# CD-independent subsets

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# CD-independent subsets in posets

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set, and let $a, b \in P$. The elements $a$ and $b$ are called *disjoint* and we write $a \perp b$ if

either $\mathbb{P}$ has least element $0 \in P$ and $\inf\{a, b\} = 0$,
or $\mathbb{P}$ is without 0 and the elements $a$ and $b$ have no common lowerbound.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$, or $x \perp y$ holds. Maximal CD-independent sets (with respect to $\subseteq$) are called *CD-bases* in $\mathbb{P}$.

# Sets of pairwise disjoint elements

**Definition**

A nonempty set $D$ of nonzero elements of $P$ is called a *set of pairwise disjoint element* in $\mathbb{P}$ if $x \perp y$ holds for all $x, y \in D$, $x \neq y$; if $\mathbb{P}$ has $0$-element, then $\{0\}$ is considered to be a set of pairwise disjont elements, too.

Remark

- $D$ is a set of pairwise disjoint elements if and only if it is a CD-independent antichain in $\mathbb{P}$.

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# Order ideals

Let $X \subseteq P$.
The order ideal $\{y \in P \mid y \leq x \text{ for some } x \in X\}$ is denoted by $\downarrow X$.
The order-ideals of any poset form a (distributive) lattice with respect to $\subseteq$.
So, the antichains of a poset can be ordered as follows:

**Definition**

If $A_1, A_2$ are antichains in $\mathbb{P}$, then we say that $A_1$ *is dominated by* $A_2$, and we denote it by $A_1 \leqslant A_2$ if $\downarrow A_1 \subseteq \downarrow A_2$.

**Remarks**

- $\leqslant$ is a partial order
- $A_1 \leqslant A_2$ is satisfied if and only if

$$\text{for each } x \in A_1 \text{ there exists an } y \in A_2, \text{ with } x \leq y. \qquad \text{(A)}$$

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# $\mathcal{D}(\mathbb{P})$

Let $\mathcal{D}(\mathbb{P})$ denote the set of all sets of pairwise disjont elements of $\mathbb{P}$.

As sets of pairwise disjont elements of $\mathbb{P}$ are also antichains, restricting $\leqslant$ to $\mathcal{D}(\mathbb{P})$, we obtain a poset $(\mathcal{D}(\mathbb{P}), \leqslant)$.

The connection between CD-bases of a poset $\mathbb{P}$ and the poset $(\mathcal{D}(\mathbb{P}), \leqslant)$ is shown by the next theorem:

# Theorem

*Let* $B$ *be a CD-base of a finite poset* $(P, \leq)$, *and* let $|B| = n$.

*Then there exists a maximal chain* $\{D_i\}_{1 \leq i \leq n}$ *in* $\mathcal{D}(P)$ *such that*
$$B = \bigcup_{i=1}^{n} D_i.$$

*Moreover, for any maximal chain* $\{D_i\}_{1 \leq i \leq m}$ *in* $\mathcal{D}(P)$ *the set* $D = \bigcup_{i=1}^{m} D_i$ *is a CD-base in* $(P, \leq)$ *with* $|D| = m$.

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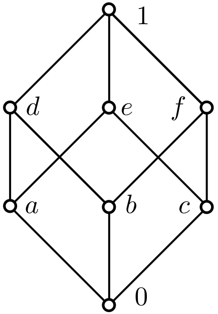
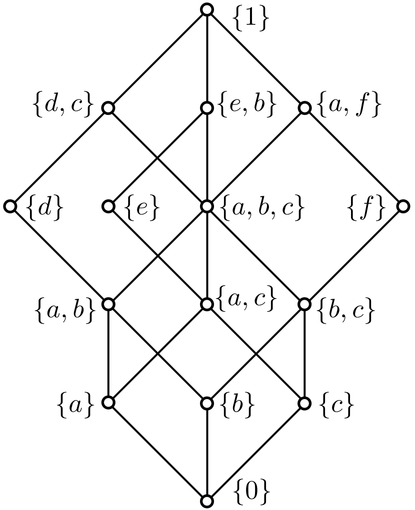
# $P$ and $\mathcal{D}(P)$



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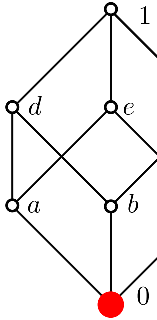


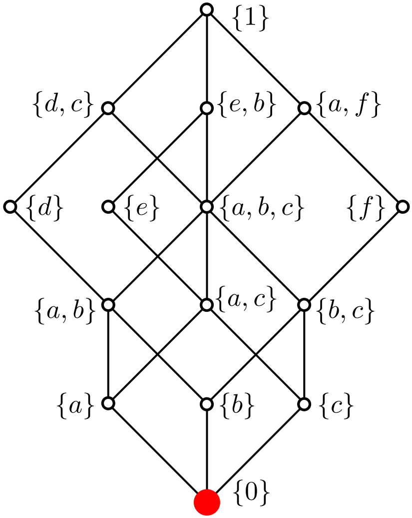

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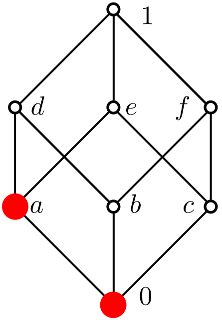


