Islands: research topic for students

Eszter K. Horváth, Szeged

Co-authors: Zoltán Németh, Gabriella Pluhár, János Barát, Péter Hajnal, Csaba Szabó, Gábor Horváth, Branimir Šešelja, Andreja Tepavčević, Attila Máder

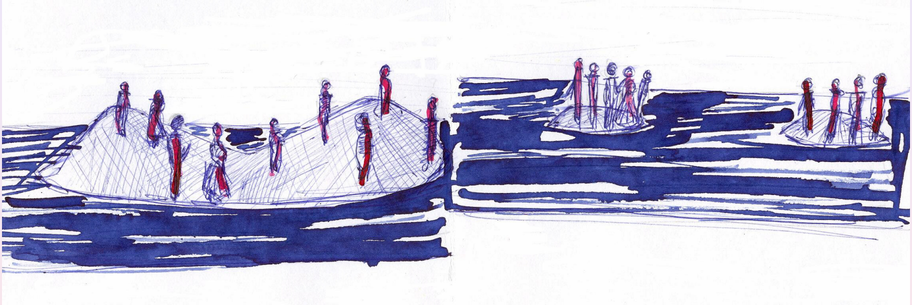
Novi Sad, 2010, 08, 24.

Project

Research is supported by the Hungary-Serbia IPA Cross-border Co-operation programme HU-SRB/0901/221/088 co-financed by the European Union.

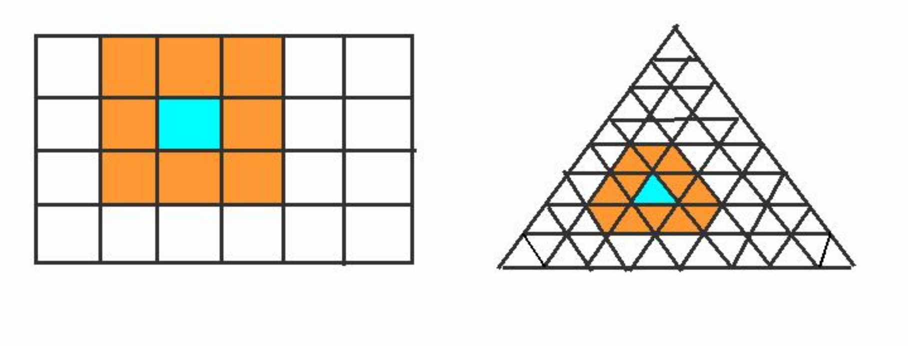
Definition/ 1

Definition/ 2



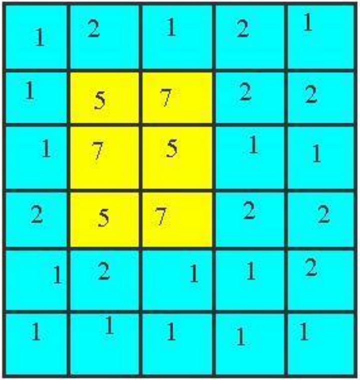
Definition/3

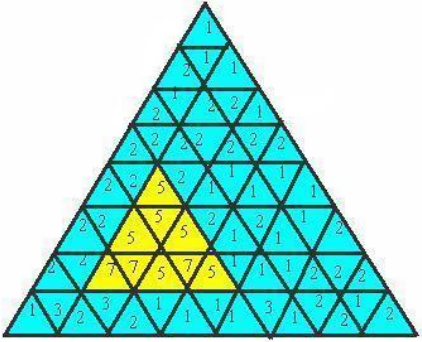
Grid, neighbourhood relation



Definition/4

We call a rectangle/triangle an *island*, if for the cell t, if we denote its height by a_t, then for each cell $\hat{t}$ neighbouring with a cell of the rectange/triangle T, the inequality $a_{\hat{t}} < min\{a_t : t \in T\}$ holds.





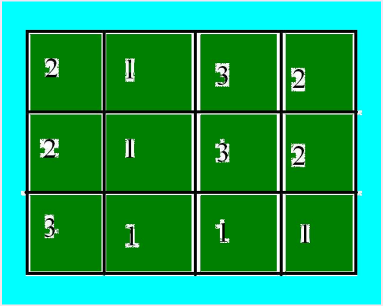
Count the islands! / 1

We put heights into the cells.
How many islands do we have?

