On long alternating non-crossing paths in 2-equicolored convex sets

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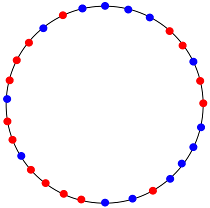
2010

Given a convex point set, $\mathcal{P}$ on the plane with even cardinality $(2k)$. Someone color half of the points red and the other half blue (it is an equicolored pointset): $\mathcal{P}$



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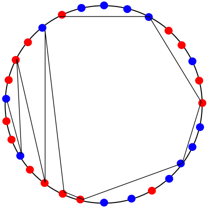

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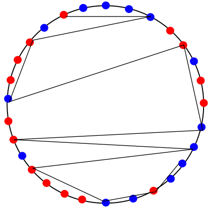

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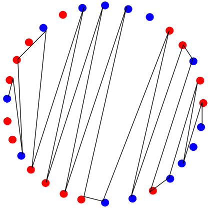

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Definition

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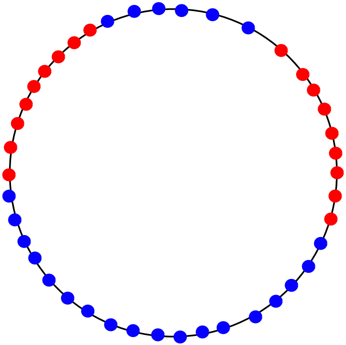
Definition (Erdős)

$$\ell(n) = \min_{\mathcal{P} \text{ is equicolored}} \{\ell(\mathcal{P})\},$$

where $\mathcal{P}$ geometrically $2n$-element convex planar point set.

Consider the following colored point set:



By easy case analysis we obtain the bound

$$\ell(n) \leq 5/4n.$$

Erdős' base camps

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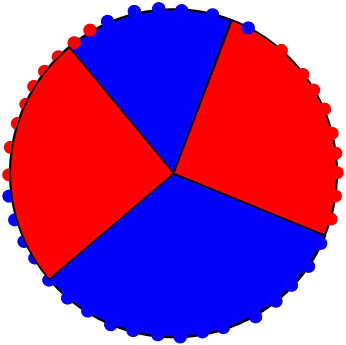

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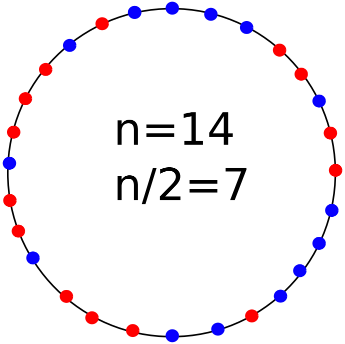

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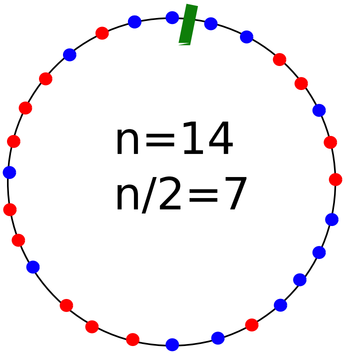

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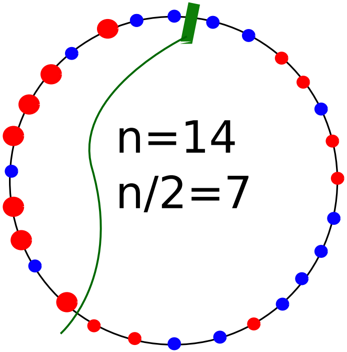

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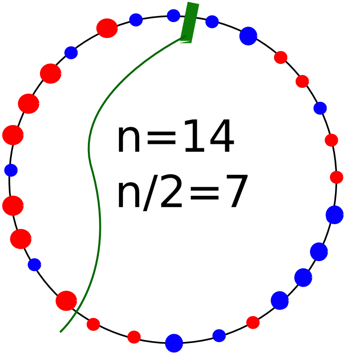

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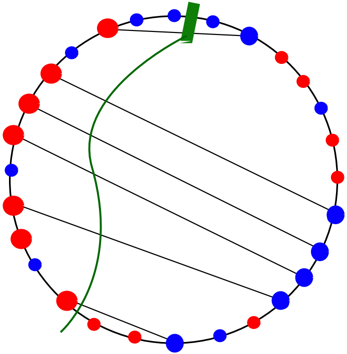

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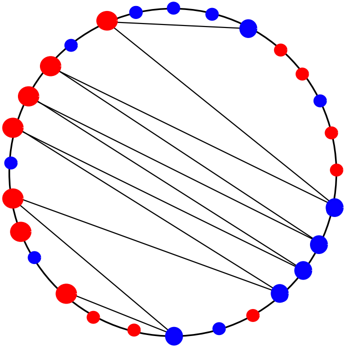

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Erdős' base camp (cont.)