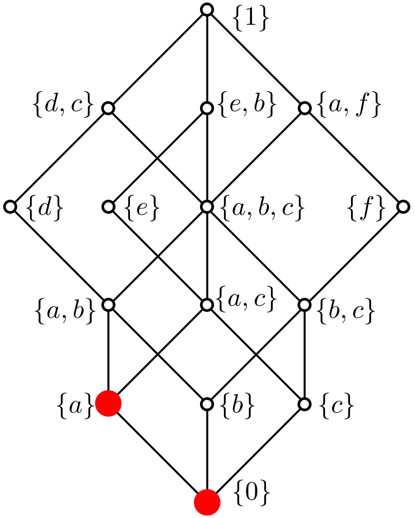

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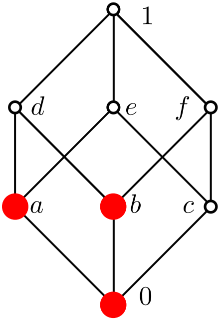


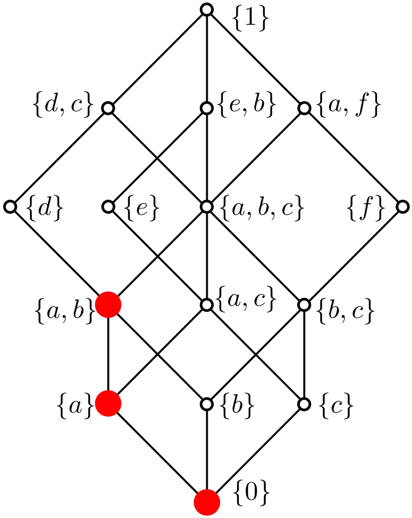

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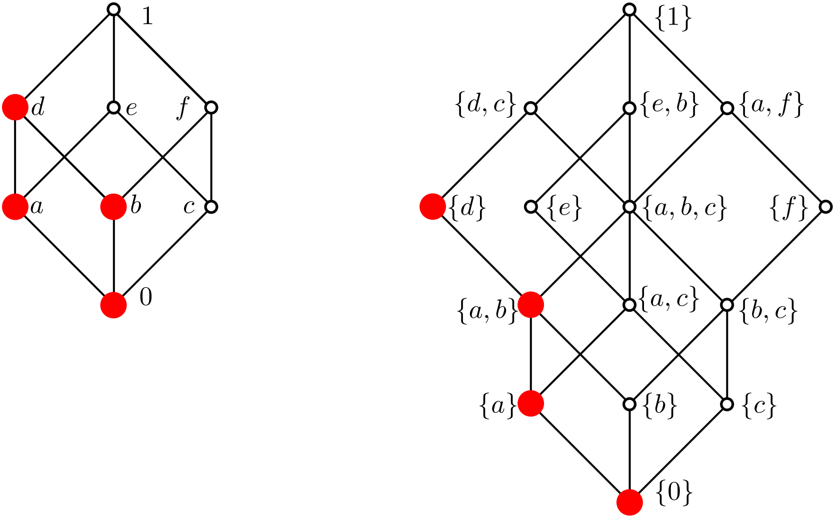

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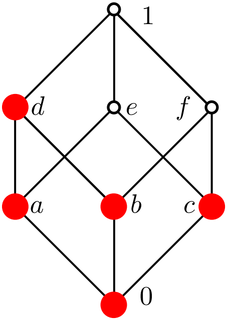


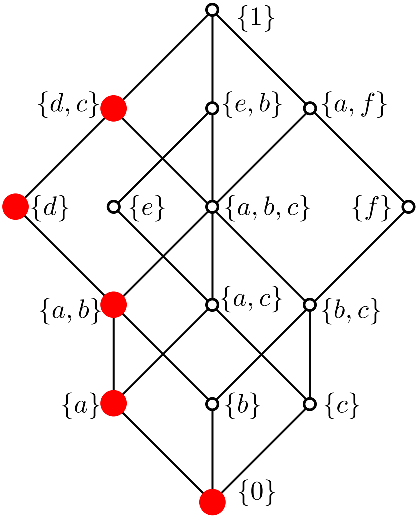

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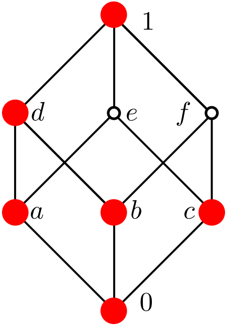


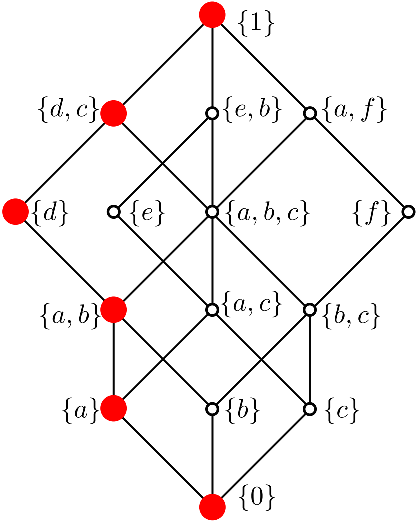

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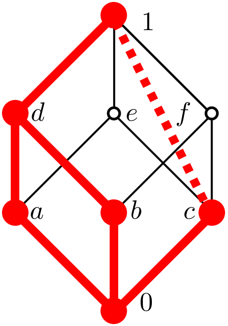


