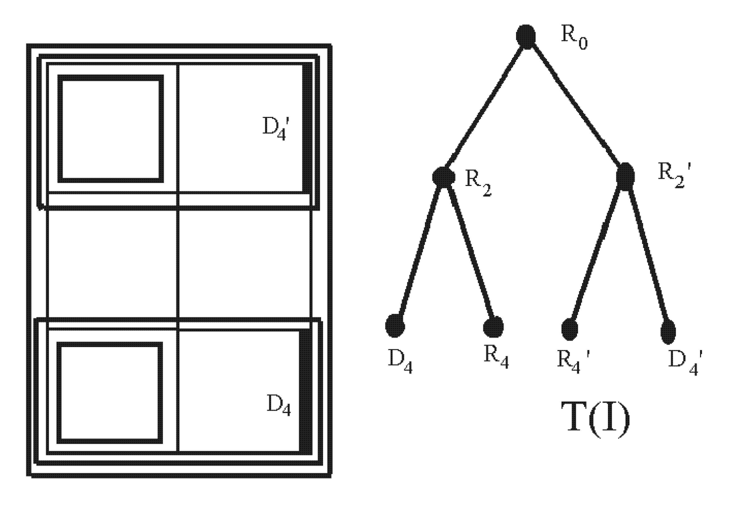

Proving methods/2

TREE-GRAPH METHOD

Lemma 2 (folklore)

(i) Let T be a binary tree with ℓ leaves. Then the number of vertices of T depends only on ℓ, moreover $|V| = 2\ell - 1$.

(ii) Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T. Then $|V| \leq 2\ell - 1$.

We have $4s + 2d \leq (n+1)(m+1)$.
The number of leaves of $T(\mathcal{I})$ is $\ell = s + d$. Hence by Lemma 2 the number of islands is

$$|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n+1)(m+1) - 1.$$

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.

Proving methods/3

ELEMENTARY METHOD

We define

$$\mu(R) = \mu(u, v) := (u + 1)(v + 1).$$

Now

$$f(m, n) = 1 + \sum_{R \in max\mathcal{I}} f(R) = 1 + \sum_{R \in max\mathcal{I}} ([\frac{(u+1)(v+1)}{2}] - 1)$$

$$= 1 + \sum_{R \in max\mathcal{I}} ([\frac{\mu(u, v)}{2}] - 1) \leq 1 - |max\mathcal{I}| + [\frac{\mu(C)}{2}].$$

If $|max\mathcal{I}| \geq 2$, then the proof is ready. Case $|max\mathcal{I}| = 1$ is an easy exercise.

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History/6

Some exact formulas

Cylindric board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_1(m, n) = [\frac{(m+1)n}{2}]$.

Cylindric board, cylindric and rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_2(m, n) = [\frac{(m+1)n}{2}] + [\frac{(m-1)}{2}]$.

Torus board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $m, n \geq 2$, then $t(m, n) = [\frac{mn}{2}]$.

Peninsulas (semi islands) (J. Barát, P. Hajnal, E.K. Horváth):
$p(m, n) = f(m, n) = [(mn + m + n - 1)/2]$.

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Further results on rectangular islands

Zs. Lengvárszky: The minimum cardinality of maximal systems of rectangular islands, European Journal of Combinatorics, **30** (2009), 216-219.

History/7

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History/8

The board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $BA = \{0,1\}^n$.

We consider two cells neighbouring if their Hamming distance is 1.

We denote the maximum number of islands in $BA = \{0,1\}^n$ by $b(n)$.

Island formula for Boolean algebras (P. Hajnal, E.K. Horváth)
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High school competition exercise

Determine the maximum number of islands on n consecutive cells, if the possible heights on the grid are the following: $0, 1, 2, \ldots, h$; where $h \geq 1$.

The solution:

$$I(n, h) = n - \left[\frac{n}{2^h} \right].$$

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Rectangular height functions/1

Joint work with Branimir Šešelja and Andreja Tepavčević

A *height function* h is a mapping from $\{1,2,...,m\} \times \{1,2,...,n\}$ to $\mathbb{N}$,
$h : \{1,2,...,m\} \times \{1,2,...,n\} \to \mathbb{N}$.

