The rooted circuits and the rooted cocircuits of convex geometries, closure operators, and monotone extensive operators

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Outline

1. Closure spaces, closure systems, matroids, and convex geometries
2. Supersolvable antimatroids
3. Lattice-embedding of convex geometries to convex geometries
4. Rooted circuits and implicational systems
5. Extensive and intensive operators
6. Relations of areas to convex geometries

1. Closure spaces, closure systems, matroids, and convex geometries

Let E be a non-empty finite set.

Def. Closure operator

$$\mu : 2^E \to 2^E$$

$A \subseteq \mu(A)$ (extensive)

$A \subseteq B \Rightarrow \mu(A) \subseteq \mu(B)$ (monotone)

$\mu(\mu(A)) = \mu(A)$ (idempotent)

Def. (μ, E) is called a closure space.

Def. Closure system

$$K \subseteq 2^E$$

- $E \in K,$
- $X, Y \in K \Rightarrow X \cap Y \in K$

A member of a closure system is called a closed set.

Def. Closure operator

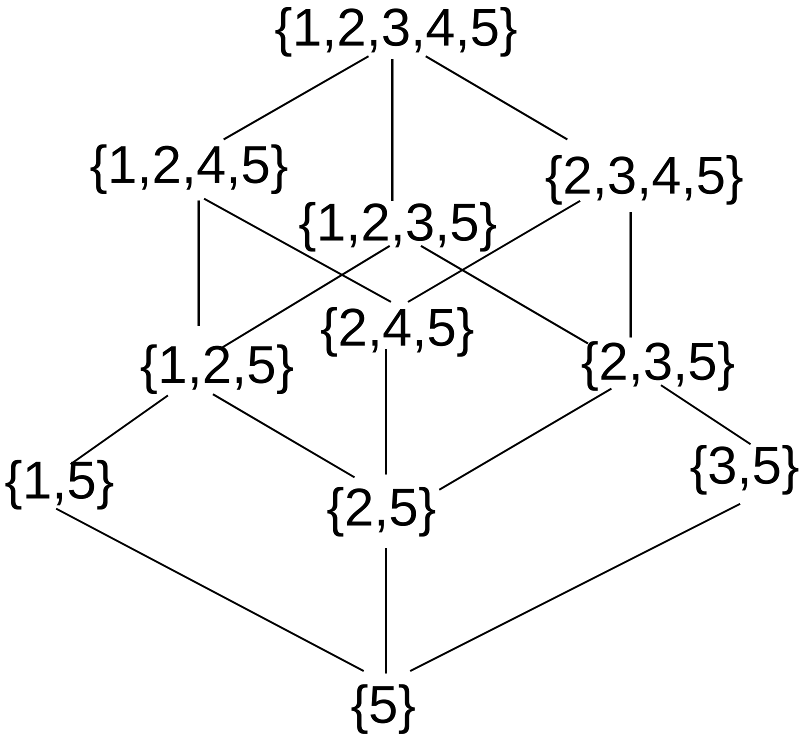
$$\mu : 2^E \rightarrow 2^E$$

- $A \subseteq \mu(A)$
- $A \subseteq B \Rightarrow \mu(A) \subseteq \mu(B)$
- $\mu(\mu(A)) = \mu(A)$

$$\mu(A) = \bigcap_{X \in K, A \subseteq X} X \longleftrightarrow K = \{ A \subseteq E : \mu(A) = A \}$$

one-to-one correspondence

Ex. A closure system is a lattice under inclusion relation.



Closure spaces (closure systems)

Def. For each element $e \in E$, suppose $X \subseteq E - e$

and X is minimal with respect to the property $e \in \mu(X)$.

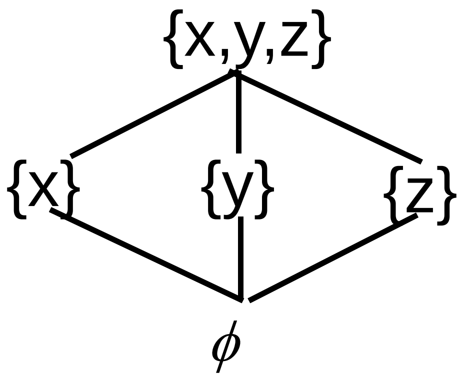
Then we say (X, e) is a rooted circuit of a closure operator μ,

(or more generally a monotone extensive operator μ).

X is called the stem, and e is called the root.

We denote the collection of the rooted circuits of μ by $C(\mu)$.

Ex.



Rooted circuits

({y,z}, x)

({x,z}, y)

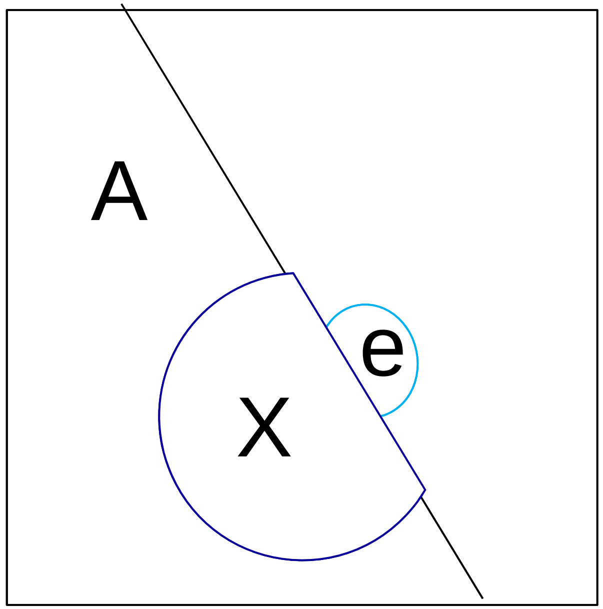
({x,y}, z)

Closure spaces (closure systems)

Conversely, the rooted-circuit system $C(\mu)$ determines the closure operator μ.

$$\mu(A) = A \bigcup \{e \in E - A : \exists (X, e) \in C(\mu), X \subseteq A\} \quad (A \subseteq E).$$

If $\mu(A)$ contains an element $e \in E - A$ here is a minimal $X \subseteq A$ such that $e \in \mu(X)$ wirh $(X, e) \in C$.

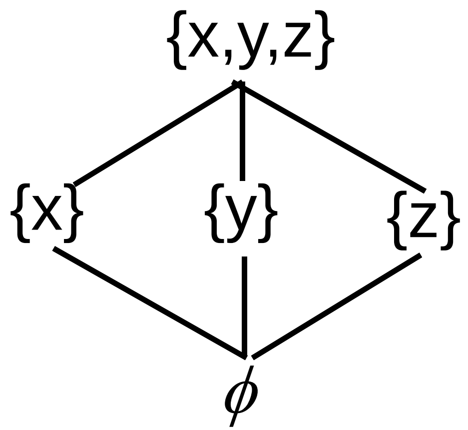


Closure spaces (closure systems)

Def. Rooted cocircuit

For $e \in E$ and $Y \subseteq E - e,$ if Y is minimal with respect to the property $e \notin \mu(E-(Y \bigcup e)),$ (Y, e) is said to be a rooted cocircuit. We denote the collection of the rooted cocircuits by $D(\mu)$.

Ex.



$a \notin \mu(E-\{y, x\}) = \mu(\{z\}) = \{z\}$

Rooted Cocircuits

({y}, x) ({z}, x)

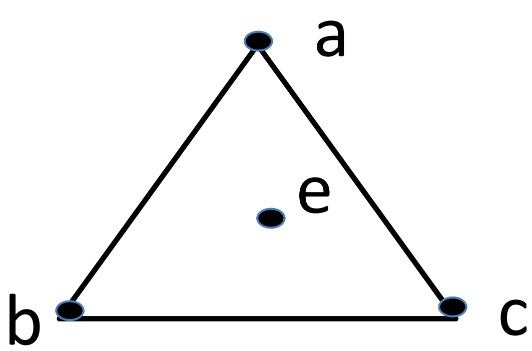
({z}, y) ({x}, y)

({x}, z) ({y}, z)

Ex. Affine-point configuration

$$E = \{a, b, c, e\} \subseteq R^2$$

$$\mu(A) = \text{conv.hull}(A) \cap E$$



Rooted Circuits

({a,b,c},e)

Rooted Cocircuits

({a},e), ({b},e), ({c},e).

Closure spaces (closure systems)

Ex. Affine-point configuration

$$E = \{a, b, c, d, e\} \subseteq R^2$$

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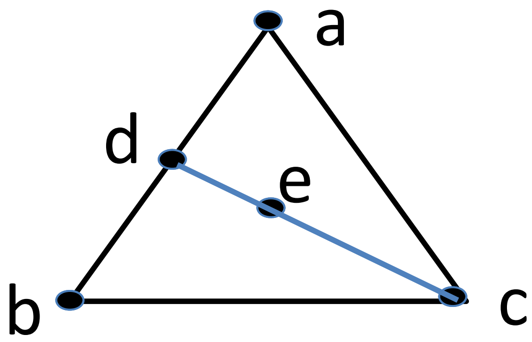


Rooted Circuits

({a,b,c},e),
({c,d},e),
({a,b},d).

Rooted Cocircuits

({c},e), ({a,d},e),
({b,d},e).
({a},d), ({b},d)

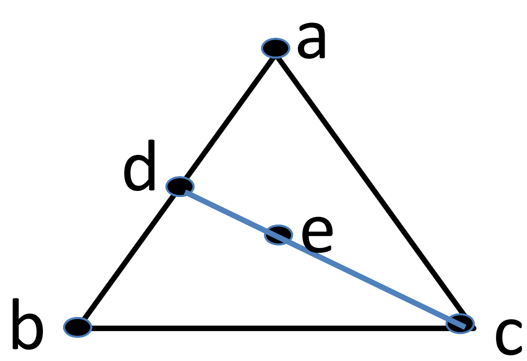
Closure spaces (closure systems)

Def. $\chi(A) = \{e \in A : e \notin \mu(A - e)\} \quad (A \subseteq E)$

Is the extreme-point operator of a closure operator μ .

