Lattices with many sublattices

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Subuniverses

For a lattice L, $\mathrm{Sub}(L)$ will denote its so-called *sublattice lattice*.

$\mathrm{Sub}(L)$ consists of all *subuniverses* of L. By a subuniverse, we mean a sublattice or the emptyset.

That is, a subset X of L is in $\mathrm{Sub}(L)$ iff X is closed with respect to join and meet.

In particular, $\emptyset \in \mathrm{Sub}(L)$. Note that for $X \in \mathrm{Sub}(L)$, X is a sublattice of L if and only if X is nonempty.

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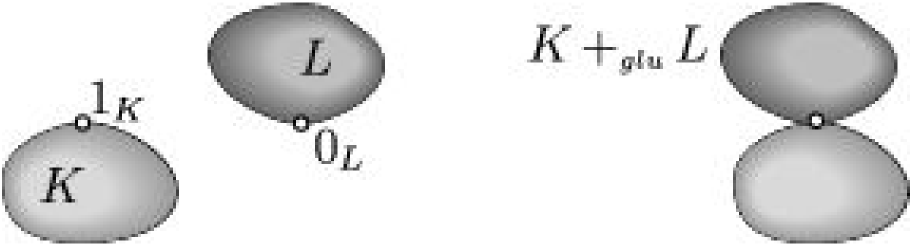
Glued sum of lattices

If K and L are finite lattices, then their *glued sum* $K +_{glu} L$ is the ordinal sum of the posets $K \setminus 1_K$, the singleton lattice, and $L \setminus \{0_L\}$, in this order.



In other words, we put L atop K and identify the elements 1_K and 0_L. For example, if each of K and L is the two-element chain, then $K +_{glu} L$ is the three-element chain.

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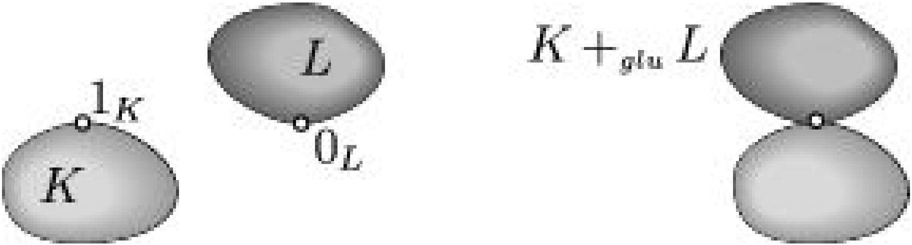

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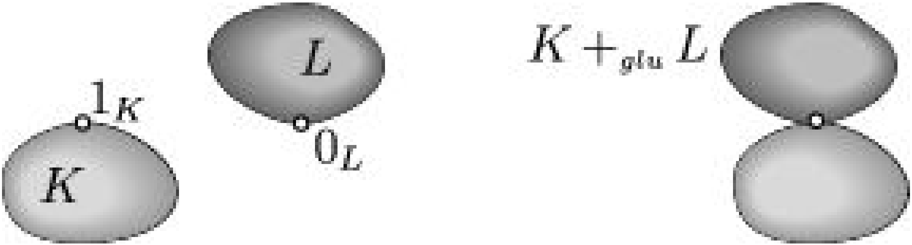

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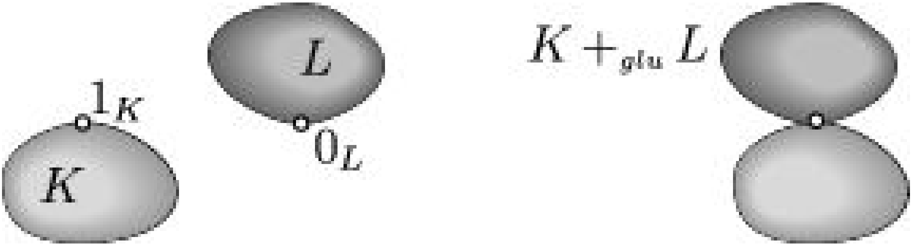

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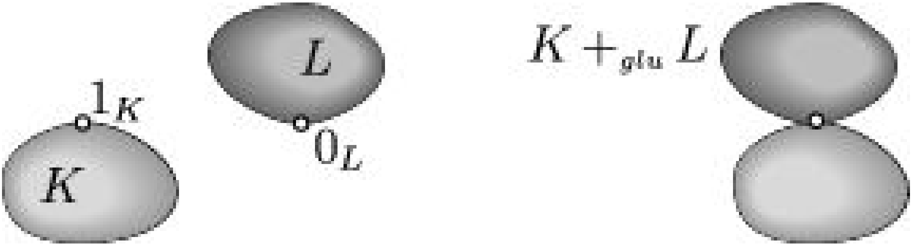

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The first three numbers

Theorem

If $5 \leq n \in \mathbb{N}^+$, then the following three assertions hold.