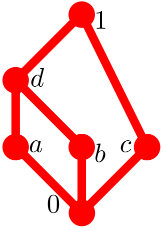


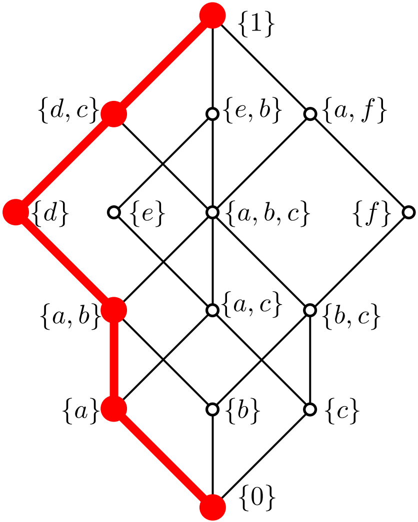

# Proof of Theorem

**Proposition** (Step 1)
If $B$ is a CD-base and $D \subseteq B$ is a set of pairwise disjoint elements in the poset $(P, \leq)$, then $\downarrow D \cap B$ is also a CD-base in the subposet $(\downarrow D, \leq)$.

**Lemma 1** (Step 2)
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**Lemma 2** (Step 2
...coming on a following page...

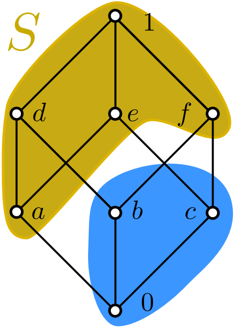
# Lemma 1

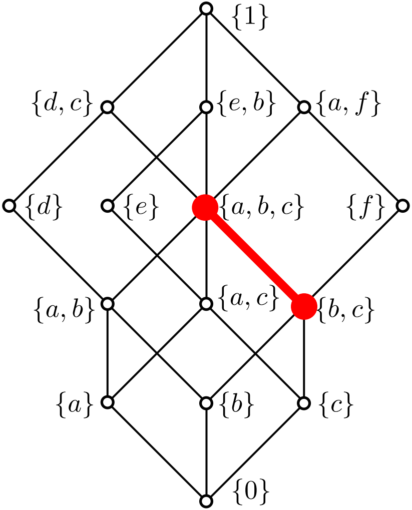
*If* $D_1 \prec D_2$ *in* $\mathcal{D}(P)$*, then* $D_2 = \{a\} \cup \big\{y \in D_1 \setminus \{0\} \mid y \perp a\big\}$ *for some minimal element* $a$ *of the set*

$$S = \big\{s \in P \setminus (D_1 \cup \{0\}) \mid y \perp s \text{ or } y < s \text{ for all } y \in D_1\big\}.$$

*Moreover,* $D_1 \prec \{a\} \cup \big\{y \in D_1 \setminus \{0\} \mid y \perp a\big\}$ *holds for any minimal element* $a$ *of the set* $S$.

# Illustration for Lemma 1



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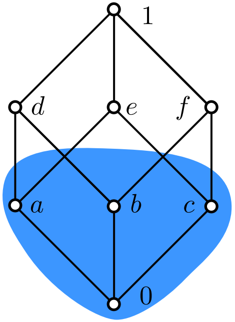


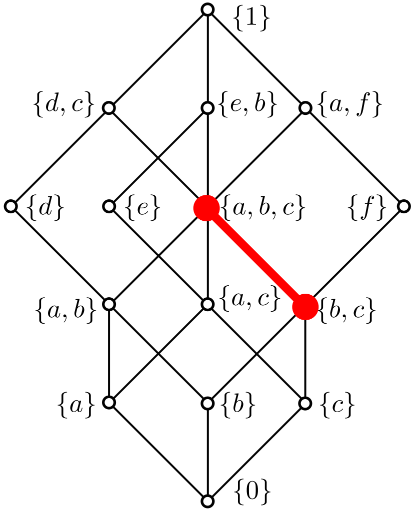

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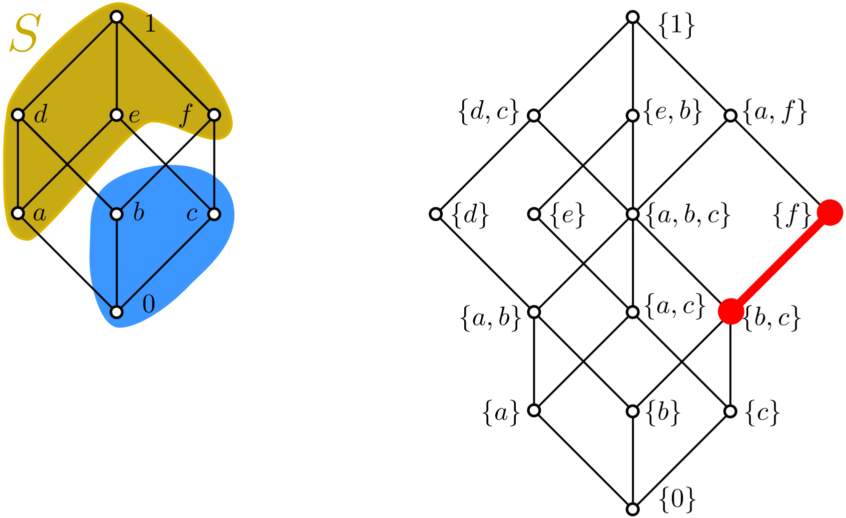

