# Enumerative mathematics and its connection to combinatorics, computer science and geometry

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# Some basic questions

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**Basic question of extremal combinatorics**

Given a finite set $\mathcal{S}_p$ and a parameter $q$.
Determine $\min / \max\{q(S) : S \in \mathcal{S}_p\}$.

# Classical examples

**Permutations**

Let $S_p$ be the set of permutations of $[p] = \{1, 2, 3, \ldots, p\}$.

**Turán's theorem**

Let $\mathcal{S}_{p,k}$ be the set of graphs over $[p]$ not containing a clique of size $k$.
Determine $\max\{|E(G)| : G \in \mathcal{S}_{p,k}\}$.

# Examples

**Permutations with excluded pattern**

Let $S_p$ be the set of permutations of $[p] = \{1, 2, 3, \ldots, p\}$ not containing $\ldots\alpha\ldots\mathrm{A}\ldots\mathrm{a}\ldots$, where $\alpha < a < A$.

**Davenport—Schinzel**

Let $\mathcal{S}_p$ be the set of words over $[p] = \{1, 2, 3, \ldots, p\}$ not containing $\ldots\mathrm{a}\ldots\alpha\ldots\mathrm{a}\ldots\alpha\ldots\mathrm{a}\ldots$ and $\ldots\mathrm{aa}\ldots$.
Let $\ell(w)$ be the length of the word $w$.
Determine $\max\{\ell(w) : w \in \mathcal{S}_p\}$.

# Examples

## Diagonals (Euler)

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$\equiv$ How many ways can you draw non intersecting diagonals into a convex $p$-gon?

## Diagonals again

What is the maximum numbers of diagonals you can draw in a regular $p$-gon without having three pairwise intersecting ones?

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Maximum how many unit distances can be determined by $p$ points in convex position?

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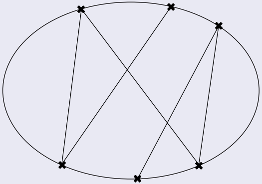
**Unit distance graph**

Given $p$ point in convex position. Connect two of them iff their distance is 1.

## Füredi's observation

A unit distance cannot have the following as a substructure:



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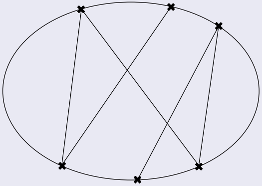

## A combinatorial question

Given a 0-1 matrix of size $p \times p$. What is the maximum number of 1's in it if it doesn't contain

$$\begin{pmatrix} 1 & 1 & \\ & & 1 \\ 1 & & 1 \end{pmatrix}?$$

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# Extremal vs enumerative questions

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## A simple enumerative question

What is the number of permutation matrices of size $p \times p$ NOT containing

$$\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}?$$

**Füredi—Hajnal conjecture**

The maximum number of 1's in a 01 matrix of size $p \times p$ not containing $F_\pi$ (a fix permutation matrix) is linear,

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Now both conjectures can be quoted as Marcus—Tardos theorem.

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# Graphs on ordered vertex set

We are considering graphs on the vertex set $[p]$.

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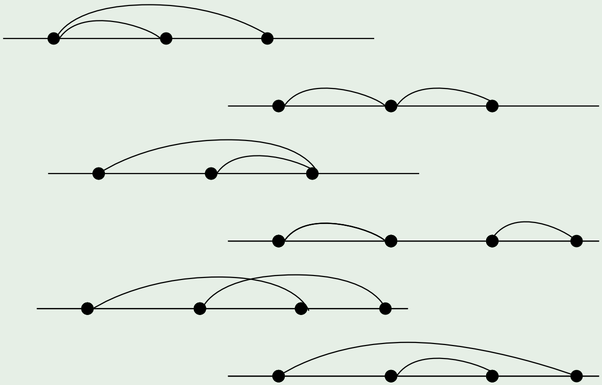
## Ordered graphs with two edges