The co-domain of the height function is the lattice $(\mathbb{N}, \leq)$, where $\mathbb{N}$ is the set of natural numbers under the usual ordering $\leq$ and suprema and infima are max and min, respectively.

For every $p \in \mathbb{N}$, the *cut of the height function*, i.e. the *p-cut* of h is an ordinary relation h_p on $\{1,2,...,m\} \times \{1,2,...,n\}$ defined by

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Rectangular height functions/2

We say that two rectangles $\{\alpha, ..., \beta\} \times \{\gamma, ..., \delta\}$ and $\{\alpha_1, ..., \beta_1\} \times \{\gamma_1, ..., \delta_1\}$ are *distant* if they are disjoint and for every two cells, namely (a, b) from the first rectangle and (c, d) from the second, we have $(a - c)^2 + (b - d)^2 \geq 4$.

The height function h is called *rectangular* if for every $p \in \mathbb{N}$, every nonempty p-cut of h is a union of distant rectangles.

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Rectangular height functions/3

5	5	3	5	5
4	4	2	4	4
2	2	1	2	2

$\Gamma_1 = \{1,2,3,4,5\} \times \{1,2,3\}$,
$\Gamma_2 = \{1,2,3,4,5\} \times \{1,2,3\} \setminus \{(3,1)\}$,
$\Gamma_3 = \{(1,2),(1,3),(2,2),(2,3),(3,3),(4,2),(4,3),(5,2),(5,3)\}$,
$\Gamma_4 = \{(1,2),(1,3),(2,2),(2,3),(4,2),(4,3),(5,2),(5,3)\}$ and
$\Gamma_5 = \{(1,3),(2,3),(4,3),(5,3)\}$

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Rectangular height functions/4
CHARACTERIZATION THEOREM

Theorem 1

A height function $h_{\mathbb{N}} : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ is rectangular if and only if for all (α, γ), $(\beta, \delta) \in \{1, 2, ..., m\} \times \{1, 2, ..., n\}$ either

- these are not neighboring cells and there is a cell (μ, ν) between (α, γ) and (β, δ) such that $h_{\mathbb{N}}(\mu, \nu) < \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}$, or
- for all $(\mu, \nu) \in [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}] \times [\min\{\gamma, \delta\}, \max\{\gamma, \delta\}]$,

$$h_{\mathbb{N}}(\mu, \nu) \geq \min\{h_{\mathbb{N}}(\alpha, \gamma), h_{\mathbb{N}}(\beta, \delta)\}.$$

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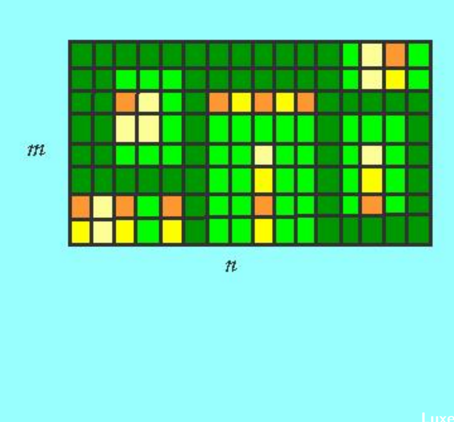
Rectangular height functions/5

Theorem 2

For every height function $h : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, there is a rectangular height function $h^* : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N}$, such that $\mathcal{I}_{rect}(h) = \mathcal{I}_{rect}(h^*)$.



Rectangular height functions/6
CONSTRUCTING ALGORITHM

1. FOR $i = t$ TO 0
2. FOR $y = 1$ TO n
3. FOR $x = 1$ TO m
4. IF $h(x, y) = a_i$ THEN
5. j:= i
6. WHILE there is no island of h which is a subset of h_{a_j} that contains (x, y) DO j:=j-1
7. ENDWHILE
8. Let $h^*(x, y) := a_j$.
9. ENDIF
10. NEXT x
11. NEXT y
12. NEXT i
13. END.