Ex. Affine-point configuration

$$E = \{a, b, c, d, e\} \subseteq R^2.$$



$$\chi(\{a, b, c, d, e\}) = \{a, b, c\}$$

$$\chi(\{a, c, d, e\}) = \{a, c, d\}$$

$$\chi(\{a, b, e\}) = \{a, b, e\}$$

Matroid

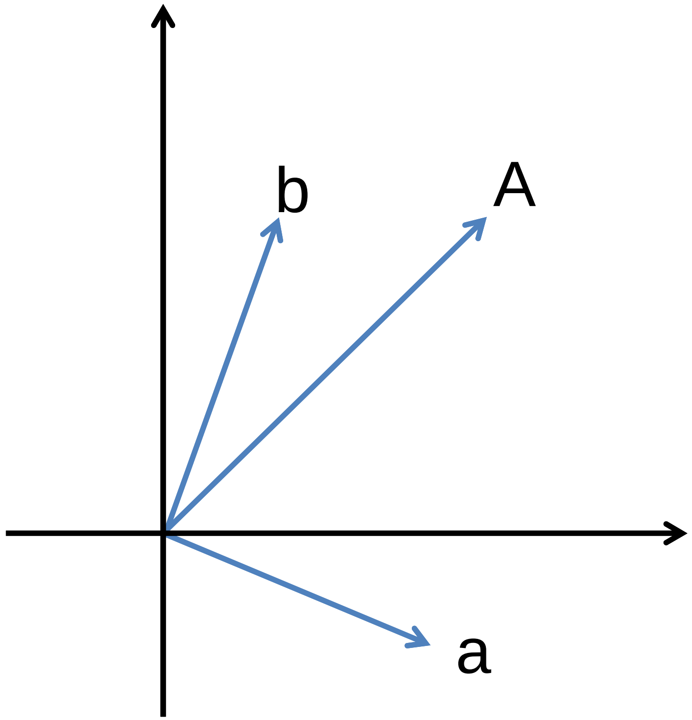
Def. A closure space (μ, E) is a **matroid** if

$$a \in \mu(A \bigcup b) \Rightarrow b \in \mu(A \bigcup a)$$

(Exchange Property)

↓

Abstraction of linear dependency

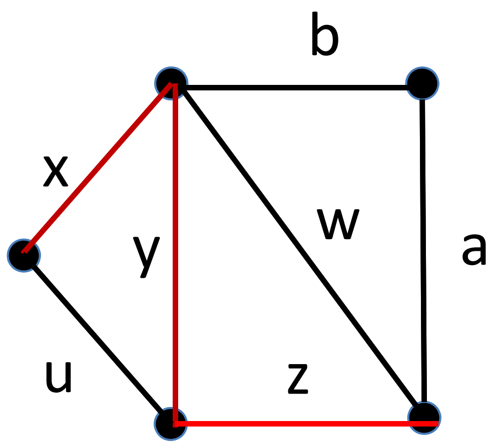


Matroid

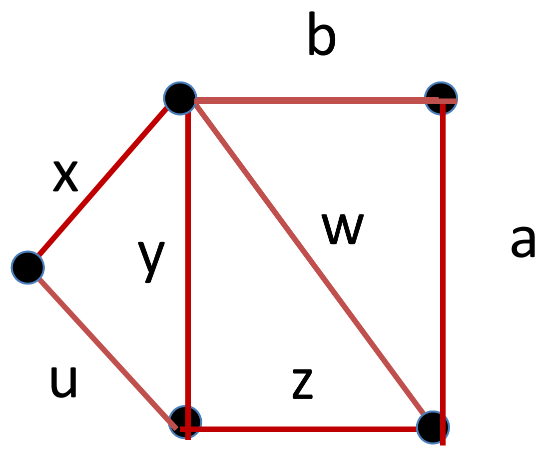
$$a \in \mu(A \cup b) \Rightarrow b \in \mu(A \cup a)$$

(Exchange Property)

Ex. A graphic matroid



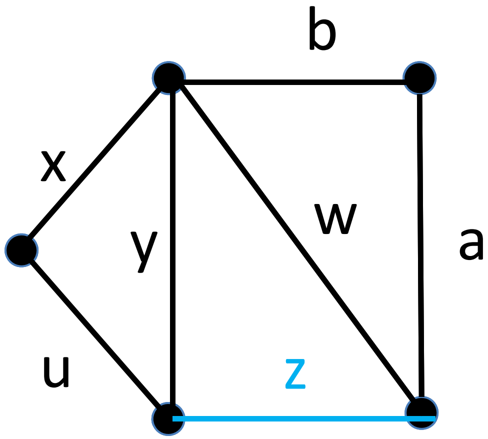
$A = \{x, y, z\}$



$\mu(A) = \{x, y, z, u, w\}$

Ex. A graphic matroid M(G) of a graph G

E=E(G)



Rooted circuits with the root z
({a, b, x, u}, z),
({x, w, u}, z),
({w, y}, z),
({a, b, y}, z).

Proposition

In a matroid M, for any circuit C and any element $e \in C$,

$(C - e, e)$ is a rooted circuit, and vice versa.

<u>Convex geometry</u>

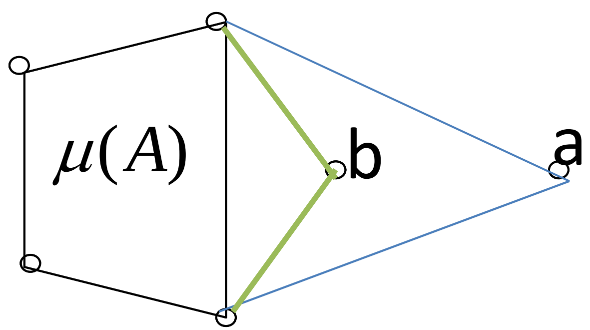
Def. A closure space (μ, E) is a convex geometry if the corresponding closure operator μ satisfies the <u>anti-exchange property</u>.

$$a \neq b,\ a, b \notin \mu(A),\ b \in \mu(A \cup a) \Rightarrow a \notin \mu(A \cup b)$$

(Anti-exchange Property)

Ex. Affine-point configuration $E \subseteq R^2$.

$$\mu(A) = \text{conv.hull}\,(A) \cap E$$



Abstraction of convexity
(Jamison 1982)

$$a \in \mu(A \cup b) \Rightarrow b \in \mu(A \cup a)$$

(Excange Property)

Convex geometry

Theorem (Edelman and Jamison 1985)

Let (μ, E) be a closure space , K be the closure system associated with it, and χ be the extreme-point operator.

Then the following are equivalent.

(1) μ is a closure operator with the anti-exchange property.

(2) For any closed set $X \in K$ with $X \neq E$, there exists $e \in E - X$ such that $X \cup e \in K$.

(3) $\chi(\mu(A)) = \chi(A) \quad for \quad A \subseteq E.$

(4) $\mu(\chi(A)) = \mu(A) \quad for \quad A \subseteq E.$ [Krein-Milman property]

Convex Geometries and Antimatroids

Def. Convex geometry

$K \subseteq 2^E$

- $E \in K,$
- $X, Y \in K \Rightarrow X \bigcap Y \in K$
- $X \in K, X \neq E \Rightarrow$

  $\exists e \in E - X : X \bigcup e \in K$

Def. Antimatroid

$F \subseteq 2^E$

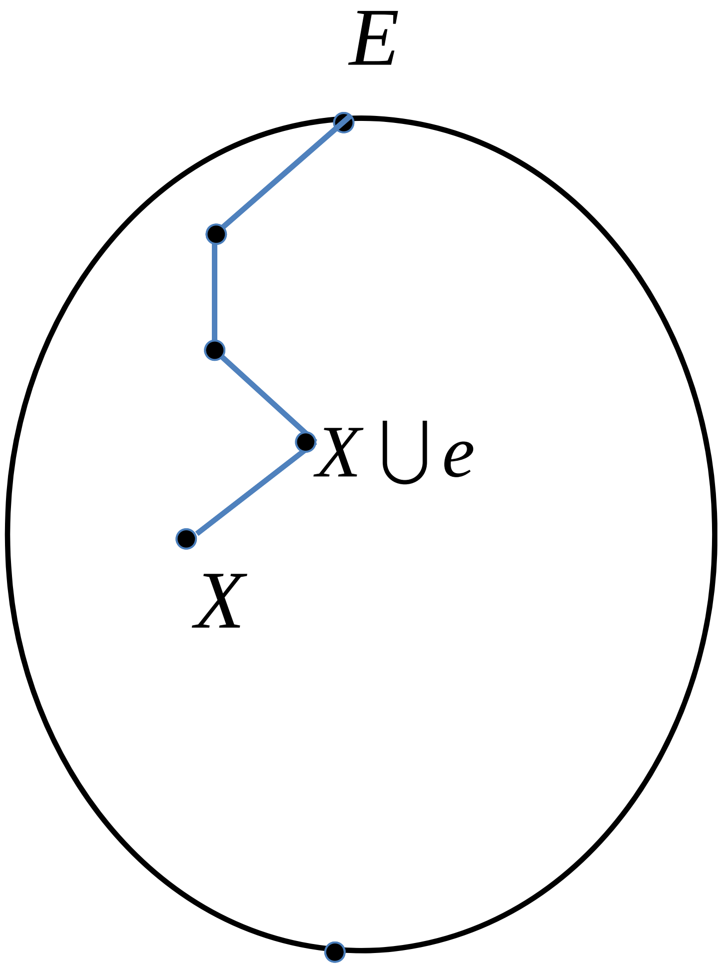
- $\phi \in F,$
- $A, B \in F \Rightarrow A \bigcup B \in F$
- $A \in F, A \neq \phi \Rightarrow$