2	1	3	2
2	1	3	2
3	1	1	1

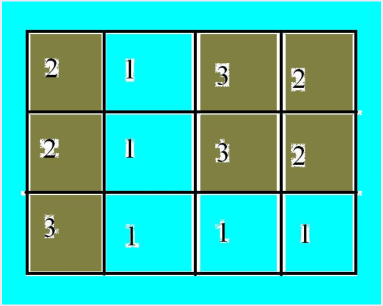
Count the islands!
Water level: 0,5
Nomber of islands: 1

2	1	3	2
2	1	3	2
3	1	1	1



Water level: 1,5
Number of islands: 2

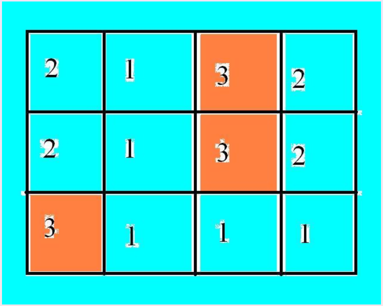
2	1	3	2
2	1	3	2
3	1	1	1



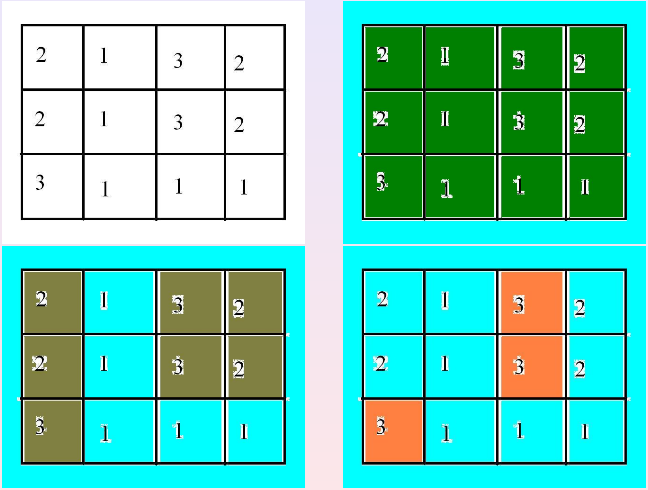
Count the islands! / 4

Water level: 2,5
Number of islands: 2

2	1	3	2
2	1	3	2
3	1	1	1

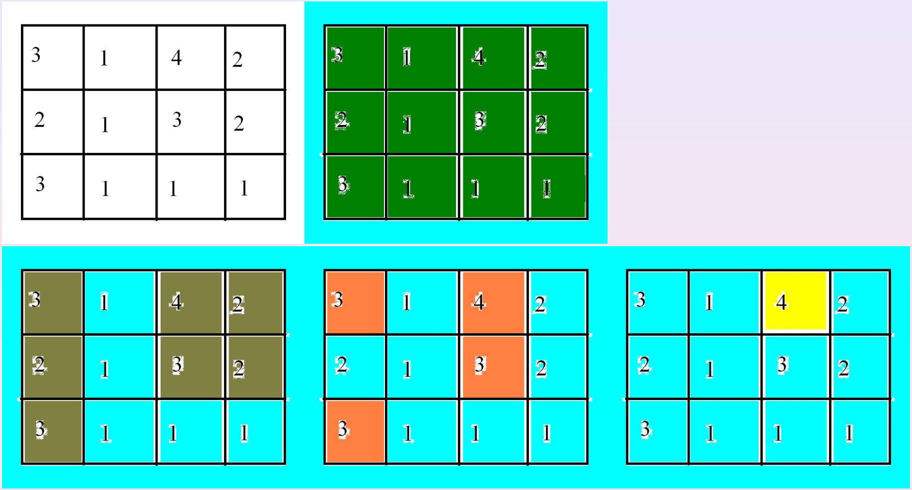


Altogether: $1+2+2=5$ islands.



Could we make more islands onto this grid? (With other heights?)

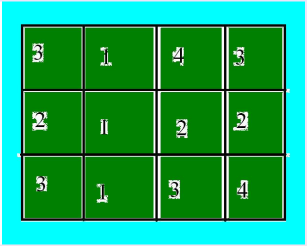
Yes, we could make more islands, here we have $1 + 2 + 3 + 1 = 7$ islands.

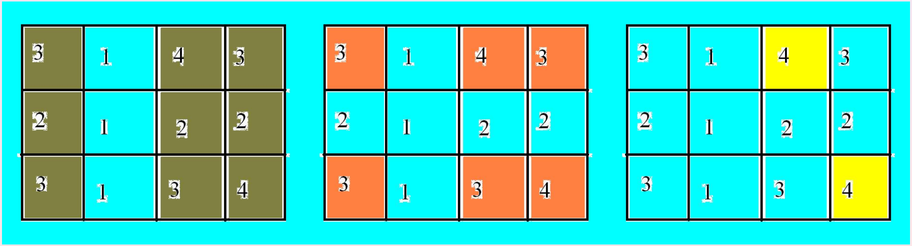


Could we make more islands onto this grid? (With other heights?)