Theorem (Erdős)

Hence $\ell(n) \geq n$.

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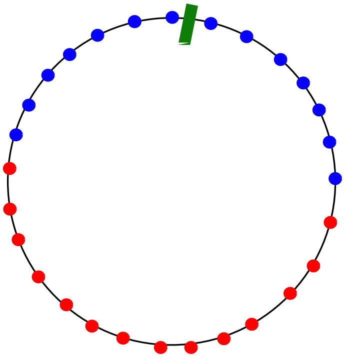
Very easy, BUT...

Sharpness of Erdős' lower bound

If the advesary party gives the initial point of the path then it is the best:



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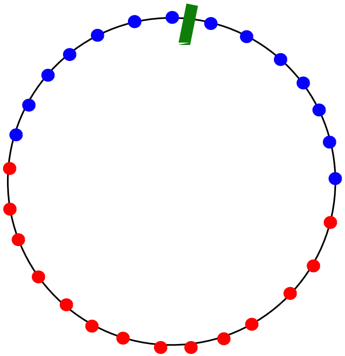
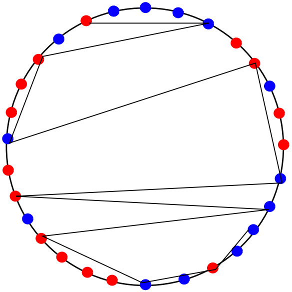


For beating the Erdős bound we must choose the initial point carefully.

The structure of non-crossing paths

Each non-crossing path has the following structure



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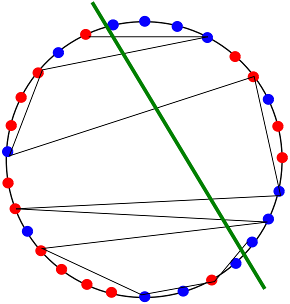

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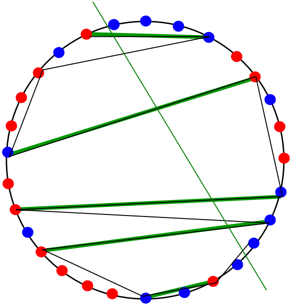

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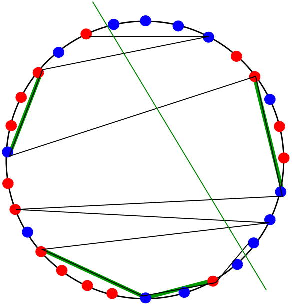

(i) an axe,

(ii) matching part,

(iii) side edges.

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Observation

In Erdős' path with matching part M, we have $alt(M) = 1$.

Sharpness of Erdős' lower bound II. (cont.)

If we insist of the non-alternating matching part, then we cannot beat the Erdős bound...

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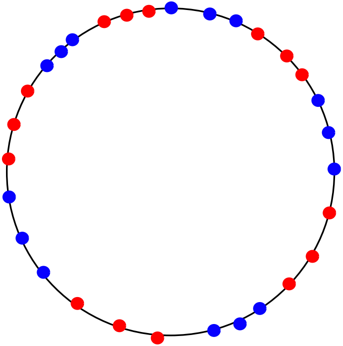

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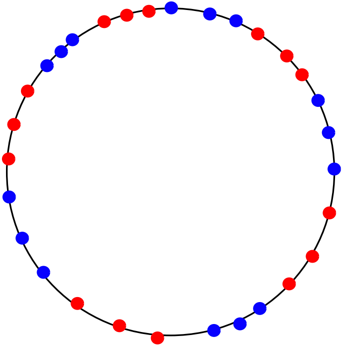

... upto a remainder term.

Kinčl–Pach–Tóth' construction

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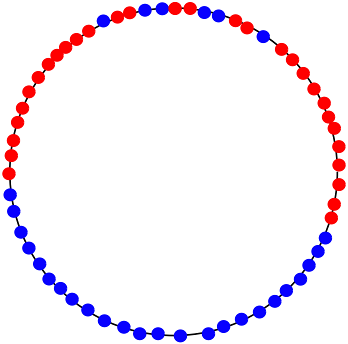

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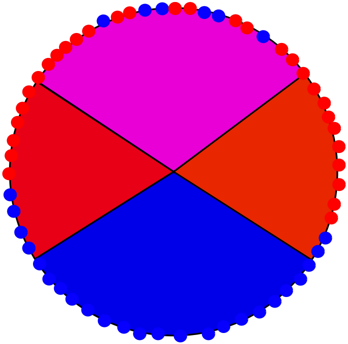