  $\exists e \in A : A - e \in F$

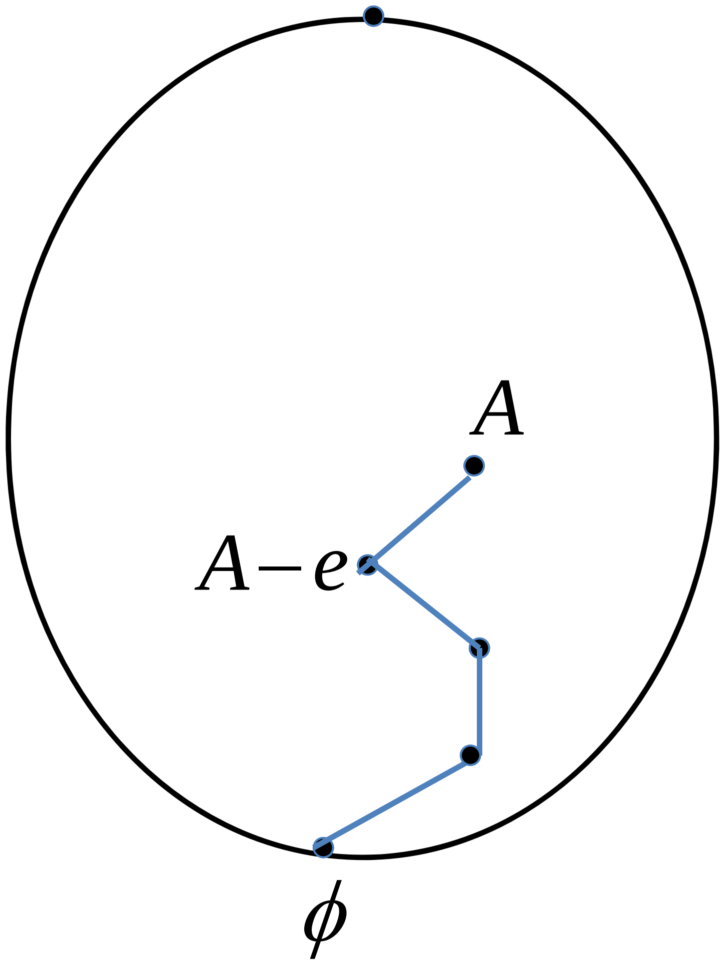
Def. Antimatroid

$$F = \{E - X : X \in K\}$$

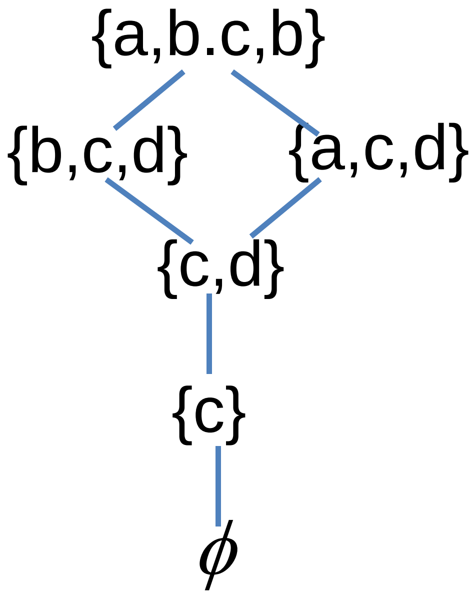
An element of K is called a convex set, and an element of F is a feasible set.



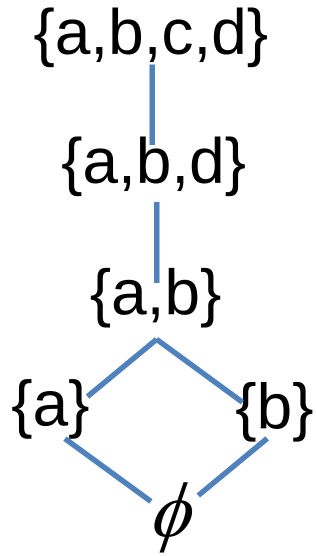
convex geometry



antimatroid



A convex geometry



An antimatroid

classes of convex geometries

- Affine convex geometry
- poset convex geometry
- double shelling of a poset
- simplicial shelling of a chordal graph
- tree node-shelling convex geometry
- graph search convex geometry

generalized affine convex geometries = all the convex geometries

Principle of shelling process of antimatroids

Once an element is removable, it remains removable until it is deleted.

$\chi(A)$ is the set of removable elements in A.

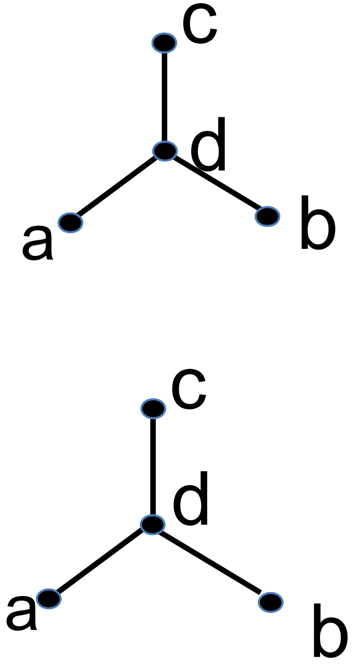
$A := E;$

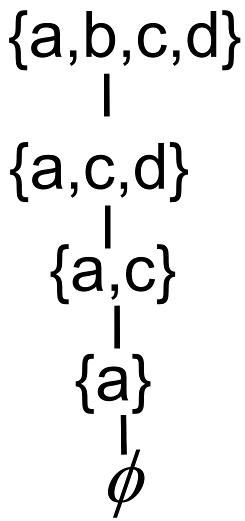
while($\chi(A)$ is non-empty) do

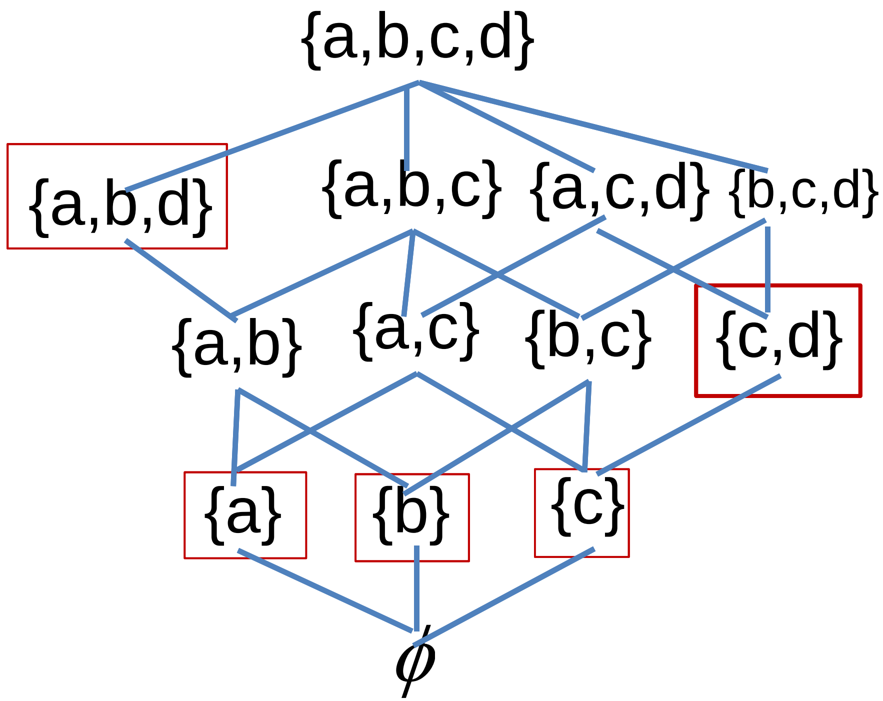
delete any element in $\chi(A)$ from A

end

Ex. Poset double shelling antimatroid – Repeating the deletion of a minimal or maximal element.

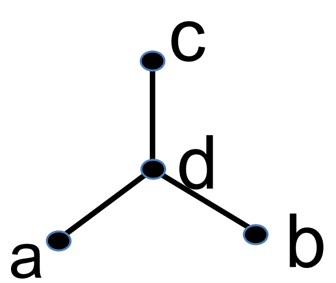




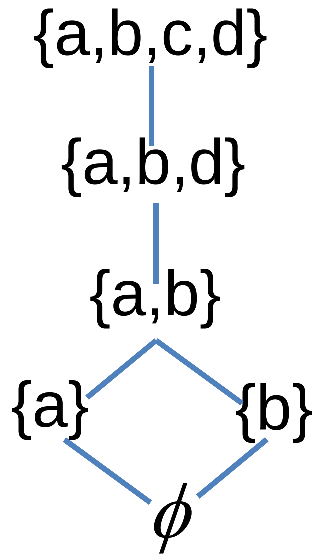


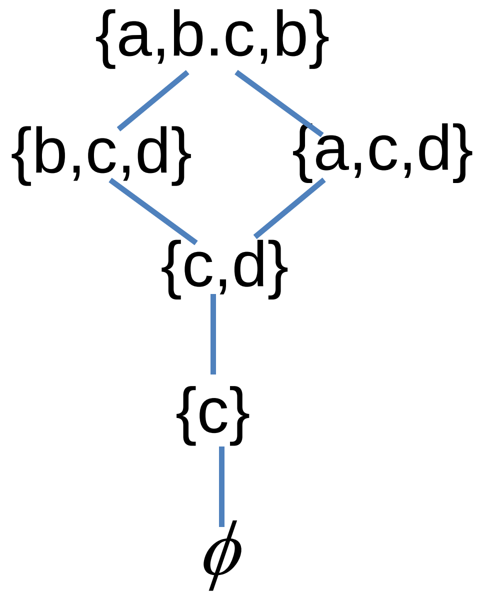
A poset double shelling antimatroid

Ex. poset convex geometries



Hasse diagram of a poset P





Deleting a minimal element → antimatroid

antimatroid of P

convex geometry of P

Def. Monophonically convex sets of a chordal graph

G : a graph, $V(G)$: the vertex set of G

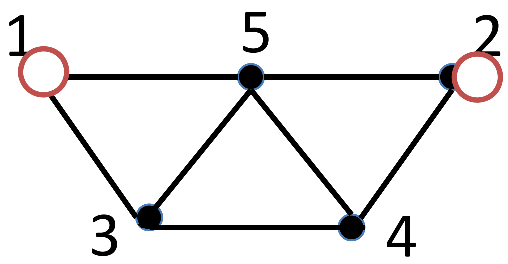
$X \subseteq V(G)$ is monophonically convex if it holds that

$$a, b \in X \text{ and } c \in V \text{ is on a chordless path between } a \text{ and } b \Rightarrow c \in X$$

<u>Theorem</u> (Edelman and Jamison 1985)

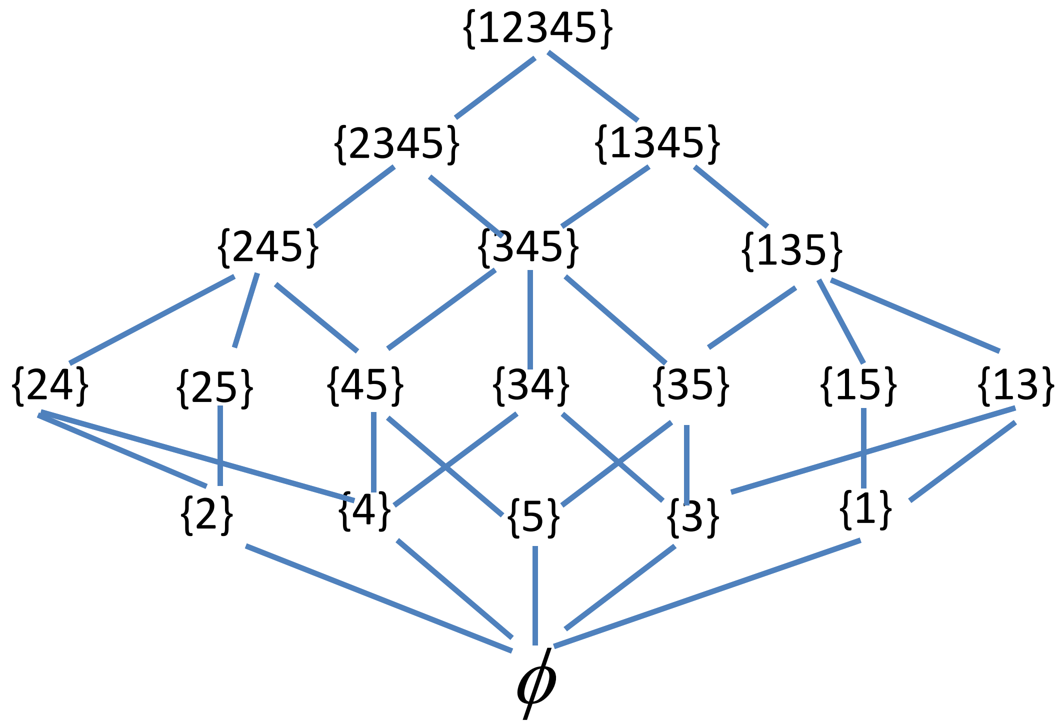
The collection of monophonically convex sets is a convex geometry if and only if G is a chordal graph.