Yes, we could make more islands, here we have $1 + 2 + 4 + 2 = 9$ islands.

3	1	4	3
2	1	2	2
3	1	3	4





HOWEWER, WE CANNOT CREATE MORE !!!

The maximum number of islands on the $m \times n$ size grid (Gábor Czédli , Szeged, 2007. june 17.)

$$f(m,n) = \left[\frac{mn + m + n - 1}{2} \right].$$

Soon we prove the formula !

History/1

Coding theory

S. Földes and N. M. Singhi: On instantaneous codes, J. of Combinatorics, Information and System Sci., 31 (2006), 317-326.

History/1

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History/2

Rectangular islands

G. Czédli: The number of rectangular islands by means of distributive lattices, European Journal of Combinatorics 30 (2009), 208-215.

The maximum number of rectangular islands in a $m \times n$ rectangular board on square grid:

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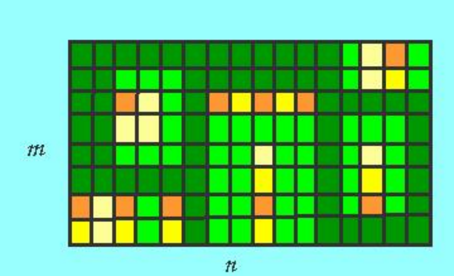

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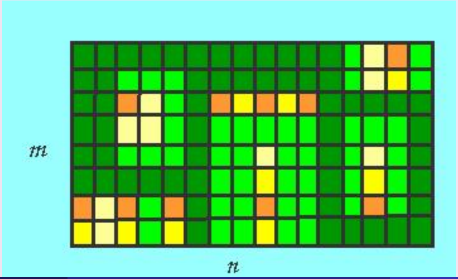

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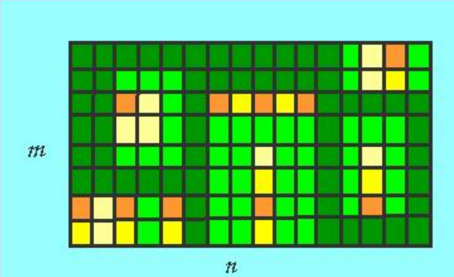

History/3

Rectangular islands in higher dimensions

G. Pluhár: The number of brick islands by means of distributive lattices, Acta Sci. Math., to appear.

History/3

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History/4

Triangular islands

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Available at http://www.math.u-szeged.hu/~horvath

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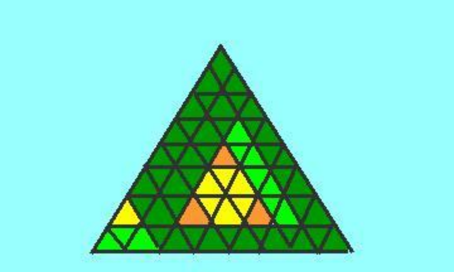

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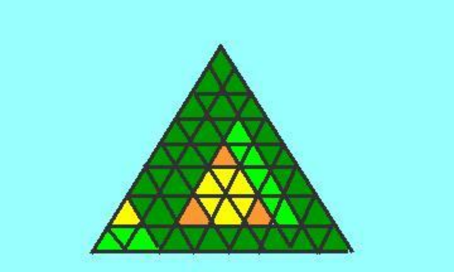

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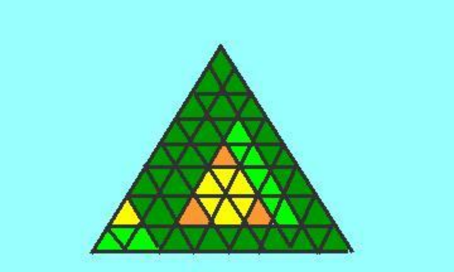

History/5

Square islands (also in higher dimensions)

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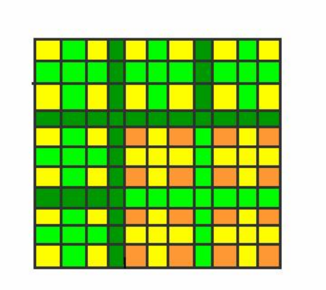

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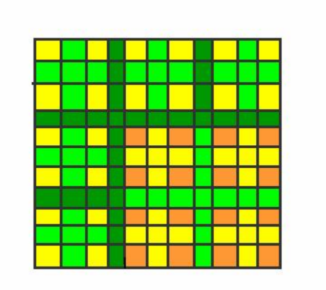