## Ozsvárt's theorem (2012)

# Surprising incidences

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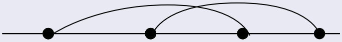
The number of ordered graphs over [$p$] without



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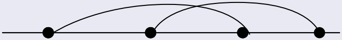

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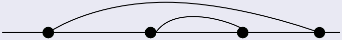

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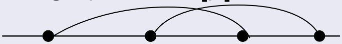

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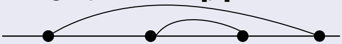

# Counting permutations according their maximum distance

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Maximal distance of a permutation on $[n]$ is

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| | $D = 0$ | $D = 1$ | $D = 2$ | $D = 3$ |
|---|---|---|---|---|
| $n = 1$ | 1 | | | |
| $n = 2$ | 1 | 1 | | |
| $n = 3$ | 1 | 2 | 3 | |
| $n = 4$ | 1 | 4 | 9 | 10 |

**The lemma**

The average maximal distance of permutations from $S_n$ is at least

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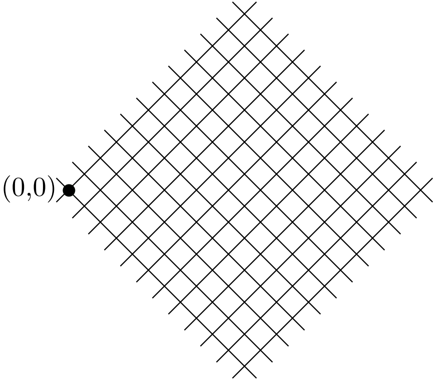
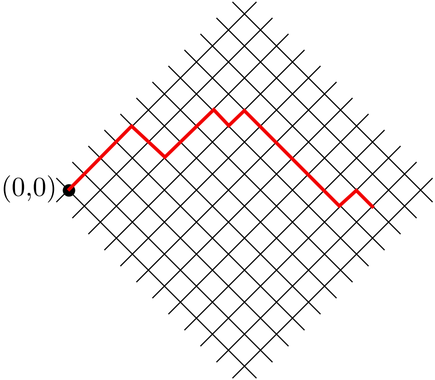
$$n - \alpha \cdot \sqrt{n}.$$

# Lattice paths

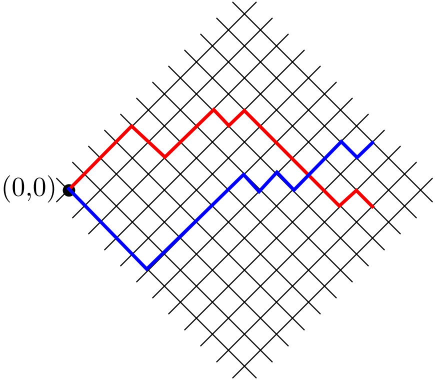



Lattice




Lattice and a lattice path

# Lattice paths




Lattice and a lattice path and another one.

# Catalan strikes again

**Theorem (Callan, Gábor V. Nagy)**

The number of lattice paths to $(0, 4n)$ not touching the points $(0, 2i+1)$ is $C_{2n}$.

# Final puzzles

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | 1 | | | | | |
| | | | | 1 | | 1 | | | | |
| | | | 1 | | 3 | | 1 | | | |
| | | 1 | | 7 | | 6 | | 1 | | |
| | 1 | | 15 | | 25 | | 10 | | 1 | |
| 1 | | 31 | | 90 | | ? | | | | |

# Final puzzles

Continue the pattern:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 3 & & 1 & & & \\ & & 1 & & 7 & & 6 & & 1 & & \\ & 1 & & 15 & & 25 & & 10 & & 1 & \\ 1 & & 31 & & 90 & & ? & & & & \end{array}$$

$$1, 2, 4, 9, 23, ?$$

# The End

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Happy Birthday to Gábor!

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Thank you for your attention!