Ex. Monophonically convex sets of a chordal graph



A chordal graph G

Removable elements
=Simplicial vertex



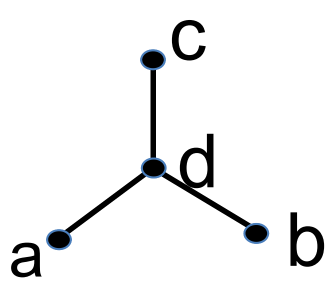
A convex geometry of the chordal graph G

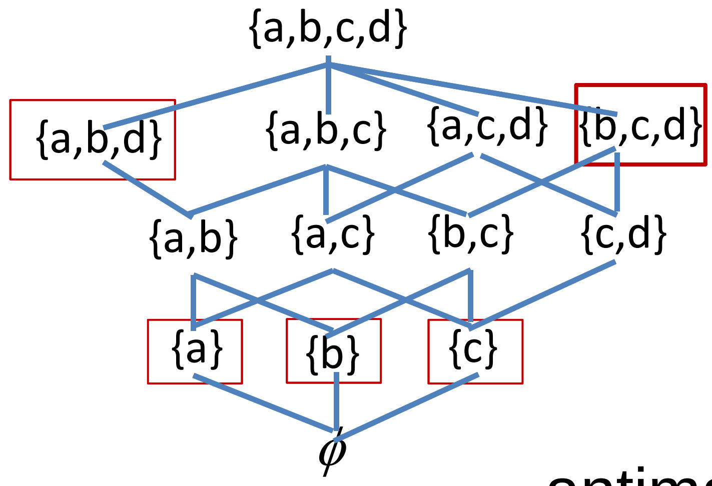
Antimatroids and rooted cocircuits

Def. Let $e \in E$. In an antimatroid F, suppose that an element $H \in F$ contains e and is minimal with respect to this property. Then we say that (Y, e) is a rooted cocircuit where $Y = H - e$. e is called the root and Y is the costem.

[Remark] H is necessarily a join-irreducible element in the lattice F.

Ex. Poset double shelling antimatroid – Repeating the deletion of a minimal or maximal element.





antimatroid

Element:	minimal feasible set
a	{a}
b	{b}
c	{c}
d	{a, b, d}, {c, d},

Rooted cocircuits
(ϕ, a),
(ϕ, b),
(ϕ, c)
({a, b}, d), ({c}, d)

clutter

Def. A clutter is a collection of sets in which no member contains another properly.

Def. For a clutter L on E, a subset of E is a transversal of L if it intersects every member of L. The collection of the minimal transversals called the blocker of L and denoted by $b(L)$.

[Note] If a set intersects every member of a clutter L, then it contains a member of the blocker $b(L)$.

clutter

Ex. clutters and their blockers

$$L = \{\{a, c\}, \{b, c\}\}, \quad b(L) = \{\{c\}, \{a, b\}\}, \quad b(b(L)) = L$$

$$L = \{\{a, b\}, \{b, c\}, \{c, a\}\}, \quad b(L) = \{\{a, b\}, \{b, c\}, \{c, a\}\},$$

$$b(b(L)) = L$$

Proposition For any clutter L, $b(b(L)) = L$ holds.

Rooted circuits:
({a,c}, d), ({b,c}, d)

Rooted cocircuits:
(ϕ, a), (ϕ, b), (ϕ, c)
({a,b}, d), ({c}, d)

root	stems	costems
a	—	ϕ
b	—	ϕ
c	—	ϕ
d	{a,c}, {b,c}	{a,b}, {c}

L={ {a, c}, {b, c} }, b(L)={ {c}, {a, b} }

Proposition (Korte, Lovasz and Schrader)
For a convex geometry (K, E), let C be the collection of the rooted circuits, and D be the collection of the rooted cocircuits. Then for each $e \in E$,
$C(e) = \{X : (X, e) \in C\}$ and $D(e) = \{Y : (Y, e) \in D\}$ are the blocker of each other.

Def. Meet-distributive lattice:

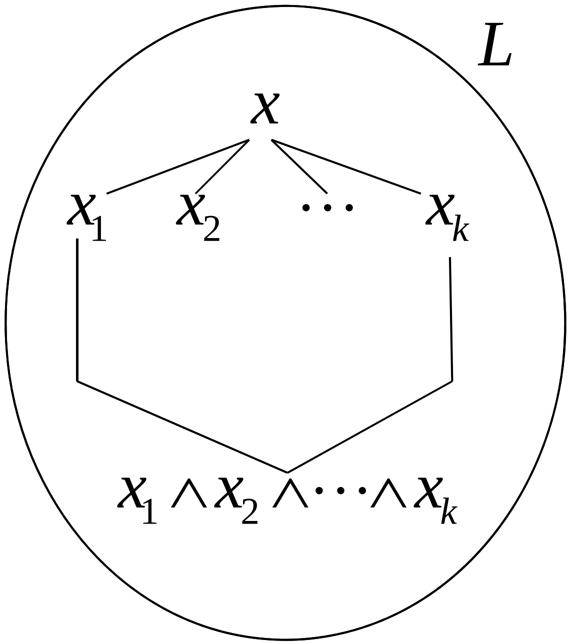
A finite lattice is meet-distributive if for any $x \in L$,

the interval $[x, x_1 \wedge x_2 \wedge \cdots \wedge x_k]$ is a Boolean lattice

where $x_1, x_2, \ldots, x_k$ are the elements covered by x.

<u>Theorem</u> (Edelman 1986)

A finite lattice is meet-distributive if and only if it is isomorphic to the lattice of a convex geometry.



Theorem (Dilworth, Boulay, Edelman, et al.)
Let L be a finite lattice. Then the following are equivalent.

(1) Every element of L has a unique irredundant join decomposition to join-irreducible elements.
(2) L is lower semimodular, and every modular sublattice is distributive.
(3) L is meet distributive.
(4) L is lower semimodular and join-semidistributive.
(5) L is isomorphic to a closure lattice of a convex geometry.

2. Supersolvable antimatroids

Def. $F \subseteq 2^E$ is an antimatroid if and only if

$K = \{E - A : A \in F\}$ is a convex geometry.

Def. $F \subseteq 2^E$ is an antimatroid if

(1) $\phi \in F$,

(2) $A, B \in F \Rightarrow A \cup B \in F$,

(3) If $A \in F$ and $A \neq \phi$, then $\exists e \in A$ such that $A - e \in F$.

Def. $F \subseteq 2^E$ is an antimatroid if

(1) $\phi \in F$,

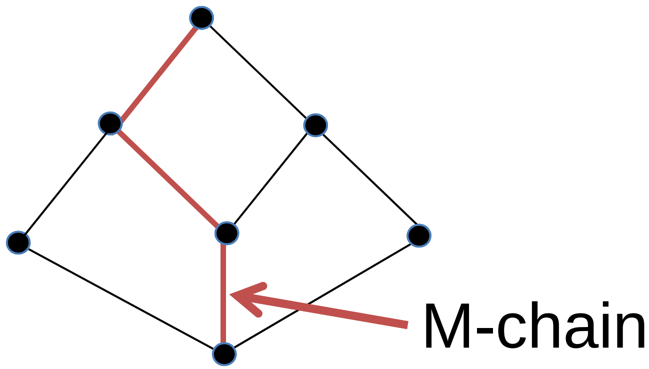
(2A) If $A, B \in F$ and $A \not\subseteq B$, then $\exists b \in B \setminus A$ such that $A \cup b \in F$.

[strong axioms of antimatroids]

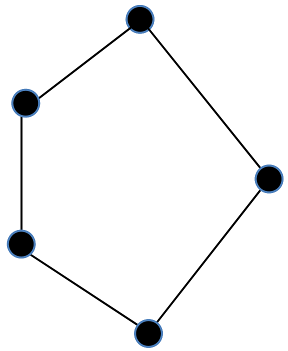
Def. A lattice is supersolvable if there is a maximal chain D (called an M-chain) such that a sublattice generated by D and any other chain is necessarily distributive.

Def. An antimatroid is supersolvable if it is supersolvable as a lattice.

Ex. Supersolvable and not supersolvable lattices



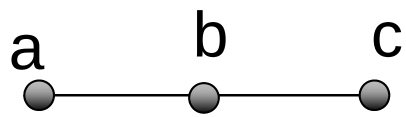
A supersolvable lattice



A non-supersolvable lattice

Def. For a convex geometry, we define a digraph, called the circuit-graph, with the vertex set being the underlying set and the edge set $\{(f,e):(X,e)\in C, f\in X\}$ where C is the family of the rooted circuits.

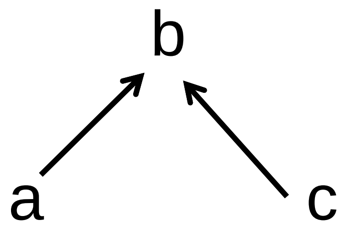
Ex.