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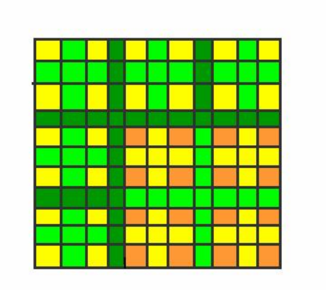

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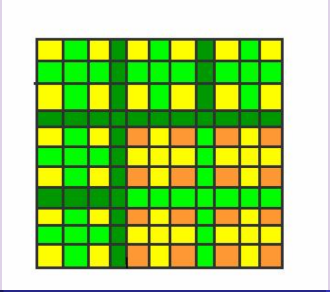

Proving methods/1

LATTICE THEORETICAL METHOD

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Any two weak bases of a finite distributive lattice have the same number of elements.

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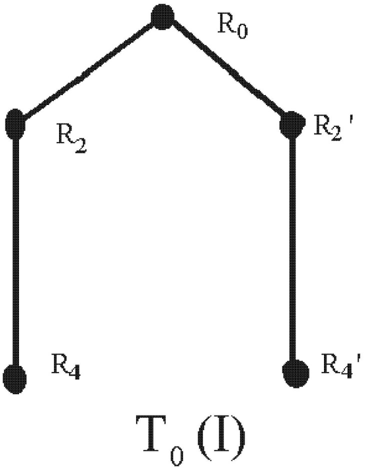
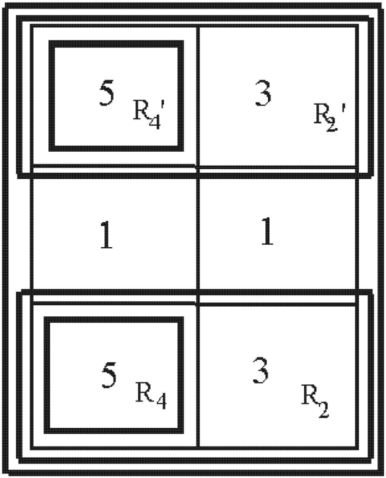
Proving methods/2

TREE-GRAPH METHOD





Proving methods/2

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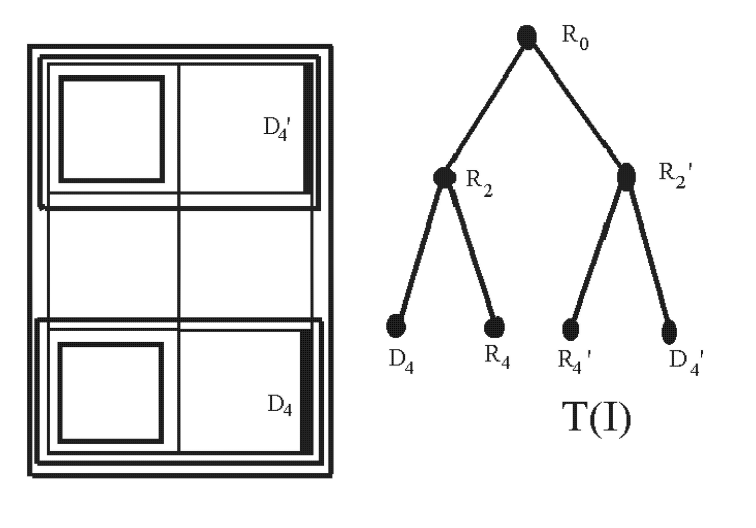

Proving methods/2

TREE-GRAPH METHOD

Lemma 2 (folklore)

(i) Let T be a binary tree with ℓ leaves. Then the number of vertices of T depends only on ℓ, moreover $|V| = 2\ell - 1$.

(ii) Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T. Then $|V| \leq 2\ell - 1$.

We have $4s + 2d \leq (n+1)(m+1)$.
The number of leaves of $T(\mathcal{I})$ is $\ell = s + d$. Hence by Lemma 2 the number of islands is

$$|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n+1)(m+1) - 1.$$

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.

Proving methods/3

ELEMENTARY METHOD

We define

$$\mu(R) = \mu(u, v) := (u + 1)(v + 1).$$

Now

$$f(m, n) = 1 + \sum_{R \in max\mathcal{I}} f(R) = 1 + \sum_{R \in max\mathcal{I}} \left(\left[\frac{(u+1)(v+1)}{2}\right] - 1\right)$$

$$= 1 + \sum_{R \in max\mathcal{I}} \left(\left[\frac{\mu(u, v)}{2}\right] - 1\right) \leq 1 - |max\mathcal{I}| + \left[\frac{\mu(\mathrm{C})}{2}\right].$$

If $|max\mathcal{I}| \geq 2$, then the proof is ready. Case $|max\mathcal{I}| = 1$ is an easy excersise.