(i) *The largest number in* $\mathrm{NS}(n)$ *is* $2^n = 32 \cdot 2^{n-5}$. *Furthermore, an n-element lattice L has exactly* 2^n *subuniverses if an only if L is a chain.*

(ii) *The second largest number in* $\mathrm{NS}(n)$ *is* $26 \cdot 2^{n-5}$. *Furthermore, an n-element lattice L has exactly* $26 \cdot 2^{n-5}$ *subuniverses if and only if* $L \cong C_1 +_{glu} B_4 +_{glu} C_2$, *where* C_1 *and* C_2 *are chains.*

(iii) *The third largest number in* $\mathrm{NS}(n)$ *is* $23 \cdot 2^{n-5}$. *Furthermore, an n-element lattice L has exactly* $23 \cdot 2^{n-5}$ *subuniverses if and only if* $L \cong C_0 +_{glu} N_5 +_{glu} C_1$, *where* C_0 *and* C_1 *are chains.*

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Basic lemma

Lemma

If K is a sublattice and H is a subset of a finite lattice L, then the following three assertions hold.

(i) *With the notation $t := |\{H \cap S : S \in \mathrm{Sub}(L)\}|$, we have that $|\,\mathrm{Sub}(L)| \leq t \cdot 2^{|L|-|H|}$.*

(ii) $|\,\mathrm{Sub}(L)| \leq |\,\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}$.

(iii) *Assume, in addition, that K has neither an isolated element, nor an isolated edge. Then $|\,\mathrm{Sub}(L)| = |\,\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}$ if and only if L is (isomorphic to) $C_0 +_{glu} K +_{glu} C_1$ for some chains C_0 and C_1.*

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If H is a subset of L, then $|\operatorname{Sub}(L)| \leq |\{H \cap S : S \in \operatorname{Sub}(L)\}| \cdot 2^{|L|-|H|}$

Proof

Let $\varphi\colon \operatorname{Sub}(L) \to \{H \cap S : S \in \operatorname{Sub}(L)\}$, $X \mapsto H \cap X$.

Each $Y \in \{H \cap S : S \in \operatorname{Sub}(L)\}$ has at most $2^{|L|-|H|}$ preimages.

Q. E. D.

If K is a sublattice of L, then $|\operatorname{Sub}(L)| \leq |\operatorname{Sub}(K)| \cdot 2^{|L|-|K|}$.

If $|\operatorname{Sub}(L)| = |\operatorname{Sub}(K)| \cdot 2^{|L|-|K|}$, then for every $S \in \operatorname{Sub}(K)$ and every subset X of $L \setminus K$, we have that $S \cup X \in \operatorname{Sub}(L)$.

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Isolated element

We call an element $u \in L$ *isolated* if u is doubly irreducible and $L = \downarrow u \cup \uparrow u$.

That is, if u is doubly irreducible and $x \parallel u$ holds for no $x \in L$.

For an element $u \in L$, u is isolated if and only if for every $X \in \mathrm{Sub}(L)$, we have that $X \cup \{u\} \in \mathrm{Sub}(L)$ and $X \setminus \{u\} \in \mathrm{Sub}(L)$.

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Assume that u is isolated and $X \in \mathrm{Sub}(L)$. Since u is doubly irreducible, $X \setminus \{u\} \in \mathrm{Sub}(L)$. Since u is comparable with all elements of X, $X \cup \{u\} \in \mathrm{Sub}(L)$.
To show the converse, assume that u is not isolated. If u is not doubly irreducible, then $u = a \vee b$ with $a, b < u$ or dually, and $X := \{a, b, u, a \wedge b\} \in \mathrm{Sub}(L)$ but $X \setminus \{u\} \notin \mathrm{Sub}(L)$. If $u \parallel v$ for some $v \in L$, then $\{v\} \in \mathrm{Sub}(L)$ but $\{v\} \cup \{u\} \notin \mathrm{Sub}(L)$.