Rooted circuits

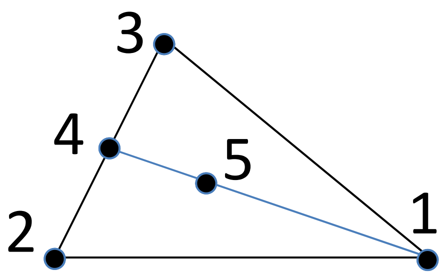
({a, c}, b)

Circuit-graph



Acyclic digraph

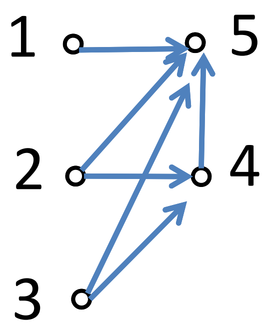
Ex. An affine point configuration

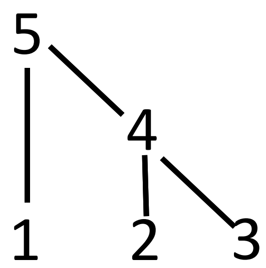


Rooted circuits

({1,2,3}, 5) , ({1,4}, 5)
({2,3}, 4)

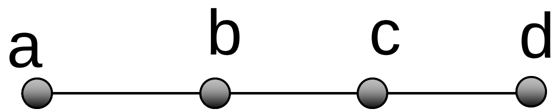
Circuit graph





(acyclic → partial order)

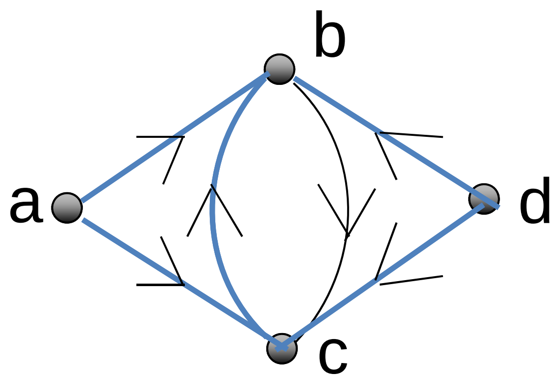
Ex.



Rooted circuits

({a, c}, b), ({a, d}, b)
({a, d}, c), ({b, d}, c)

The circuit graph is not acyclic in this case.



Theorem

For an antimatroid $F \subseteq 2^E$, the following are equivalent.

(1) F is supersolvable.

(2) There exists a total ordering ω on E such that

for $A, B \in F$ with $A \not\subseteq B$ and $b \in \min_\omega (B \setminus A)$,

$A \cup b \in F$. (Armstrong 2009) (cf. (2A))

(3) The rooted-circuit digraph of the convex geometry

$K = F^C = \{E - A : A \in F\}$ is acyclic.

Suppose that the circuit digraph of $K = F^C = \{E - A : A \in F\}$ is acyclic. Then it determines a partial order P on E. Then

Corollary

Any linear extension ω of P satisfies (2) above.

3. Lattice-embedding of convex geometries to convex geometries

Lemma

Let K_1, K_2 be convex geometries on E, and μ_1, μ_2 be their closure operators. Then the following are equivalent.

(1) $K_1 \subseteq K_2$.

(2) $\mu_2(A) \subseteq \mu_1(A)$ for $A \subseteq E$.

(3) For any rooted circuit (X_2, a) of K_2 , there is a rooted circuit (X_1, a) of K_1 such that $X_2 \subseteq X_1$.

Theorem

Let K_1 , $K_2 \subseteq 2^E$ be convex geometries on E such that $K_1 \subseteq K_2$.
Then the maximum size of stems of K_1 is at most the maximum size of stems of K_2 .

Conjecture Let (K_1, E_1), (K_2, E_2) be convex geometries.
Suppose there exists a lattice-embedding $f : K_1 \to K_2$.
Then the maximum size of stems of K_1 is at most the maximum size of stems of K_2 .

A result from the conjecture

Let (K_1, E_1), (K_2, E_2) be convex geometries. If the maximum size of stems of K_2 is larger than that of K_1, then there exists no lattice-embedding $f: K_1 \to K_2$.

Theorem (Adaricheva et al. (2003))

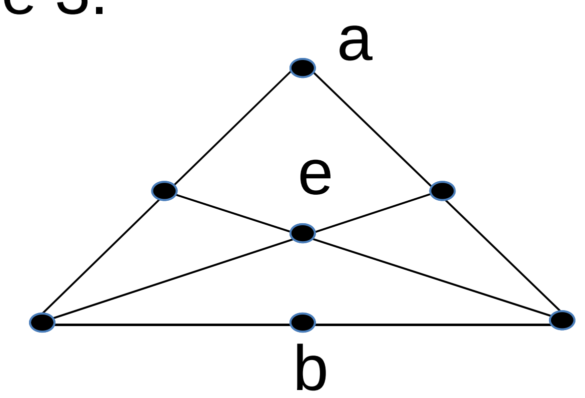
A finite join-semidistributive lattice cannot be necessarily embedded into a finite biatomic atomistic convex geometry.

Def. A lattice with zero is biatomic if

- every element is above an atom (atomic),
- for every atom p of L and all element $a, b \in L$, if $p \le a \vee b$, then there are atoms $q \le a$ and $r \le b$ such that $p \le q \vee r$.

We misunderstood that the size of a stem of a biatomic convex geometry is at most two.

Ex. 7 points configuration in an affine space below. Deleting a rooted circuit ({a,b}, e), then the other rooted circuits defines a convex geometry which is biatomic and has a stem of size 3.



5. Extensive and intensive operators

Def. An operator is a map $f: 2^E \rightarrow 2^E$

Def. An operator f is extensive if $A \subseteq f(A)$.

Def. An operator f is intensive if $f(A) \subseteq A$.

extensive $f \longmapsto f^*(A) = \{ a \in A : a \notin f(A-a) \}$ intensive

intensive $f \longmapsto f^*(A) = A \bigcup \{ b \in E - A : b \notin f(A \bigcup b) \}$ extensive

<u>Ex.</u>

closure operator $\mu \longmapsto \mu^*(A) = \chi(A)$

$= \{ a \in A : a \notin \mu(A-a) \}$ extreme-point operator

Def. Ext(E) : the collection of all the extensive operators.

Def. Int(E) : the collection of all the intensive operators.

Theorem (Danilov and Koshevoy 2009)

$$\mathrm{Ext}(E) \overset{*}{\longleftrightarrow} \mathrm{Int}(E)$$ is a bijection.

Def. An extensive operator is monotone if

$$B \subseteq A \Rightarrow \mu(B) \subseteq \mu(A)$$

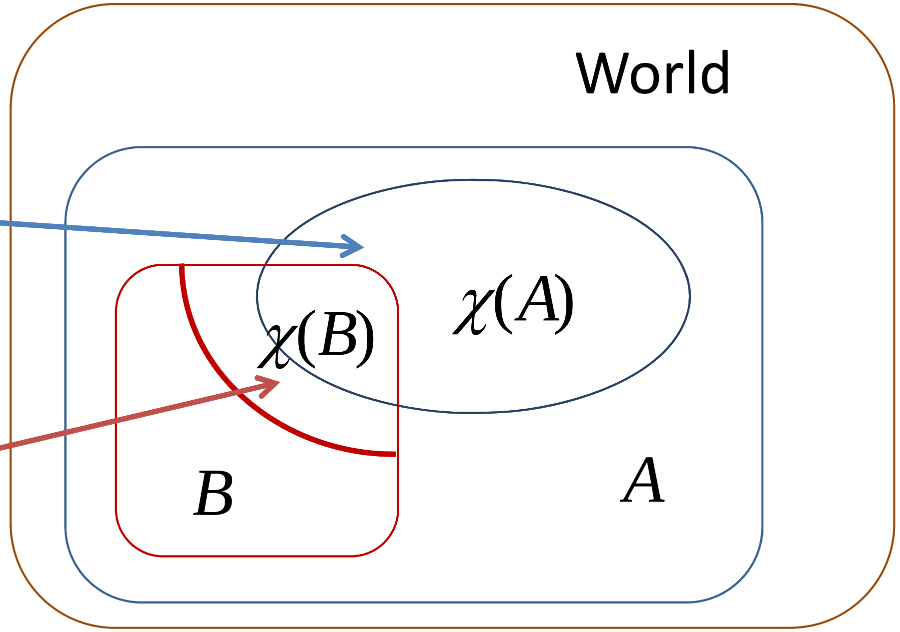
Def. An intensive operator χ is hereditary if

$$B \subseteq A \Rightarrow \chi(A) \cap B \subseteq \chi(B)$$

Ex. Selection of representatives

$\chi(A)$ is the national football team of A (Hungary)

$\chi(B)$ is the representative team of football of B (Szeged)



`hereditary' is a nice property as a choice function.

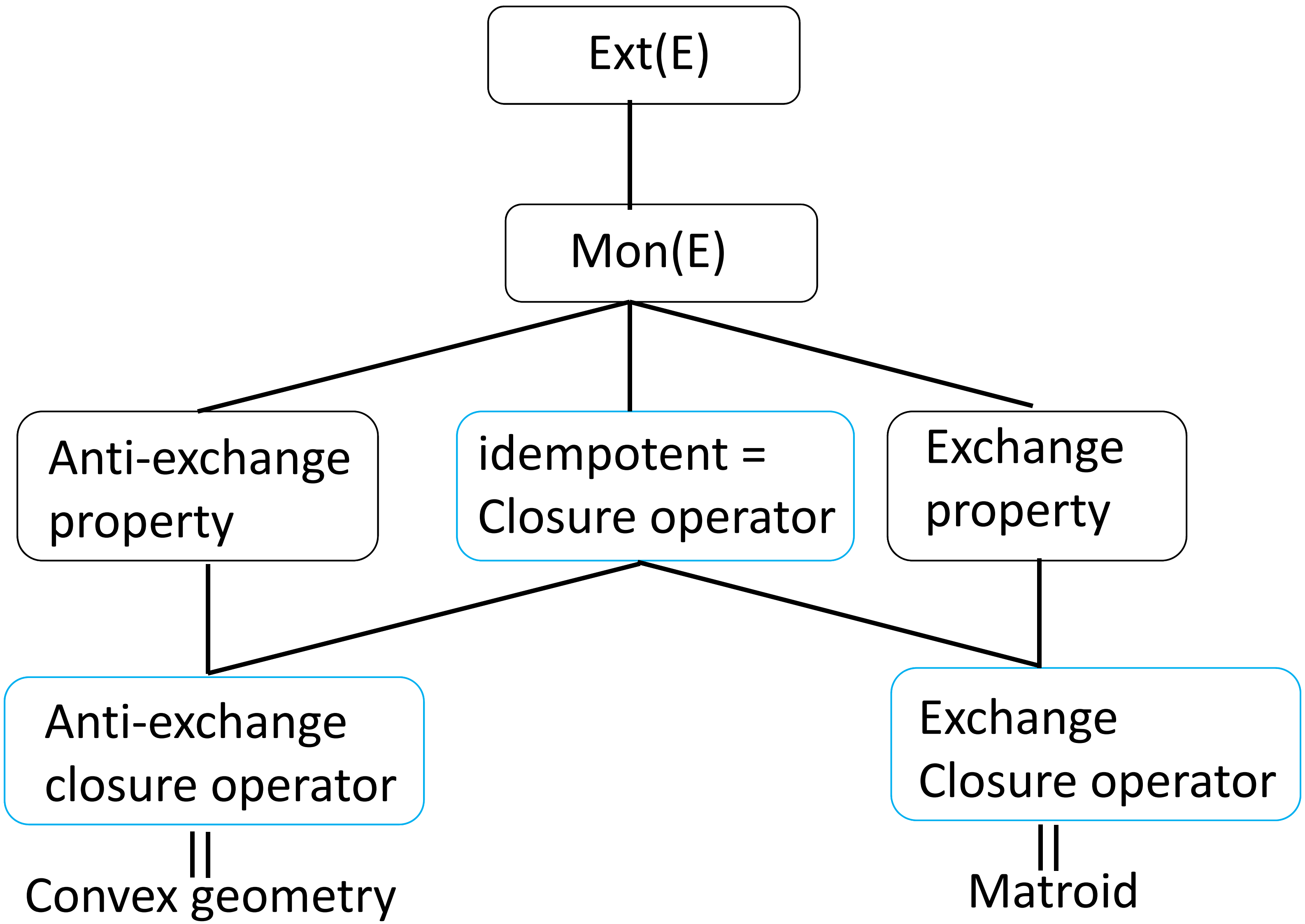
Def. Mon(E) : the collection of all the monotone extensive operators on E