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History/6

Some exact formulas

Cylindric board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_1(m,n) = [\frac{(m+1)n}{2}]$.

Cylindric board, cylindric and rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $n \geq 2$, then $h_2(m,n) = [\frac{(m+1)n}{2}] + [\frac{(m-1)}{2}]$.

Torus board, rectangular islands (J. Barát, P. Hajnal, E.K. Horváth):
If $m, n \geq 2$, then $t(m,n) = [\frac{mn}{2}]$.

Peninsulas (semi islands) (J. Barát, P. Hajnal, E.K. Horváth):
$p(m,n) = f(m,n) = [(mn + m + n - 1)/2]$.

History/6

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History/7

Further results on rectangular islands

Zs. Lengvárszky: The minimum cardinality of maximal systems of rectangular islands, European Journal of Combinatorics, **30** (2009), 216-219.

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History/8

The board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $BA = \{0,1\}^n$.

We consider two cells neighbouring if their Hamming distance is 1.

We denote the maximum number of islands in $BA = \{0,1\}^n$ by $b(n)$.

Island formula for Boolean algebras (P. Hajnal, E.K. Horváth)
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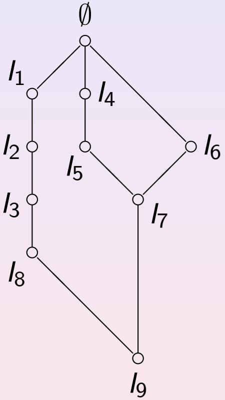
Constructing algorithm

Joint work with Branimir Šešelja and Andreja Tepavčević

CONSTRUCTING ALGORITHM

1. FOR $i = t$ TO 0
2. FOR $y = 1$ TO n
3. FOR $x = 1$ TO m
4. IF $h(x, y) = a_i$ THEN
5. j:= i
6. WHILE there is no island of h which is a subset of h_{a_j} that contains (x, y) DO j:=j-1
7. ENDWHILE
8. Let $h^*(x, y) := a_j$.
9. ENDIF
10. NEXT x
11. NEXT y
12. NEXT i
13. END.

The lattice of islands



$L = (\mathcal{I}_0(\Gamma), \supseteq)$

Height of the hills

We denote by $\Lambda_{max}(m, n)$ the maximum number of different nonempty p-cuts of a standard rectangular height function on the rectangular table of size $m \times n$.

Theorem 5 $\Lambda_{max}(m, n) = m + n - 1.$

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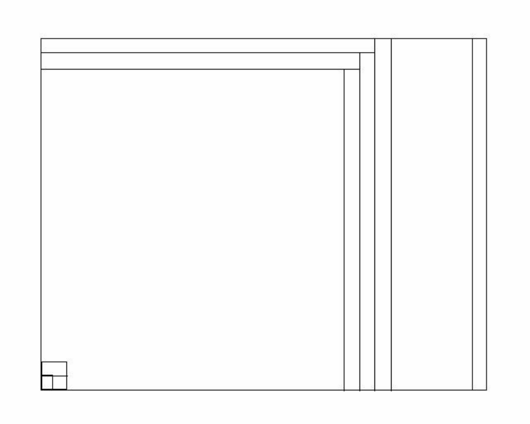

The maximum number of different nonempty p-cuts of a standard rectangular height function is equal to the minimum cardinality of maximal systems of islands.

Height of the hills

We denote by $\Lambda_h^{cz}(m,n)$ the number of different nonempty cuts of a standard rectangular height function h in the case h has maximally many islands, i.e., when the number of islands is

$$f(m,n) = \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor .$$

Theorem

Let $h : \{1,2,...,m\} \times \{1,2,...,n\} \to \mathbb{N}$ be a standard rectangular height function having maximally many islands $f(m,n)$. Then, $\Lambda_h^{cz}(m,n) \geq \lceil log_2(m+1) \rceil + \lceil log_2(n+1) \rceil - 1$.

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High school competition exercise

Determine the maximum number of islands on n consecutive cells, if the possible heights on the grid are the following: $0, 1, 2, \ldots, h$; where $h \geq 1$.

The solution:

$I(n,h) = n - \left[\frac{n}{2^h}\right]$.

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