$\ell(\mathcal{P}) = \frac{4}{3}n + \alpha\sqrt{n},$

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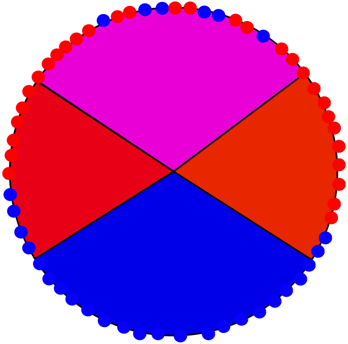

$\ell(\mathcal{P}) = \frac{4}{3}n + \alpha\sqrt{n}$, by case analysis.

Kinčl–Pach–Tóth' lower bound

Find two neighboring arcs of equal length k, I and J such that

$$\sharp B - \sharp R(I), \sharp R - \sharp B(J) \geq L.$$

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HENCE they find an alternating, non-crossing path of length $n + L$.

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Their matching, M is such that $alt(M) \leq 2$. HENCE they have a limit: $L \leq \sqrt{n}$.

Folklore conjecture

The upper bound gives the correct order of magnitude. For suitable $\alpha > 0$

$$\ell(n) \geq 4/3n + \alpha\sqrt{n}.$$

Further constructions

Remember Kinčel-Tóth-Pach construction

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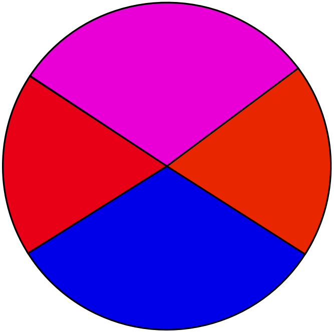

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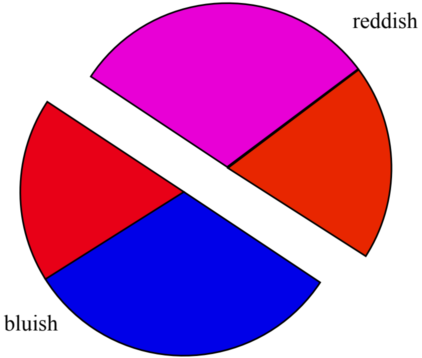

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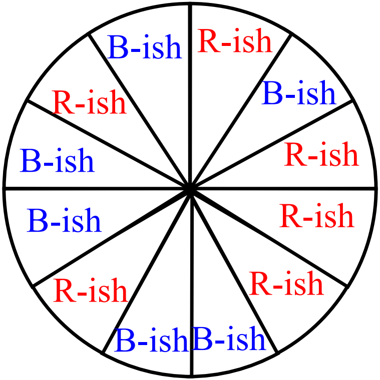

It has a structure.

That can be combined to give colored point sets with ℓ-parameter $4/3n + \mathcal{O}(\sqrt{n})$

Further constructions (cont.)

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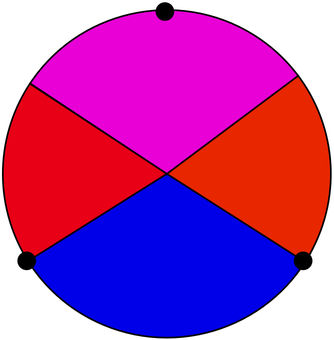

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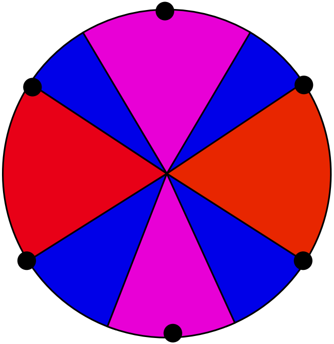

An other utilization of the observation. Different type of examples

Further constructions (cont.)

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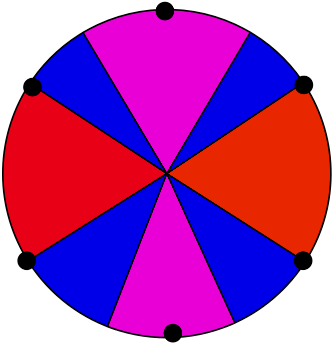
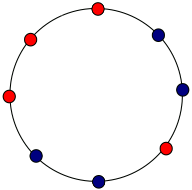


An other utilization of the observation. Different type of examples (extremal?)