Def. Her(E) : the collection of all the hereditary intensive operators on E

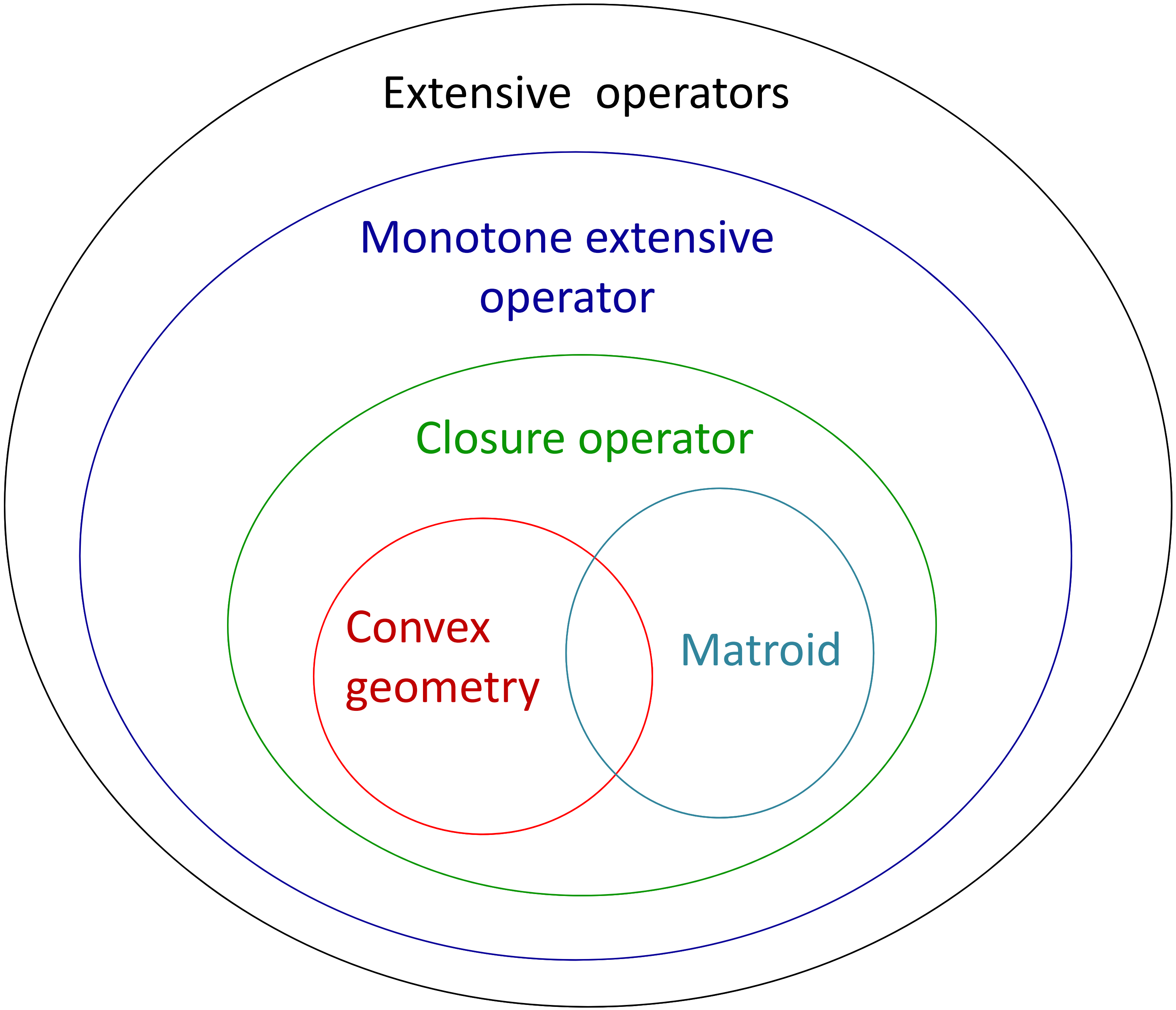
$$
\begin{array}{ccc}
\mathrm{Ext}(E) & \overset{*}{\longleftrightarrow} & \mathrm{Int}(E) \\
\cup & & \cup \\
\mathrm{Mon}(E) & \overset{*}{\longleftrightarrow} & \mathrm{Her}(E)
\end{array}
$$

Theorem (Danilov and Koshevoy 2009)

$\mathrm{Mon}(E) \overset{*}{\longleftrightarrow} \mathrm{Her}(E)$ is a bijection.



Ext(E)
Mon(E)
Anti-exchange property
idempotent = Closure operator
Exchange property
Anti-exchange closure operator
||
Convex geometry
Exchange Closure operator
||
Matroid

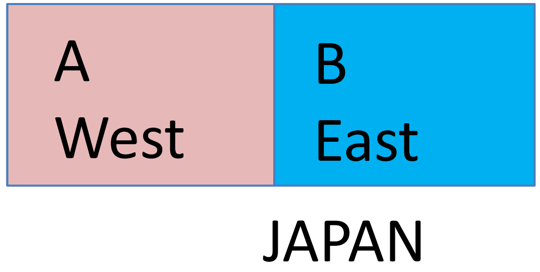

Extensive operators
Monotone extensive operator
Closure operator
Convex geometry
Matroid

Def. An intensive operator χ is path-independent if

$$\chi(A \cup B) = \chi(\chi(A) \cup \chi(B)) \text{ for any } A, B \subseteq E.$$

$$\Updownarrow$$

$$\chi(A \cup B \cup \cdots \cup D) = \chi(\chi(A) \cup \chi(B) \cup \cdots \cup \chi(D))$$



<u>Theorem</u> (Koshevoy 1999)

An intensive operator (choice function) is path-independent if and only if it is an extreme-point operator of a convex geometry.

Def. A rooted set on E is a pair (X,e) such that $e \in E, X \subseteq E-e$.

Def. A rooted clutter C on E is a family of rooted sets such that for each $e \in E$, $C(e)=\{X:(X,e)\in C\}$ is a clutter.

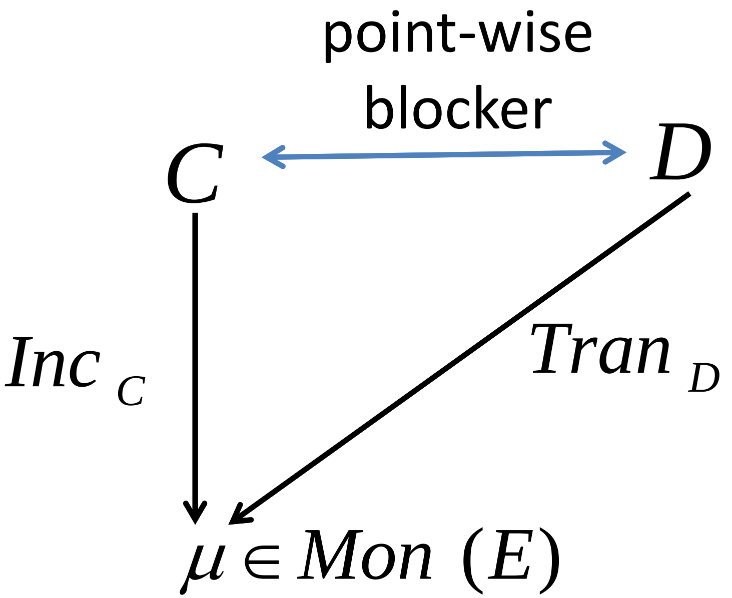
Def. A pair (C,D) of rooted clutters is a rooted circuit-cocircuit system if for each $e \in E$, $C(e)=\{X:(X,e)\in C\}$ and $D(e)=\{Y:(Y,e)\in D\}$ are the blocker of each other. We say that C and D are point-wise blocker of each other.

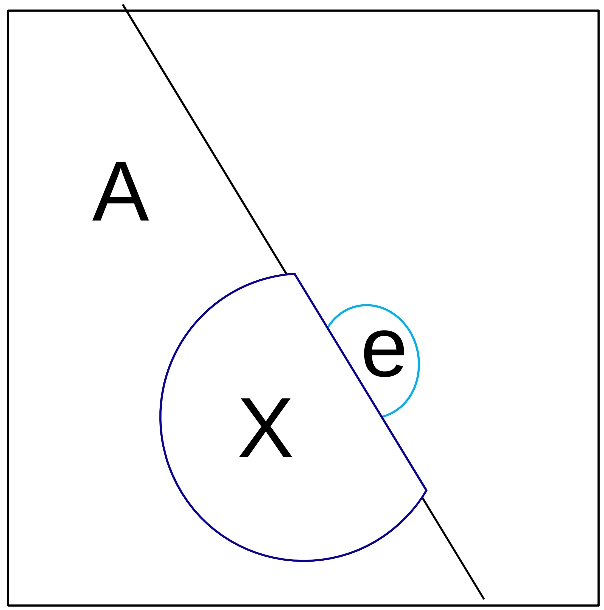
Conversely, a monotone extensive operator determines a rooted circuit-cocircuit system.