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Numbers

Lemma

The following seven assertions hold.

(i) $|\,\mathrm{Sub}(B_4)| = 13 = 26 \cdot 2^{4-5}$.

(ii) $|\,\mathrm{Sub}(N_5)| = 23 = 23 \cdot 2^{5-5}$.

(iii) $|\,\mathrm{Sub}(C^{(2)} \times C^{(3)})| = 38 = 19 \cdot 2^{6-5}$.

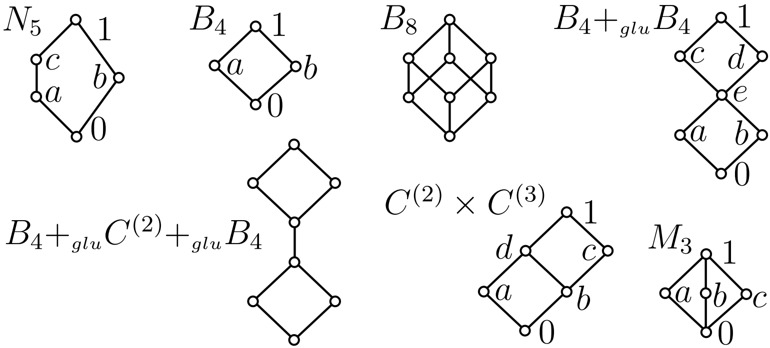
(iv) $|\,\mathrm{Sub}(B_4 +_{glu} B_4)| = 85 = 21.25 \cdot 2^{7-5}$.

(v) $|\mathrm{Sub}(B_4 +_{glu} C^{(2)} +_{glu} B_4)| = 169 = 21.125 \cdot 2^{8-5}$.

(vi) $|\,\mathrm{Sub}(M_3)| = 20 = 20 \cdot 2^{5-5}$.

(vii) $|\,\mathrm{Sub}(B_8)| = 74 = 9.25 \cdot 2^{8-5}$.

Lattices



N5
1
c
b
a
0
B4
1
a
b
0
B8
B4+gluB4
1
c
d
e
a
b
0
B4+gluC(2)+gluB4
C(2) × C(3)
1
d
c
a
b
0
M3
1
a
b
c
0

$|\operatorname{Sub}(N_5)| = 23 = 23 \cdot 2^{5-5}$

Observe that

$$\begin{aligned} &|\{S \in \operatorname{Sub}(N_5) : \{a,c\} \cap S = \emptyset\}| = 8, \qquad \text{by } (\textbf{??}),\\ &|\{S \in \operatorname{Sub}(N_5) : \{a,c\} \cap S \neq \emptyset\},\ b \notin S| = 3 \cdot 4 = 12, \text{ and}\\ &|\{S \in \operatorname{Sub}(N_5) : \{a,c\} \cap S \neq \emptyset\},\ b \in S| = 3, \end{aligned}$$

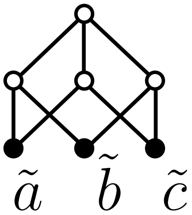
whereby $|\operatorname{Sub}(N_5)| = 8 + 12 + 3 = 23.$

Free join-semilattice

Lemma

For every join-semilattice S generated by $\{a, b, c\}$*, there is a unique surjective homomorphism* φ *from the free join-semilattice* $F_{\mathrm{jsl}}(\tilde{a}, \tilde{b}, \tilde{c})$ *onto S such that* $\varphi(\tilde{a}) = a$*,* $\varphi(\tilde{b}) = b$*, and* $\varphi(\tilde{c}) = c$*.*

$$F_{\mathrm{jsl}}(\tilde{a}, \tilde{b}, \tilde{c})$$

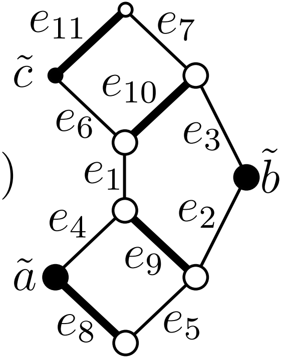


Free lattice

Lemma (I. Rival and R. Wille)

For every lattice K generated by $\{a, b, c\}$ such that $a < c$, there is a unique surjective homomorphism φ from the finitely presented lattice $F_{\mathrm{lat}}(\tilde{a}, \tilde{b}, \tilde{c})$ onto K such that $\varphi(\tilde{a}) = a$, $\varphi(\tilde{b}) = b$, and $\varphi(\tilde{c}) = c$.