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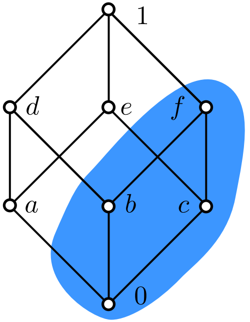


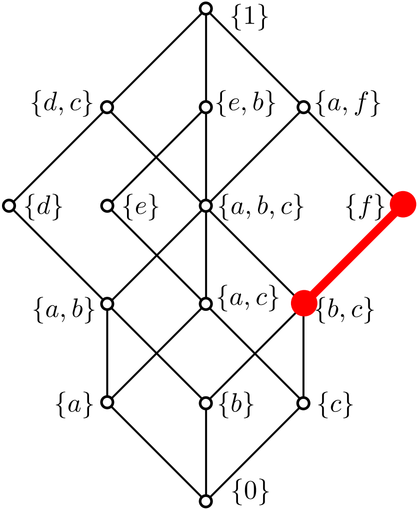

# Lemma 2

*Assume that $B$ is a CD-base with at least two elements in a finite poset $\mathbb{P} = (P, \leq)$, $M = \max(B)$, and $m \in M$. Then $M$ and $N := \max(B \setminus \{m\})$ are disjoint sets.*

*Moreover $M$ is a maximal element in $\mathcal{D}(P)$, and $N \prec M$ holds in $\mathcal{D}(P)$.*

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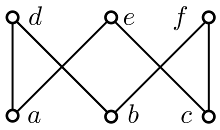


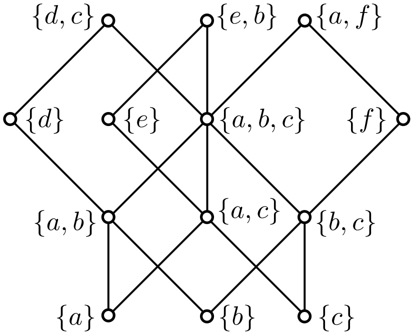

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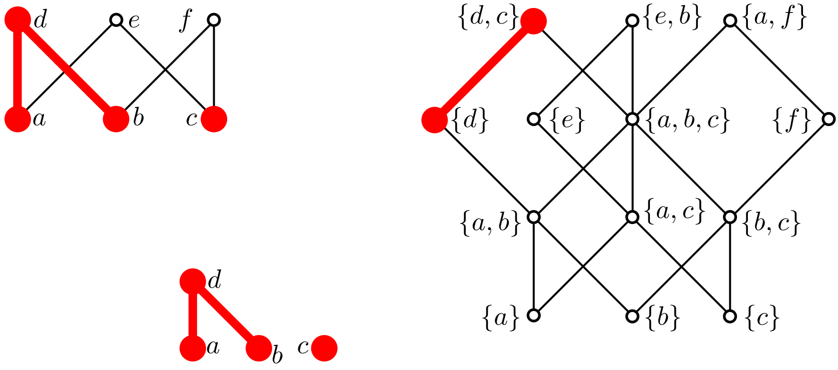

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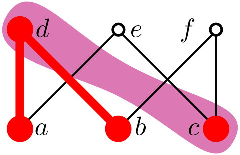


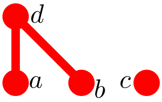


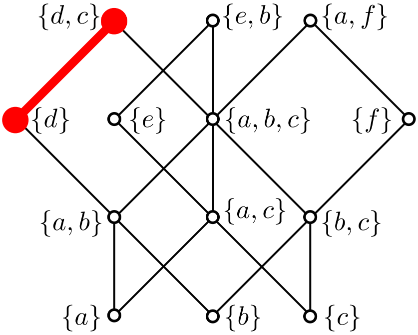

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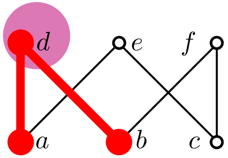


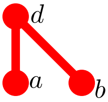


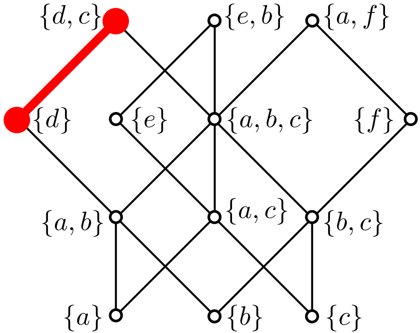

# Corollary

Let $\mathbb{P} = (P, \leq)$ be a finite poset.

The CD-bases of $\mathbb{P}$ have the same number of elements if and only if the poset $\mathcal{D}(P)$ is graded.

Let $B \subseteq P$ be a CD-base of $\mathbb{P}$ , and $(B, \leq)$ the poset under the restricted ordering. Then any maximal chain $\mathcal{C} = \{D_i\}_{1 \leq i \leq m}$ in $\mathcal{D}(B)$ is also a maximal chain in $\mathcal{D}(P)$.