Def. For a rooted circuit-cocircuit system (C,D), let

$$\mu(A) = Inc_C(A) = A \cup \{e \in E - A : \exists (X,e) \in C,\ X \subseteq A\} \qquad (A \subseteq E),$$

$$= Tran_D(A) = A \cup \{e \in E - A : \forall (Y,e) \in D, Y \cap A \neq \phi\} \quad (A \subseteq E).$$





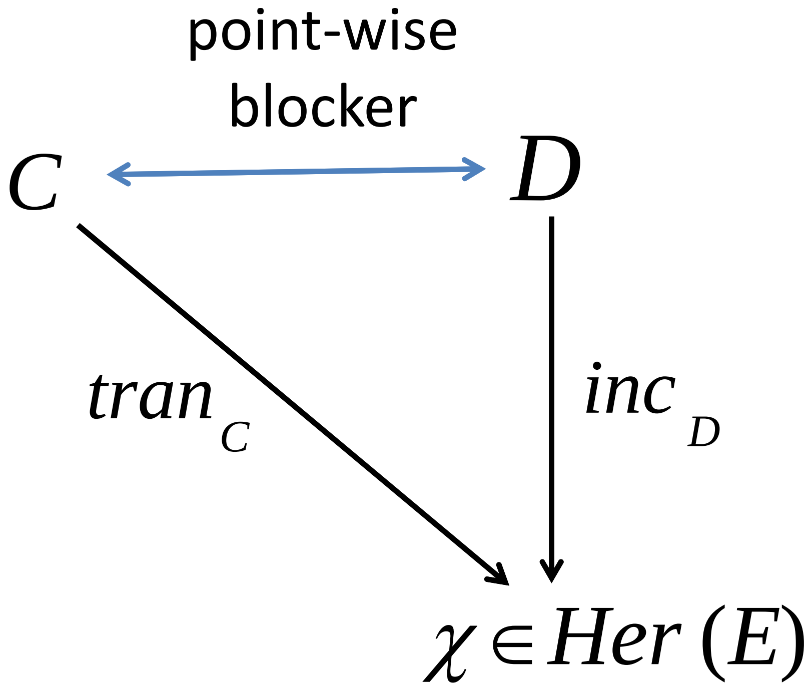
$$e \in \mu(X) \quad for \ (X,e) \in C$$

<u>Theorem</u> The diagram is commutative.

and

$$\chi(A) = tran_C(A) = \{e \in A : \forall (X,e) \in C,\ X \cap A \neq \phi\}$$

$$= inc_D(A) = \{e \in A : \exists\,(Y,e) \in D,\ Y \subseteq A\}$$



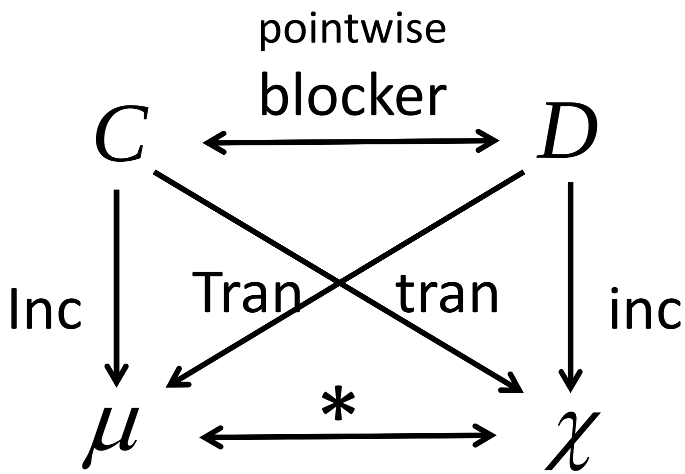
<u>Theorem</u>

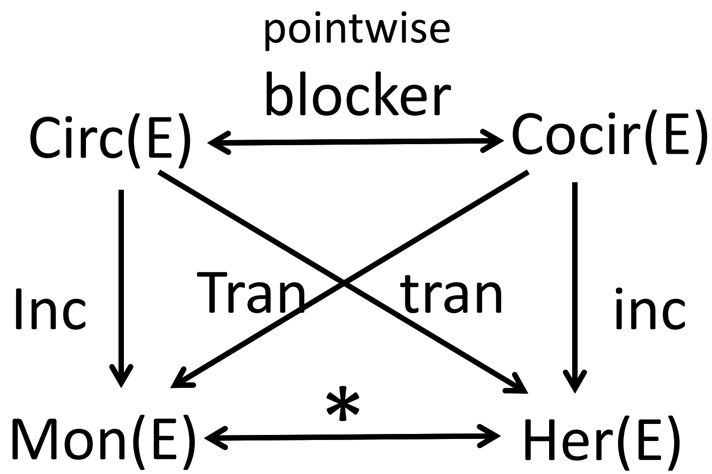
This diagram is commutative.

<u>Proposition</u> For a rooted circuit-cocircuit system,

$Inc_C = Tran_D = \mu \in$ Mon(E), $tran_C = inc_D = \chi \in$ Her(E),

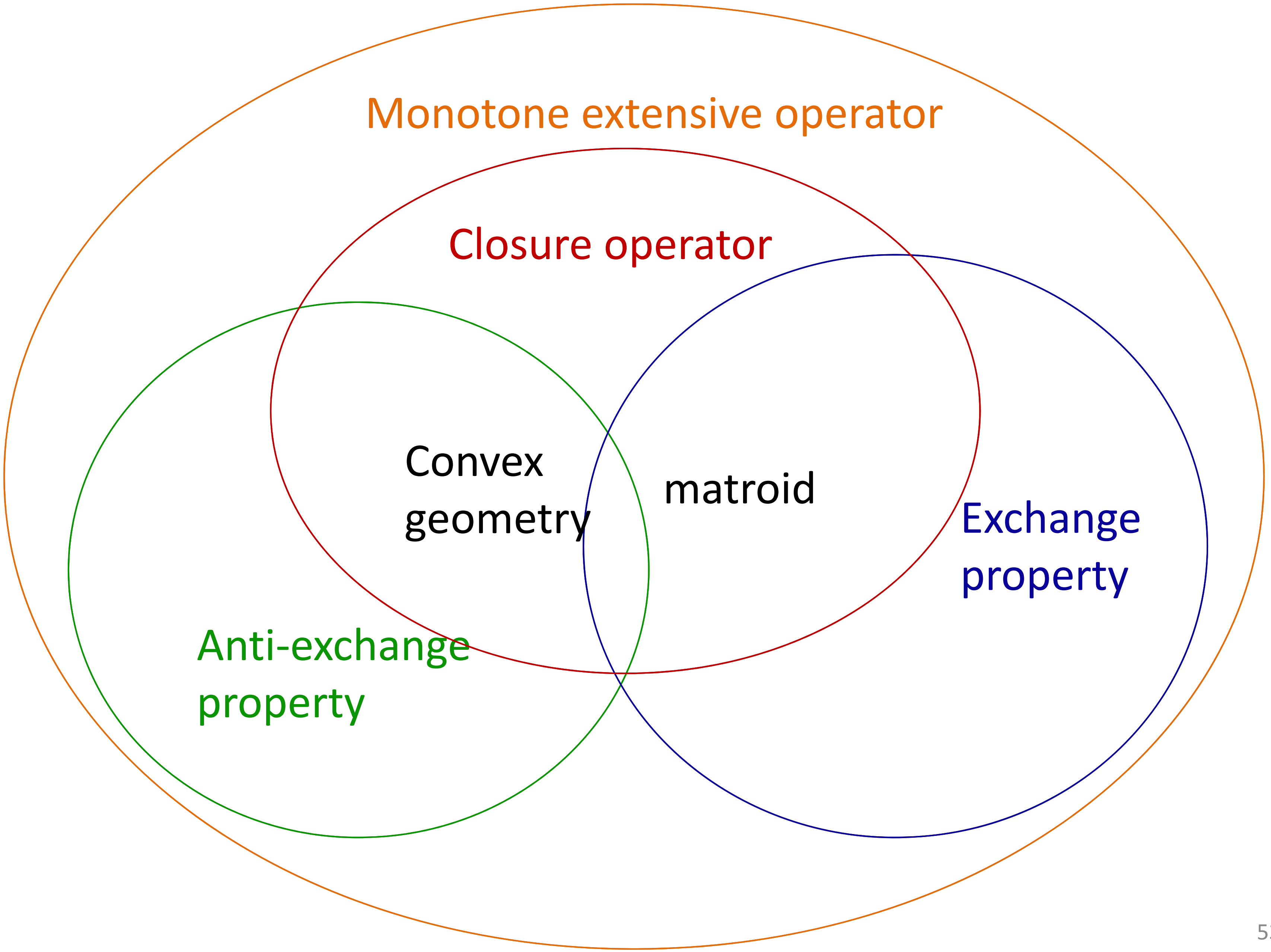
$\mu^* = \chi$, and $\chi^* = \mu$.





<u>Theorem</u>

The diagram is commutative, and every arrow is a bijection.


Monotone extensive operator
Closure operator
Convex
geometry
matroid
Exchange
property
Anti-exchange
property

Anti-exchange monotone extensive operator

Theorem Let (C,D) be a circuit-cocircuit system, and

$\mu = Inc_C = Tran_D \in$ Mon(E).

Then the following are equivalent.

(AEx) μ satisfies the anti-exchange property.

(CA) If $(X,x) \in C$ and $(Y,y) \in C$, then either $(X',x) \in C$ for some $X' \subseteq (X \bigcup Y) - y$ or $(Y',y) \in C$ for some $Y' \subseteq (X \bigcup Y) - x$.

(DA) If $(X,x) \in D$ and $(Y,y) \in D$, then either $(X',x) \in D$ for some $X' \subseteq (X \bigcup Y) - y$ or $(Y',y) \in D$ for some $Y' \subseteq (X \bigcup Y) - x$.

Note: Switching C and D gives nothing changed.

Closure operator

Theorem Let (C,D) be a circuit-cocircuit system, and

$\mu = Inc_C = Tran_D \in$ Mon(E).

Then the following are equivalent.

(Clo) μ is idempotent, i.e. μ is a closure operator.

(CO) If $(X,x) \in C$, $(Y,y) \in C$ and $y \notin X$, then there exists

$(W,y) \in C$ such that $W \subseteq (X \cup Y) - x$.

(DO) If $(X,x) \in C$ and $f \in X$, then there exists $(Y,f) \in D$

such that $Y \subseteq (X - f) \cup x$.

Exchange monotone extensive operator

Theorem Let (C, D) be a circuit-cocircuit system, and

$\mu = Inc_C = Tran_D \in$ Mon(E).

Then the following are equivalent.

(Ex) μ satisfies the exchange property.