Rectangular height functions/7
LATTICE-VALUED REPRESENTATION

Theorem 3

Let $h: \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ be a rectangular height function. Then there is a lattice L and an L-valued mapping Φ, such that the cuts of Φ are precisely all islands of h.

Rectangular height functions/8

Let $h : \{1,2,3,4,5\} \times \{1,2,3,4\} \to \mathbb{N}$ be a height function.

4	9	8	7	1	5
3	8	8	7	1	4
2	7	7	7	1	5
1	2	2	2	1	6
	1	2	3	4	5

Rectangular height functions/9

h is a rectangular height function. Its islands are:

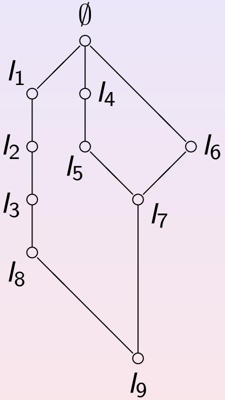
$I_1 = \{(1,4)\}$,
$I_2 = \{(1,3),(1,4),(2,3),(2,4)\}$,
$I_3 = \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$,
$I_4 = \{(5,1)\}$,
$I_5 = \{(5,1),(5,2)\}$,
$I_6 = \{(5,4)\}$,
$I_7 = \{(5,1),(5,2),(5,3),(5,4)\}$,
$I_8 = \{(1,2),(1,3),(1,4),(2,2),(2,3),$
$(2,4),(3,2),(3,3),(3,4),(1,1),(2,1),(3,1)\}$,
$I_9 = \{1,2,3,4,5\} \times \{1,2,3,4\}$.

Rectangular height functions/10

Its cut relations are:

$h_{10} = \emptyset$
$h_9 = I_1$ (one-element island)
$h_8 = I_2$ (four-element square island)
$h_7 = I_3$ (nine-element square island)
$h_6 = I_3 \cup I_4$ (this cut is a disjoint union of two islands)
$h_5 = I_3 \cup I_5 \cup I_6$ (union of three islands)
$h_4 = I_3 \cup I_7$ (union of two islands)
$h_2 = I_7 \cup I_8$ (union of two islands)
$h_1 = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4\} = I_9$ (the whole domain)

Rectangular height functions/11



$L = (\mathcal{I}_0(\Gamma), \supseteq)$

Theorem 4

For every rectangular height function

$$h^* : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N},$$

there is a rectangular height function

$$h^{**} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N},$$

such that $\mathcal{I}_{rect}(h^*) = \mathcal{I}_{rect}(h^{**})$ and in h^{**} every island appears exactly in one cut.

If a rectangular height function h^{**} has the property that each island appears exactly in one cut, then we call it *standard rectangular height function.*

Rectangular height functions/12

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Rectangular height functions/13

We denote by $\Lambda_{max}(m, n)$ the maximum number of different nonempty p-cuts of a standard rectangular height function on the rectangular table of size $m \times n$.

Theorem 5 $\Lambda_{max}(m, n) = m + n - 1.$

Rectangular height functions/13

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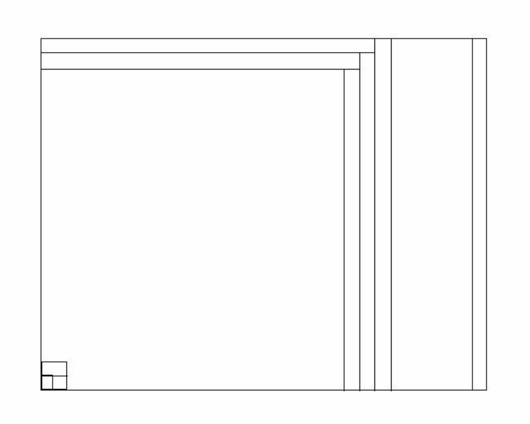
Rectangular height functions/14



The maximum number of different nonempty p-cuts of a standard rectangular height function is equal to the minimum cardinality of maximal systems of islands.