If $D$ is a disjoint set in $\mathbb{P}$ and the CD-bases of $\mathbb{P}$ have the same number of elements, then the CD-bases of the subposet $(I(D), \leq)$ also have the same number of elements.

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# Graded posets

The poset $\mathbb{P}$ is called *graded*, if all its maximal chains have the same cardinality.

*Let* $\mathbb{P} = (P, \leq)$ *be a finite poset with* 0. *Then the following conditions are equivalent:*

(i) *The CD-bases of* $\mathbb{P}$ *have the same number of elements,*

(ii) $\mathcal{D}(P)$ *is graded.*

A set of pairwise disjoint elements $D$ of a poset $(P, \leq)$ is called *complete,* if there is no $p \in P \setminus D$ such that $D \cup \{p\}$ is also a set of pairwise disjoint elements.

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# Weakly 0-modular lattices

Now, we will see that under some weak conditions, the condition that $\mathcal{D}(\mathbb{P})$ graded implies that $\mathbb{P}$ itself is graded.
A poset with least element 0 and greatest element 1 is called *bounded*. A lattice $\mathbb{L} = (L, \leq)$ with 0 is called 0-*modular* if for all $a, b, c \in L$

$$a \leq b \text{ and } b \wedge c = 0 \text{ imply } b \wedge (a \vee c) = a \qquad (\mathrm{M}_0,)$$

where (and everywhere later) we use the usual notation $x \vee y = \sup\{a, b\}$ and $x \wedge y = \inf\{a, b\}$. Equivalently, $L$ has no pentagon sublattice $N_5$ that contains $0 = 0_L$.
We know that that the tolerance lattices of algebras belonging to congruence distributive varieties are 0-*modular* (but not necessarily modular).
If $(\mathrm{M}_0)$ is satisfied under the assumptions that $a$ is an atom and $c \prec b \vee c$, then $L$ is called *weakly* 0-*modular*.

# Weakly 0-modular lattices

$L$ is *lower-semimodular* if for all $a, b, c \in L$, $b \prec c$ implies $a \wedge b \preceq a \wedge c$.

It belongs to the folklore that join-semidistributivity and lower semimodularity characterize the closure lattices of finite convex geometries.

It is easy to see that any lower-semimodular lattice and any 0-modular lattice is weakly 0-modular.

We say that a poset $\mathbb{P}$ with 0 is *weakly* 0-*modular* if the above weak form of $(\mathrm{M}_0)$ holds whenever $\sup\{a, c\}$ and $\sup\{b, c\}$ exist in $\mathbb{P}$.

# If $\mathbb{P}$ is a finite bounded poset

Let $\mathbb{P}$ be a finite bounded poset.

If all the principal ideals $\downarrow a$ of $\mathbb{P}$ are weakly 0-modular, then $A(\mathbb{P}) \cup C$ is a CD-base for every maximal chain $C$ in $\mathbb{P}$.

If each principal ideal of $\mathbb{P}$ is weakly 0-modular and $\mathcal{D}(\mathbb{P})$ is graded, then $\mathbb{P}$ is also graded, and any CD-base of $\mathbb{P}$ contains $|A(\mathbb{P})| + l(\mathbb{P})$ elements.

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If all the principal ideals $\downarrow a$ of $\mathbb{P}$ are weakly 0-modular, then $A(\mathbb{P}) \cup C$ is a CD-base for every maximal chain $C$ in $\mathbb{P}$.

If each principal ideal of $\mathbb{P}$ is weakly 0-modular and $\mathcal{D}(\mathbb{P})$ is graded, then $\mathbb{P}$ is also graded, and any CD-base of $\mathbb{P}$ contains $|A(\mathbb{P})| + l(\mathbb{P})$ elements.

# CD-bases in semilattices and lattices / 1

**Lemma**

We will see that if $\mathbb{P}$ is a $\wedge$-semilattice, $B$ is a CD-base of it, and $\mathbb{B} = (B, \leq)$ is the poset under the restricted ordering, then for appropriately chosen elements of $\mathcal{D}(\mathbb{B})$ and $\mathcal{D}(\mathbb{P})$ a special kind of distributivity holds. The next Lemma shows a property of $\mathcal{D}(\mathbb{P})$ for any poset $\mathbb{P}$ with $0$.

Let $\mathbb{P}$ be a poset with $0$. Assume that $K \neq \varnothing$ is an index set and, for each $k \in K$, $D_k$ is a set of pairwise disjoint elements in $\mathbb{P}$. If for every choice function $f \in \prod_{k \in K} D_k$ the meet $\bigwedge_{k \in K} f(k)$ exists in $\mathbb{P}$, then $\bigwedge\limits_{k \in K} D_k$ exists in $\mathcal{D}(\mathbb{P})$. In particular, for $K = \{1, 2\}$ and $D_1 = \{a_i \mid i \in I\}$, $D_2 = \{b_j \mid j \in J\} \in \mathcal{D}(\mathbb{P})$ such that all the $a_i \wedge b_j$ exists, we have

$$D_1 \wedge D_2 = \begin{cases} M := \{a_i \wedge b_j \mid i \in I, j \in J, a_i \wedge b_j \neq 0\} & \textit{if } M \neq \emptyset; \\ \{0\} & \textit{otherwise.} \end{cases}$$