(CE) If $(X, x) \in C$ and $f \in X$, then there exists $(Y, f) \in C$

such that $Y \subseteq (X - f) \cup x.$

(DE) If $(X, x) \in D,\ (Y, y) \in D$ and $y \notin X,$ then there exists

$(W, y) \in D$ such that $W \subseteq (X \cup Y) - x.$

For a circuit-cocircuit system (C, D), we shall call (D, C) the dual system. Under taking the dual, the anti-exchange property is invariant.

Taking the dual switches the exchange property and the idempotent (closure operator).

Hence the concept of matroid is invariant under taking the dual. When (C, D) gives rise to a matroid, the dual system (D, C) gives the dual matroid in the ordinary sense.

Exchange property + closure = matroid

Exchange Property

(CE) If $(X,x)\in C$ and $f\in X$, then there exists $(Y,f)\in C$

such that $Y'\subseteq(X-f)\bigcup e.$

(DE) If $(X,x)\in D, (Y,y)\in D$ and $y\notin X,$ then there exists

$(W,y)\in D$ such that $W\subseteq(X\bigcup Y)-x.$

Closure

(CO) If $(X,x)\in C, (Y,y)\in C$ and $y\notin X,$ then there exists

$(W,y)\in C$ such that $W\subseteq(X\bigcup Y)-x.$

(DO) If $(X,x)\in C$ and $f\in X$, then there exists $(Y,f)\in D$

such that $Y'\subseteq(X-f)\bigcup e.$

New self-dual axioms of Matroids

Axioms of Matroid

(CE) If $(X,x) \in C$ and $f \in X$, then there exists $(Y', f) \in C$ such that $Y' \subseteq (X-f) \cup x$.

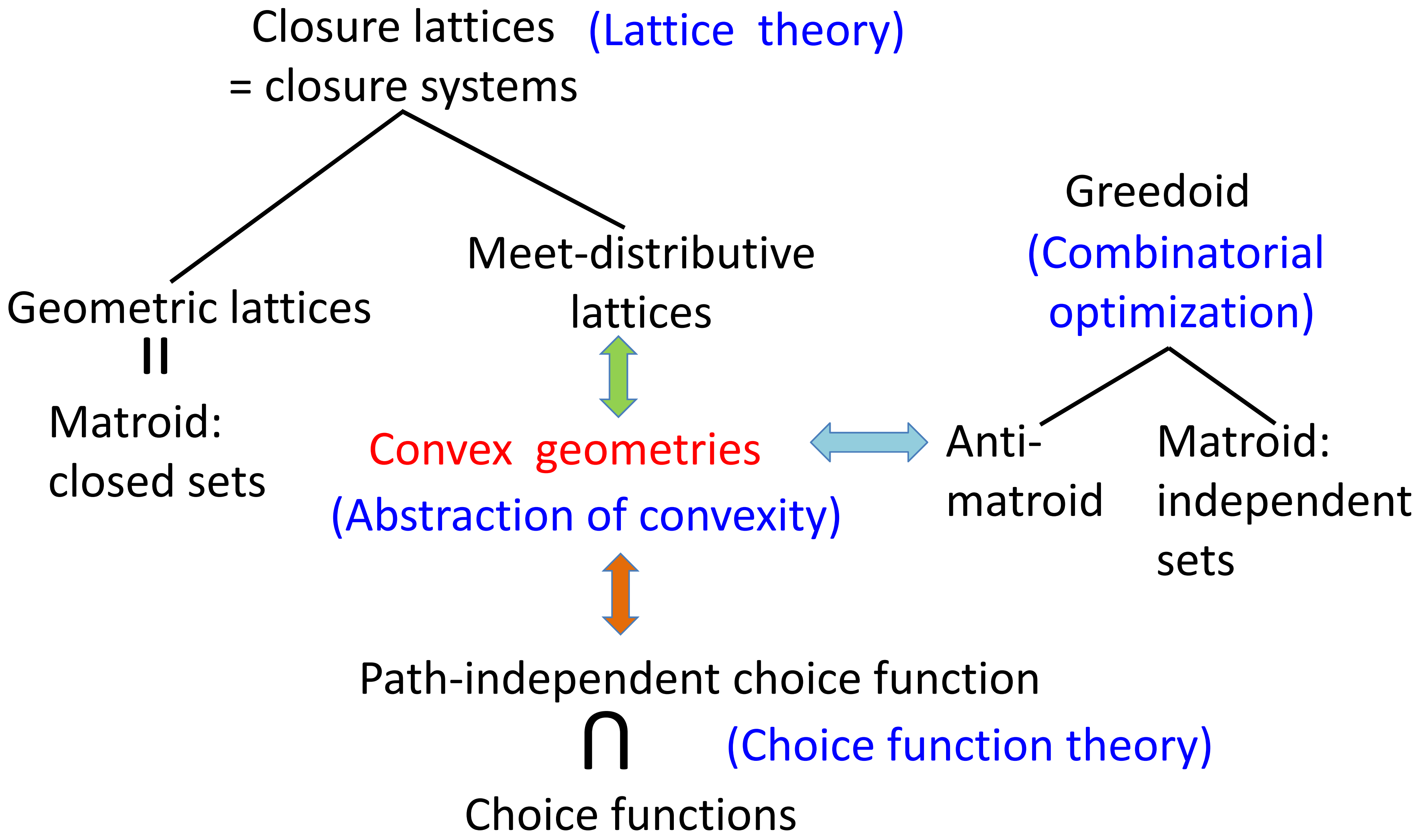
(DO) If $(Y,y) \in D$ and $f \in Y$, then there exists $(Y', y) \in D$ such that $Y' \subseteq (Y-f) \cup x$.

Axioms of Matroid

(CO) If $(X,x) \in C$, $(Y,y) \in C$ and $y \notin X$, then there exists $(W,y) \in C$ such that $W \subseteq (X \cup Y) - x$.

(DE) If $(X,x) \in D$, $(Y,y) \in D$ and $y \notin X$, then there exists $(W,y) \in D$ such that $W \subseteq (X \cup Y) - x$.

6. Related areas of convex geometries



4. Rooted circuits and implicational systems

Def. An implicational system

$$S = \{ (A_i, B_i) : i \in I \} \qquad (A_i, B_i \subseteq E)$$

We write $A \to B \in S$ instead of $(A, B) \in S$.

Def. $X \subseteq E$ fulfills if $A \subseteq X \Rightarrow B \subseteq X$.

<u>Lemma</u>

$$K(S) = \{ X \subseteq E : X \ \mathit{fulfills\ every} A \to B \in S \}$$

is a closure system.

Def. For a closure system K if $K = K(S)$ then S is a generating system. If $K(S) = K(S')$, then we say that S and S' are equivalent.

A minimal generating system is called a basis.

Def. A basis S of K is minimum if $|S| \leq |S'|$ for any basis S' of K.

Def. A basis S of K is optimal if

$$\sum_{A \to B \in S} (|A| + |B|) \leq \sum_{A' \to B' \in S'} (|A'| + |B'|)$$ for any basis S' of K.

Def. Let C be the collection of rooted circuits of K

We put $Stem(K) = \{ X : (X, e) \in C \}$

<u>Lemma</u>

Let K be a closure system, and C be the collection of rooted circuits. Then $S(C) = \{ X \to e : (X, e) \in C \}$ is a generating system of K.

Def. quasiclosed set: $For\, A \subseteq E,$

$$A^{\circ} = A \cup \bigcup \{ \mu(X) : X \subset A, \mu(X) \subset \mu(A) \},$$

$$A^{\bullet} = A^{\circ} \cup A^{\circ\circ} \cup A^{\circ\circ\circ} \cup \cdots \quad .$$

$A \subseteq E$ is quasiclosed if $A = A^{\bullet}$ and $A = \mu(A)$.

Def. pseudoclosed set:

$P \subseteq E$ is a pseudoclosed set if P is a minimal quasiclosed set among those quasiclosed sets Q satisfying $\mu(P) = \mu(Q)$.

Theorem (Guigues and Duquenne 1986, Wild 1994)

Let (μ, E) be a closure system.

(1) The basis $S_P = \{P \to (\mu(P) - P) : P \text{ is pseudoclosed}\}$ is a minimum basis of (μ, E)

(1) An optimal basis is always a minimum basis.

(2) For each pseudoclosed set P, every optimal basis includes an implication $X_P \to Y_P$ with $X_P \subseteq P$. The cardinality of X_P is uniquely determined as

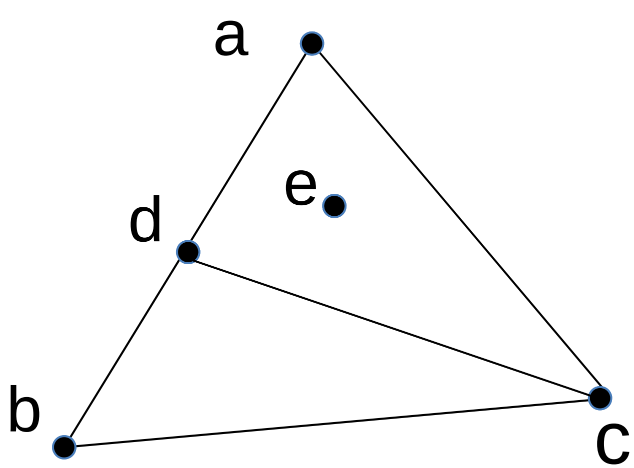
$$|X_P| = \min\{|X| : X \subseteq P, \mu(X) = \mu(P)\}.$$

Def. For a stem X, we put $Int(X) = \{ f : (X, f) \in C \}$,

$bd(X) = \mu(X) - (X \cup \text{int}(X))$.

If $bd(X) = \phi$, then X is called a prime stem.

Ex. Affine point configuration.



Stems

{a,b,c} not prime

{a,c,d} prime

{a,d) prime

Theorem

Suppose that the sizes of stems of a closure space are all the same. Then X is a stem if and only if it is a pseudoclosed set.

Theorem

Under the same condition as above. Then $S = \{X \to (\mu(X) - X) : X \ is\ a\ stem\}$ is a minimum basis.

Theorem

For an affine convex geometry (μ, E),

$S = \{X \rightarrow (\mu(X) - X) : X \text{ is a prime stem}\}$

is a minimum basis.

Theorem

For an affine convex geometry (μ, E),

$S = \{X \rightarrow e_X : X \text{ is a prime stem}\}$

is an optimal basis.

Thank you.

Koeszenem szepen.

What is study?

The Master said,

`Is it not a pleasure to learn and practice what is learned?

Is it not a joy to have friends come together from far?

Is it not gentlemanly not to take offence though men may take no note of him?'

--- The Analects of Confucius