Lemma 1

If $m \geq 3$ and $n \geq 3$ and a height function $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ has maximally many islands, then it has exactly two maximal islands.

Lemma 2

If $m \geq 3$ or $n \geq 3$, then for any odd number $t = 2k + 1$ with $1 \leq t \leq max\{m - 2, n - 2\}$, there is a standard rectangular height function $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ having the maximum number of islands f(m,n), such that one of the side-lengths of one of the maximal islands is equal to t.
(Remark: The statement is not true for even side-lengths, one can construct counterexample easily!)

Rectangular height functions/15

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Rectangular height functions/16

We denote by $\Lambda_h^{cz}(m,n)$ the number of different nonempty cuts of a standard rectangular height function h in the case h has maximally many islands, i.e., when the number of islands is

$$f(m,n) = \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor.$$

Theorem 6

Let $h : \{1,2,...,m\} \times \{1,2,...,n\} \to \mathbb{N}$ be a standard rectangular height function having maximally many islands $f(m,n)$. Then, $\Lambda_h^{cz}(m,n) \geq \lceil log_2(m+1) \rceil + \lceil log_2(n+1) \rceil - 1$.

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CD-independent subsets in distributive lattices

G. Czédli, M. Hartmann and E. T. Schmidt: CD-independent subsets in distributive lattices, Publicationes Mathematicae Debrecen, 74/1-2 (2009).

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CD-independent subsets in posets

Definitions

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set and $a, b \in P$. The elements a and b are called *disjoint* and we write $a \perp b$ if

either $\mathbb{P}$ has least element $0 \in P$ and $\inf\{a, b\} = 0$,
or if $\mathbb{P}$ is without 0, then a and b have no common lowerbound.

- Notice, that $a \perp b$ implies $x \perp y$ for all $x, y \in P$ with $x \leq a$ and $y \leq b$.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$ or $x \perp y$ holds.

Maximal CD-independent sets (with respect to $\subseteq$) are called *CD-bases* in $\mathbb{P}$.

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Disjoint systems

Definition

A nonempty set D of nonzero elements of P is called a *disjoint system* in $\mathbb{P}$ if $x \perp y$ holds for all $x, y \in D$, $x \neq y$.

Remarks

- Any disjoint system $D \subseteq P$ and any chain $C \subseteq P$ is a CD-independent set.
- D is a disjoint system, if and only if it is a CD-independent antichain in $\mathbb{P}$.
- *If X is a CD-independent set in $\mathbb{P}$, then any antichain $A \subseteq X$ is a disjoint system in $\mathbb{P}$.*

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Order ideals

Any antichain $A = \{a_i \mid i \in I\}$ of a poset $\mathbb{P}$ determines a unique order-ideal $I(A)$ of $\mathbb{P}$:

$$I(A) = \bigcup_{i \in I} (a_i] = \{x \in P \mid x \leq a_i, \text{ for some } i \in I\},$$

where $(a]$ stands for the principal ideal of an element $a \in P$.

Definition

If A_1, A_2 are antichains in $\mathbb{P}$, then we say that A_1 *is dominated by* A_2, and we denote it by $A_1 \leqslant A_2$ if

$$I(A_1) \subseteq I(A_2).$$

Remarks

- $\leqslant$ is a partial order
- $A_1 \leqslant A_2$ is satisfied if and only if

$$\text{for each } x \in A_1 \text{ there exists an } y \in A_2, \text{ with } x \leq y. \qquad \text{(A)}$$

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- If $D_1 \leqslant D_2$, then for any $x \in D_1$ and $y \in D_2$ either $x \leq y$ or $x \perp y$ is satisfied.
- The poset $(P, \leq)$ can be order-embedded into $(\mathcal{D}(P), \leqslant)$.