# CD-bases in semilattices and lattices / 2

Let $\mathbb{P} = (P, \wedge)$ be a semilattice with 0. Now, for all $a, b \in P$, the relation $a \perp b$ means that $a \wedge b = 0$. Hence, a set $\{a_i \mid i \in I\}$ of nonzero elements is a set of pairwise disjoint elements if and only if $a_i \wedge a_j = 0$ for all $i, j \in I$, $i \neq j$. A pair $a, b \in P$ with least upperbound $a \vee b$ in $\mathbb{P}$ is called a *distributive pair* if $(c \wedge a) \vee (c \wedge b)$ exists in $\mathbb{P}$ for all $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$. We say that $(P, \wedge)$ is *dp-distributive* (*distributive with respect to disjoint pairs*) if any pair $a, b \in P$ with $a \wedge b = 0$ is a distributive pair.

**Theorem**

(i) If $\mathbb{P} = (P, \wedge)$ is a semilattice with 0, then $\mathcal{D}(\mathbb{P})$ is a dp-distributive semilattice. Furthermore, for all $D_1, D_2 \in \mathcal{D}(\mathbb{P})$, if $D_1 \cup D_2$ is a CD-independent set, then $D_1, D_2$ is a distributive pair in $\mathcal{D}(\mathbb{P})$.

(ii) If $\mathbb{P}$ is a complete lattice, then $\mathcal{D}(\mathbb{P})$ is a dp-distributive complete lattice.

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(ii) If $\mathbb{P}$ is a complete lattice, then $\mathcal{D}(\mathbb{P})$ is a dp-distributive complete lattice.

Let $(P, \leq)$ be a poset and $A \subseteq P$. $(A, \leq)$ is called a *sublattice* of $(P, \leq)$, if $(A, \leq)$ is a lattice such that for any $a, b \in A$ the infimum and the supremum of $\{a, b\}$ is the same in the subposet $(A, \leq)$ and in $(P, \leq)$. If the relation $x \prec y$ in $(A, \leq)$ for some $x, y \in A$ implies $x \prec y$ in the poset $(P, \leq)$, then we say that $(A, \leq)$ is a *cover-preserving subposet* of $(P, \leq)$.

**Theorem**

*Let* $\mathbb{P} = (P, \leq)$ *be a poset with* $0$ *and* $B$ *a CD-base of it. Then* $(\mathcal{D}(B), \leqslant)$ *is a distributive cover-preserving sublattice of the poset* $(\mathcal{D}(P), \leqslant)$*. If* $\mathbb{P}$ *is a* $\wedge$*-semilattice, then for any* $D \in \mathcal{D}(P)$ *and* $D_1, D_2 \in \mathcal{D}(B)$ *we have* $(D_1 \vee D_2) \wedge D = (D_1 \wedge D) \vee (D_2 \wedge D)$ *in* $(\mathcal{D}(P), \leqslant)$.

# CD-bases in particular lattice classes

We investigate CD-bases in two particular classes of lattices. The properties of the first class generalize the properties of tolerance lattices of majority algebras. It was proved that the tolerance lattice of arbitrary majority algebra is a pseudocomplemented, 0-modular and dp-distributive lattice. These properties are not independent, since, for instance, it can be shown that dp-distributivity implies 0-modularity.

Let $\mathbb{L} = (L, \leq)$ a lattice. A lattice $\mathbb{L}$ with 0 is called *pseudocomplemented* if for each $x \in L$ there exists an element $x^* \in L$ such that, for all $y \in L$, $y \wedge x = 0 \Leftrightarrow y \leq x^*$. It is known that an algebraic lattice $L$ is pseudocomplemented if and only if it is 0-*distributive*, that is, for all $a, b, x \in L$, $x \wedge a = 0$ and $x \wedge b = 0$ imply $x \wedge (a \vee b) = 0$. We say that $\mathbb{L}$ is *weakly* 0-*distributive* if this implication holds under the condition $a \wedge b = 0$. Clearly, any 0-distributive lattice is weakly 0-distributive. If $D$ is a set of pairwise disjoint elements in a weakly 0-distributive lattice and $|D| \geq 2$, then it is easy to see that replacing two different elements $d_1, d_2 \in D$ by their join $d_1 \vee d_2$, we obtain again a disjoint set.

**Lemma**

Let $\mathbb{L}$ be a finite weakly 0-distributive lattice and $D$ a dual atom of the poset $\mathcal{D}(\mathbb{L})$. Then either $D = \{d\}$ for some $d \in L$ with $d \prec 1$, or $D$ consist of two different elements $d_1, d_2 \in L$ with $d_1 \vee d_2 = 1$.

Let $\mathbb{L}$ be a graded lattice, and $a \in L$. Then the *height* of $a$ is the length of the interval $[0, a]$, denoted by $l(a)$. (In the literature, it is also denoted by $h(a)$.) A graded lattice always has 0 and 1.

A graded lattice $\mathbb{L}$ is 0-modular, whenever $l(a) + l(b) = l(a \vee b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.