$F_{\mathrm{lat}}(\tilde{a}, \tilde{b}, \tilde{c})$



3-antichain

Lemma

If an n-element lattice L has a 3-antichain, then we have that $|\operatorname{Sub}(L)| \leq 20 \cdot 2^{n-5}$.

Proof. Let $\{a, b, c\}$ be a 3-antichain in L. Lemma 4 yields a unique join-homomorphism from $F_{\mathrm{jsl}}((\tilde{a}, \tilde{b}, \tilde{c}))$ to $S := \{a, b, c, a \vee b, a \vee c, b \vee c, a \vee b \vee c\}$ such that φ maps to $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ to a, b, and c, respectively.

Since $\{a, b, c\}$ is an antichain, none of the six lower edges of $F_{\mathrm{jsl}}((\tilde{a}, \tilde{b}, \tilde{c}))$ is collapsed by the kernel $\Theta := \ker(\varphi)$ of φ. Hence, there are only four cases for the join-subsemilattice $S \cong F_{\mathrm{jsl}}((\tilde{a}, \tilde{b}, \tilde{c}))/\Theta$ of L, depending on the number the upper edges collapsed by Θ.

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Case 1

None of the three upper edges is collapsed by Θ.

Then S is isomorphic to $F_{\text{jsl}}((\tilde{a}, \tilde{b}, \tilde{c}))$, whereby $\{a \vee b, a \vee c, b \vee c\}$ is a 3-antichain.

We know that this 3-antichain generates a sublattice isomorphic to B_8. Hence, $|\text{Sub}(L)| \leq 9.25 \cdot 2^{n-5} \leq 20 \cdot 2^{n-5}$.

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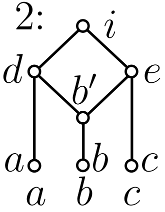
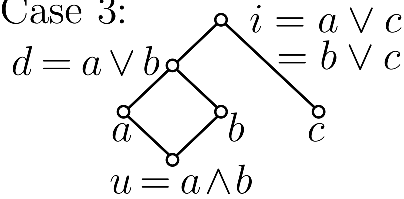


Case 3:



Case 4

All the three upper edges are collapsed.

Clearly, $a \vee b = a \vee c = b \vee c = a \vee b \vee c =: i$.

If $a \wedge b = a \wedge c = b \wedge c = a \wedge b \wedge c$ failed, then the dual of one of the previous three cases would apply.

Hence, we can assume that the sublattice $[\{a, b, c\}]$ generated by $\{a, b, c\}$ is isomorphic to M_3.

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Third number

If $L \cong C_0 +_{glu} N_5 +_{glu} C_1$ for finite chains C_0 and C_1,
then $|\mathrm{Sub}(L)| = 23 \cdot 2^{n-5}$.

It suffices to exclude the existence of a lattice L such that $|L| = n$, $23 \cdot 2^{n-5} \leq |\mathrm{Sub}(L)| < 26 \cdot 2^{n-5}$, but L is not of this form above.

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Sketch of proof

Suppose, for a contradiction, that L is a lattice satisfying $|L| = n$, $23 \cdot 2^{n-5} \leq |\,\mathrm{Sub}(L)| < 26 \cdot 2^{n-5}$, but L is not of form

$$L \cong C_0 +_{glu} N_5 +_{glu} C_1.$$

then, L has at least two 2-antichains but it has no 3-antichain.

We show that cannot have two *non-disjoint* 2-antichains.

$$\text{If } x, y, z \in L \text{ such that } |\{x, y, z\}| = 3 \text{ and } x \parallel y, \text{ then either } \{x, y\} \subseteq {\downarrow}z, \text{ or } \{x, y\} \subseteq {\uparrow}z. \tag{1}$$

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End of proof

We have a four-element subset $\{a, b, c, d\}$ of L such that $a \parallel b$ and $c \parallel d$.

Let $S := \{a \wedge b, a, b, u, v, c, d, c \vee d\}$. Depending on $u = v$ or $u < v$, S is a sublattice isomorphic to $B_4 +_{glu} B_4$ or $B_4 +_{glu} C^{(2)} +_{glu} B_4$.

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