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- $I(A_1) \prec I(A_2) \Leftrightarrow A_1 \prec A_2$, for any antichains $A_1, A_2 \subseteq P$.
- If D_1, D_2 are disjoint systems in P, then $D_1 \subseteq D_2$ implies $D_1 \leqslant D_2$.
- If $D_1 \leqslant D_2$, then for any $x \in D_1$ and $y \in D_2$ either $x \leq y$ or $x \perp y$ is satisfied.
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Tolerance relation

Definition

Let $\rho \subseteq P \times P$.

For any $x, y \in P$, $(x, y) \in \rho \Leftrightarrow$ either $x \leq y$ or $y \leq x$ or $x \perp y$.

Remarks

- ρ is a tolerance relation on P.
- The CD-bases of $\mathbb{P}$ are exactly the tolerance classes (tolerance blocks) of ρ.
- *Any poset $\mathbb{P} = (P, \leq)$ hast at least one CD-base, and the set P is covered by the CD-bases of $\mathbb{P}$.*

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Theorem

Let B *be a CD-base of a finite poset* $(P, \leq)$, *and* let $|B| = n$.

Then there exists a maximal chain $\{D_i\}_{1 \leq i \leq n}$ *in* $\mathcal{D}(P)$ *such that* $B = \bigcup_{i=1}^{n} D_i$.

Moreover, for any maximal chain $\{D_i\}_{1 \leq i \leq m}$ *in* $\mathcal{D}(P)$ *the set* $D = \bigcup_{i=1}^{m} D_i$ *is a CD-base in* $(P, \leq)$ *with* $|D| = m$.

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Proof of the Theorem

Proposition

If B is a CD-base and $D \subseteq B$ is a disjoint system in the poset $(P, \leq)$, then $I(D) \cap B$ is also a CD-base in the subposet $(I(D), \leq)$.

Lemma

If $D_1 \prec D_2$ in $\mathcal{D}(P)$, then $D_2 = \{a\} \cup \{y \in D_1 \setminus \{0\} \mid y \perp a\}$ for some minimal element a of the set
$S = \{s \in P \setminus (D_1 \cup \{0\}) \mid y \perp s \text{ or } y < s \text{ for all } y \in D_1\}$.
Moreover, $D_1 \prec \{a\} \cup \{y \in D_1 \setminus \{0\} \mid y \perp a\}$ holds for any minimal element a of the set S.

Lemma

Assume that B is a CD-base with at least two elements in a finite poset $\mathbb{P} = (P, \leq)$, $M = \max(B)$, and $m \in M$. Then M and $N := \max(B \setminus \{m\})$ are disjoint sets. Moreover M is a maximal element in $\mathcal{D}(P)$, and $N \prec M$ holds in $\mathcal{D}(P)$.

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Let $\mathbb{P}$ =$(P, \leq)$ be a finite poset.

Let $\mathbb{P} = (P, \leq)$ be a finite poset. Then the CD-bases of $\mathbb{P}$ have the same number of elements if and only if the poset $\mathcal{D}(P)$ is graded.

Let $B \subseteq P$ be a CD-base of $\mathbb{P}$, and $(B, \leq)$ the poset under the restricted ordering. Then any maximal chain $\mathcal{C} = \{D_i\}_{1 \leq i \leq m}$ in $\mathcal{D}(B)$ is also a maximal chain in $\mathcal{D}(P)$.

If D is a disjoint set in $\mathbb{P}$ and the CD-bases of $\mathbb{P}$ have the same number of elements, then the CD-bases of the subposet $(I(D), \leq)$ also have the same number of elements.

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$\mathcal{D}(P)$ is graded

The poset $\mathbb{P}$ is called *graded*, if all its maximal chains have the same cardinality.

Let $\mathbb{P} = (P, \leq)$ *be a finite poset with* 0. *Then the following conditions are equivalent:*

(i) *The CD-bases of* $\mathbb{P}$ *have the same number of elements,*

(ii) $\mathcal{D}(P)$ *is graded.*

A disjoint system D of a poset $(P, \leq)$ is called *complete,* if there is no $p \in P \setminus D$ such that $D \cup \{p\}$ is also a disjoint system.