**Theorem**

*Let* $\mathsf{L}$ *be a finite, weakly* $\mathsf{0}$*-distributive lattice. Then the following are equivalent:*

- (i) $L$ *is graded, and* $l(a) + l(b) = l(a \vee b)$ *holds for all* $a, b \in L$ *with* $a \wedge b = 0$.
- (ii) $L$ *is* $0$*-modular, and the CD-bases of* $L$ *have the same number of elements.*

**Theorem**

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- (ii) *$L$ is $0$-modular, and the CD-bases of $L$ have the same number of elements.*

# Modular pair

We say that two elements $a, b \in L$ form a *modular pair* in the lattice $L$ and we write $(a, b)M$ if for all $x \in L$, $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$.

Also, $a$ and $b$ form a *dual-modular pair* if for all $x \in L$, $x \geq b$ implies $x \wedge (a \vee b) = (x \wedge a) \vee b$. This is denoted by $(a, b)M^*$.

Clearly, if $a$ and $b$ form a distributive pair, then $(a, b)M^*$ is satisfied.

By means of modular pairs, the 0-modularity condition can be reformulated as follows): For all $a, b \in L$,

**Lemma (M. Stern)** In a graded lattice of finite length, $(a, b)M$ implies $l(a) + l(b) \leq l(a \wedge b) + l(a \vee b)$.

# CD-bases

With the help of Lemma of M. Stern above, using an $N_5$ sublattice containing 0 as well as the dual lattice, we obtain

**Proposition** If $\mathbb{L}$ is a lattice with 0 such that $(a,b)M^*$ holds for all $a, b \in L$ with $a \wedge b = 0$, then $L$ is 0-modular. If in addition $\mathbb{L}$ is a graded lattice of finite length, then $l(a \vee b) = l(a) + l(b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.

**Corollary** (i) Let $\mathbb{L}$ be a finite, weakly 0-distributive lattice such that for each $a, b \in L$ with $a \wedge b = 0$, condition $(a,b)M^*$ holds. Then the CD-bases of $\mathbb{L}$ have the same number of elements if and only if $\mathbb{L}$ is graded.
(ii) If $\mathbb{L}$ is a finite pseudocomplemented modular lattice, then the CD-bases of $\mathbb{L}$ have the same number of elements.

# dp-distributive lattices

As any dp-distributive lattice $\mathbb{L}$ is weakly 0-distributive, and $(a, b)M^*$ holds for all $a, b \in L$ with $a \wedge b = 0$ since $(a, b)$ is a distributive pair, we obtain

**Corollary**

(i) Any dp-distributive lattice is 0-modular. If $\mathbb{L}$ is a dp-distributive graded lattice with finite length, then $l(a \vee b) = l(a) + l(b)$ holds for all $a, b \in L$ with $a \wedge b = 0$.

(ii) The CD-bases in a finite dp-distributive lattice $\mathbb{L}$ have the same number of elements if and only if $\mathbb{L}$ is graded.

# Interval system

An *interval system* $(V, \mathcal{I})$ is an algebraic closure system satisfying the axioms:

($I_0$) $\{x\} \in \mathcal{I}$ for all $x \in V$, and $\emptyset \in \mathcal{I}$;

($I_1$) $A, B \in \mathcal{I}$ and $A \cap B \neq \emptyset$ imply $A \cup B \in \mathcal{I}$;

($I_2$) For any $A, B \in \mathcal{I}$ the relations $A \cap B \neq \emptyset$, $A \nsubseteq B$ and $B \nsubseteq A$ imply $A \setminus B \in \mathcal{I}$ (and $B \setminus A \in \mathcal{I}$).

The *modules* ($X$*-sets, or autonomous sets*) of an undirected graph $G = (V, E)$, the *intervals* of an $n$-ary relation $R \subseteq V^n$ on the set $V$ for $n \geq 2$ – in particular, the usual intervals of a linearly ordered set $(V, \leq)$ – form interval systems.

# Generalizing interval systems

Let us consider now the condition:

$$(\mathcal{II}) \text{ If } a \wedge b \neq 0, \text{ then } (x \leq a \vee b \text{ and } x \wedge a = 0) \Rightarrow x \leq b \text{ for all } a, b, x \in L.$$

Lattices with 0 satisfying condition $(\mathcal{II})$ and with the property that $\uparrow a$ is a modular lattice for all $a \in L$, $a \neq 0$, can be considered as a generalization of the lattice $(\mathcal{I}, \subseteq)$ of an interval system $(V, \mathcal{I})$. To study their CD-bases, first we proved:

**Lemma** Let $\mathbb{L}$ be an atomic lattice satisfying condition $(\mathcal{II})$. Assume $D \in \mathcal{D}(\mathbb{L})$ and define $S_D = \big\{ s \in L \setminus (D \cup \{0\}) \mid d \wedge s = 0$ or $d < s$, for all $d \in D \big\}$. Then for all $b, c \in S_D$ with $b \wedge c \neq 0$ and all $d \in D$, $d \wedge (b \vee c) \neq 0$ if and only if $0 < d < b$ or $0 < d < c$ holds.