Lower bound: Coding a equicolored pointset



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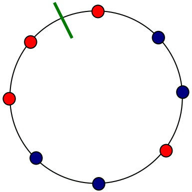

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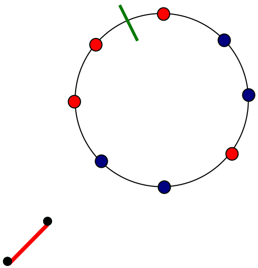

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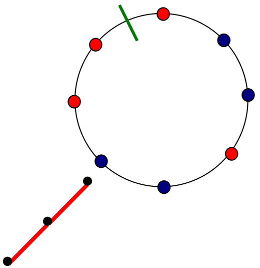

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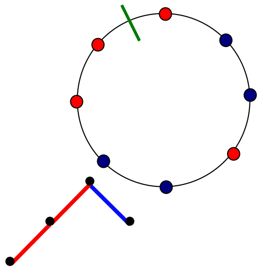

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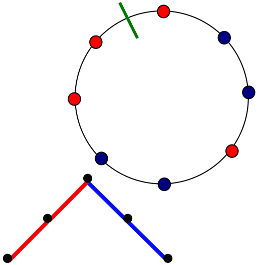

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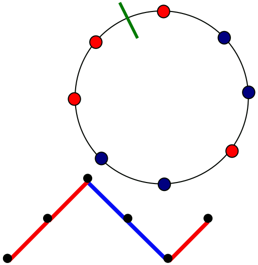

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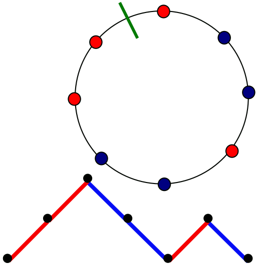

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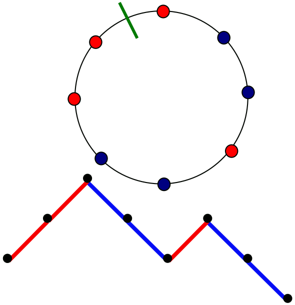

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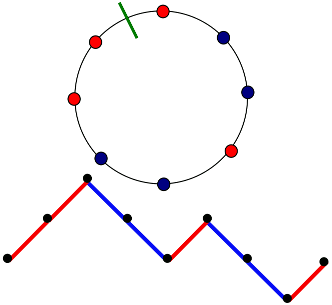

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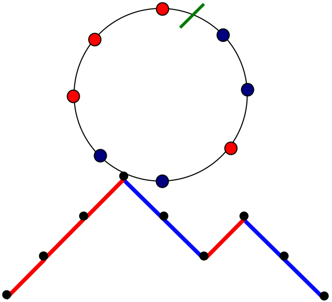

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Lower bound: Finding good I, J arcs

Let r be the number of runs.

Observation

At each level of the Dyck-path there are at most r steps.

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Corollary

More than half of the steps are at higher level than $\mathcal{O}(n/r)$

Lower bound: Finding good I, J arcs (cont.)

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There are two SYMMETRIC point of the coding Dyck path of height at least $\mathcal{O}(n/r)$

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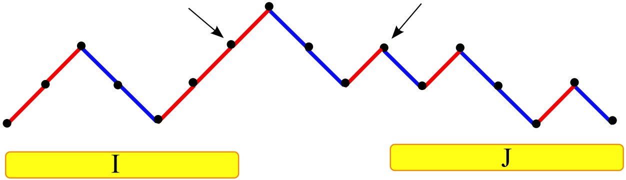

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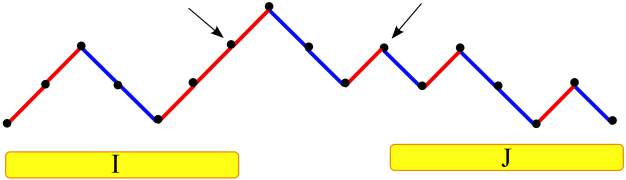

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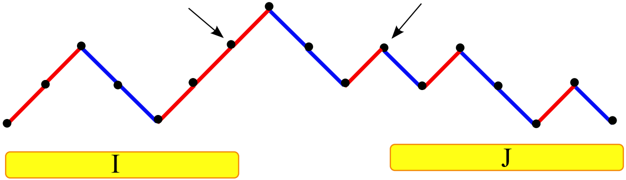

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Theorem

$$\ell(n) \geq n + \sqrt{n}.$$

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So far we recognized local densities in the color distribution. We MUST recognize global similarities.

Discrepancy of the coloring

Thank you for Laci's rendering his educational work with great ENTHUSIASM.

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Thank you for your attention