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If all the principal ideals $(a]$ of $\mathbb{P}$ *are weakly 0-modular, then* $A(P) \cup C$ *is a CD-base for every maximal chain* C *in* $\mathbb{P}$.

If $\mathbb{P}$ *has weakly 0-modular principal ideals and* $\mathcal{D}(P)$ *is graded, then* $\mathbb{P}$ *is also graded, and any CD-base of* $\mathbb{P}$ *contains* $|A(P)| + l(P)$ *elements.*

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CD-bases in semilattices and lattices / 1

Lemma

Let $\mathbb{P}$ *be a poset with* 0 *and* D_k, $k \in K$ $(K \neq \emptyset)$ *disjoint sets in* $\mathbb{P}$. *If the meet* $\bigwedge_{k \in K} a^{(k)}$ *of any system of elements* $a^{(k)} \in D_k$, $k \in K$ *exist in* $\mathbb{P}$, *then* $\bigwedge_{k \in K} D_k$ *also exists in* $\mathcal{D}(P)$.

CD-bases in semilattices and lattices / 2

A pair $a, b \in P$ with least upperbound $a \vee b$ in $\mathbb{P}$ is called a *distributive pair*, if $(c \wedge a) \vee (c \wedge b)$ exists in $\mathbb{P}$ for any $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$.

We say that $(P, \wedge)$ is *dp-distributive*, if any $a, b \in P$ with $a \wedge b = 0$ is a distributive pair.

Theorem

(i) *If* $\mathbb{P} = (P, \wedge)$ *is a semilattice with* 0, *then* $\mathcal{D}(P)$ *is a dp-distributive semilattice; if* $D_1 \cup D_2$ *is a CD-independent set for some* $D_1, D_2 \in \mathcal{D}(P)$, *then* D_1, D_2 *is a distributive pair in* $\mathcal{D}(P)$.

(ii) *If* $\mathbb{P}$ *is a complete lattice, then* $\mathcal{D}(P)$ *is a dp-distributive complete lattice.*

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(ii) *If* $\mathbb{P}$ *is a complete lattice, then* $\mathcal{D}(P)$ *is a dp-distributive complete lattice.*

Let $(P, \leq)$ be a poset and $A \subseteq P$. $(A, \leq)$ is called a *sublattice* of $(P, \leq)$, if $(A, \leq)$ is a lattice such that for any $a, b \in A$ the infimum and the supremum of $\{a, b\}$ is the same in the subposet $(A, \leq)$ and in $(P, \leq)$. If the relation $x \prec y$ in $(A, \leq)$ for some $x, y \in A$ implies $x \prec y$ in the poset $(P, \leq)$, then we say that $(A, \leq)$ is a *cover-preserving subposet* of $(P, \leq)$.

Theorem

Let $\mathbb{P} = (P, \leq)$ *be a poset with* 0 *and* B *a CD-base of it. Then* $(\mathcal{D}(B), \leqslant)$ *is a distributive cover-preserving sublattice of the poset* $(\mathcal{D}(P), \leqslant)$. *If* $\mathbb{P}$ *is a* $\wedge$*-semilattice, then for any* $D \in \mathcal{D}(P)$ *and* $D_1, D_2 \in \mathcal{D}(B)$ *we have* $(D_1 \vee D_2) \wedge D = (D_1 \wedge D) \vee (D_2 \wedge D)$ *in* $(\mathcal{D}(P), \leqslant)$.

CD-bases in particular lattice classes

Lemma

Let L be a finite weakly 0-distributive lattice and D a dual atom in $\mathcal{D}(L)$. Then either $D = \{d\}$, for some $d \in L$ with $d \prec 1$, or D consist of two different elements $d_1, d_2 \in L$ and $d_1 \vee d_2 = 1$.

Theorem

Let L be a finite, weakly 0-distributive lattice. Then the following are equivalent:

- (i) *L is graded, and $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.*
- (ii) *L is 0-modular, and the CD-bases of L have the same number of elements.*

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