**Remark** Let $\mathbb{L}$ be a finite lattice and $D = \{d_j \mid j \in J\} \in \mathcal{DC}(\mathbb{L})$. If $D \prec D'$ for some $D' \in D(\mathbb{L})$; then, in view of Lemma **??**, there is a minimal element $a \in S_D$ such that

$D' = \{a\} \cup \{d_j \in D \setminus \{0\} \mid d_j \wedge a = 0\}$. In this case there exists a set $K \subset J$ such that

# Birkhoff's condition

It is well-known that a finite lattice $\mathbb{L}$ is semimodular if and only if it satisfies *Birkhoff's condition*, namely, for all $a, b \in L$

(Bi) $\qquad a \wedge b \prec a, b$ implies $a, b \prec a \vee b$.

We also say that a pair $a, b \in L$ satisfies Birkhoff's condition if the above implication (Bi) is valid for $a, b$. It is known that any distributive pair $a, b \in L$ satisfies Birkhoff's condition.

**Theorem** Let $\mathbb{L}$ be a finite lattice satisfying condition $(\mathcal{II})$ such that any principal filter $\uparrow a$ with $a \in L \setminus \{0\}$ is a modular lattice. Then $\mathcal{DC}(\mathbb{L})$ is a semimodular lattice.

# Thank you for your attention!

Thank you for your attention!

**Corollary** (i) If $\mathbb{L}$ is a finite distributive lattice, then $\mathcal{DC}(\mathbb{L})$ is a semimodular lattice.
(ii) If $\mathbb{L}$ is a finite lattice that satisfies the conditions in Theorem **??**, then its CD-bases have the same number of elements.

By applying this to interval systems we obtain:

**Corollary**

*If $(V, \mathcal{I})$ is a finite interval system, then the* CD-*bases of the lattice* $(\mathcal{I}, \subseteq)$ *contain the same number of elements.*

# Island domain

$U \in \mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{P}(U)$

Let $h\colon U \to \mathbb{R}$ be a height function and let $S \in \mathcal{C}$ be a nonempty set.

We say that $S$ is an *pre-island* with respect to the triple $(\mathcal{C}, \mathcal{K}, h)$, if every $K \in \mathcal{K}$ with $S \prec K$ satisfies

$$\min h(K) < \min h(S).$$

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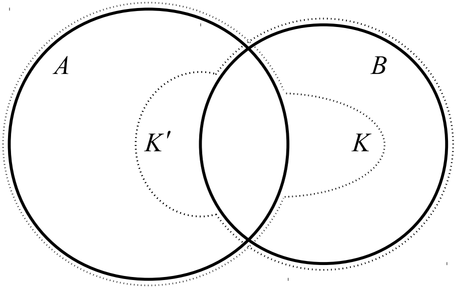
# Connective island domains

## Definition

A pair $(\mathcal{C}, \mathcal{K})$ is an *connective island domain* if

$$\forall A, B \in \mathcal{C}: \quad (A \cap B \neq \emptyset \text{ and } B \not\subseteq A) \implies \exists K \in \mathcal{K} : A \subset K \subseteq A \cup B.$$



# CDW-independence

**Definition** A family $\mathcal{H} \subseteq \mathcal{P}(U)$ is *weakly independent* if

$$H \subseteq \bigcup_{i \in I} H_i \implies \exists i \in I : H \subseteq H_i \qquad (1)$$

holds for all $H \in \mathcal{H}, H_i \in \mathcal{H}\,(i \in I)$. If $\mathcal{H}$ is both CD-independent and weakly independent, then we say that $\mathcal{H}$ is *CDW-independent*.

**Lemma**
If $(\mathcal{C}, \mathcal{K})$ is a connective island domain, then every admissible subfamily of $\mathcal{C}$ is CDW-independent. [[But not conversely.]]

# Connective island domains

**Theorem**

The following three conditions are equivalent for any pair $(\mathcal{C}, \mathcal{K})$:

(i) $(\mathcal{C}, \mathcal{K})$ is a connective island domain.

(ii) Every system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CD-independent.

(iii) Every system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CDW-independent.

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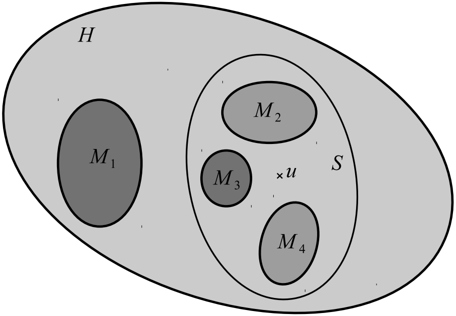
**Theorem**

If $(\mathcal{C}, \mathcal{K})$ is a connective island domain, then a subfamily of $\mathcal{C}$ is a system of pre-islands if and only if it is admissible.



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**Theorem**

If $(\mathcal{C}, \mathcal{K})$ is a connective island domain and $\mathcal{S}$ is a system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$, then $|S| \leq |U|$.

**Proof.**

Let $(\mathcal{C}, \mathcal{K})$ be a connective island domain and let $\mathcal{S} \subseteq \mathcal{C} \setminus \{\emptyset\}$ be a system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$. $\mathcal{S}$ is CDW-independent, and hence $\mathcal{S} \cup \{\emptyset\}$ is also CDW-independent. From the results of G. Czédli and E. T. Schmidt it follows that every maximal CDW-independent subset of $\mathcal{P}(U)$ has $|U| + 1$ elements. Thus we have $|\mathcal{S}| + 1 \leq |U| + 1$.

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