Orthogonal Structures in Directed Graphs

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1 Introduction

Let P be a finite partially ordered set. Dilworth's theorem states that *the maximal size of an antichain equals the minimal number of chains partitioning the elements of P.* Trivially every chain partition provides a bound on the maximal size of an antichain. One commonly known proof for the existence of a chain partition - antichain pair meeting equality is due to Fulkerson [6]. He derives the result from the well-known König-Egerváry theorem which says that *in a bipartite graph a maximal matching and a minimal vertex cover have the same size.*

Apply this theorem to the bipartite graph $B(P)$ having as vertices two copies P', P'' of P and an edge (x', y'') whenever $x < y$ in P. A matching M corresponds to a partition of P into $|P| - |M|$ chains. Begin with the partition of P into 1–element chains. For each edge $(x', y'') \in M$ hook the tail of the chain ending with x to the beginning of the chain starting with y thus reducing the number of chains by one. To a vertex cover U of $B(P)$ take the antichain $A = \{x \in P \mid x', x'' \notin U\}$. Dilworth's theorem follows from $\max |M| = \min |U|$ since $|A| = |P| - |U|$ can be shown.

A slight modification of the sketched proof associates with U a function $\chi : P \to \{-1, 0, 1\}$ defined by $\chi(x) = 1 - |\{x', x''\} \cap U|$. Extend χ to subsets of P by $\chi(X) = \sum_{x \in X} \chi(x)$. Properties of χ then are: $\chi(C) \leq 1$ for all chains C and $\chi(P) = |P| - |U|$. Now in turn we define a 1–*weighting* of P to be a function $\chi : P \to \{-1, 0, 1\}$ with $\chi(C) \leq 1$ for all chains C and pose the problem of maximizing the *value* $\chi(P)$. As in the case of antichains, every chain partition provides a bound on $\max \chi(P)$, too.

From the König-Egerváry theorem we get the duality: *The maximum value of an 1–weighting of P equals the minimal number of chains partitioning P.* To derive Dilworth's theorem two observations suffice:

1) For any 1–weighting $\chi^{-1}(1)$ is an antichain.
2) For 1–weightings of maximal value $\chi^{-1}(-1) = \emptyset$.

Thus a 1–weighting of maximal value is the characteristic function of an antichain.

Greene and Kleitman [9] found a nice generalization of Dilworth's result. Define a k *antichain family* to be a family of k pairwise disjoint antichains.

Theorem 1 *For any partially ordered set P and any positive integer k*

$$\max \sum_{A \in \mathcal{A}} |A| = \min \sum_{C \in \mathcal{C}} \min(|C|, k)$$

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where the maximum is taken over all k antichain families $\mathcal{A}$ and the minimum over all chain partitions $\mathcal{C}$ of P.

A chain partition $\mathcal{C}$ which minimizes the right hand side is called *k-saturated.* In fact a somewhat stronger result was obtained in [9].

Theorem 2 *For any $k \geq 1$ there is a chain partition which is simultaneously k-saturated and $(k+1)$-saturated.*

Greene [8] stated the duals of these theorems. Let a *ℓ chain family* be a family of ℓ pairwise disjoint chains.

Theorem 3 *For any partially ordered set P and any positive integer ℓ*

$$\max \sum_{C \in \mathcal{C}} |C| = \min \sum_{A \in \mathcal{A}} \min(|A|, \ell)$$

where the maximum is taken over all ℓ chain families $\mathcal{C}$ and the minimum over all antichain partitions $\mathcal{A}$ of P.

Again a partition which minimizes the right hand side is called *ℓ-saturated.* A transition phenomenon similar to that of theorem 1 holds.

Theorem 4 *For any $\ell \geq 1$ there is an antichain partition which is simultaneously ℓ-saturated and $(\ell+1)$-saturated.*

A further theorem of Greene [8] can be interpreted as a generalization of the Robinson-Shensted correspondence and its interpretation given by Greene [7].

To a partially ordered set P with n elements a partition λ of n is associated, such that for the Ferrers diagramm $G(P)$ corresponding to λ we get:

Theorem 5 *The number of squares in the ℓ longest columns of $G(P)$ equals the maximal number of elements covered by a ℓ chain family of P and the number of squares in the k longest rows of $G(P)$ equals the maximal number of elements covered by a k antichain family.*

Since then several proofs of the cited results have been proposed [3], [5], [10] and [12]. The Greene-Kleitman theorem has been generalized to acyclic directed graphs in [11], [2], [1] and [13]. An excellent survey is given by West [14].

The proof by András Frank [5] is particularly elegant. Following Frank we call a chain family $\mathcal{C}$ and an antichain family $\mathcal{A}$ an *orthogonal pair* iff

1. $P = \Big(\bigcup_{A \in \mathcal{A}} A\Big) \cup \Big(\bigcup_{C \in \mathcal{C}} C\Big),$
2. $|A \cap C| = 1 \quad \text{for all } A \in \mathcal{A},\ C \in \mathcal{C}.$

If $\mathcal{C}^+$ is obtained from $\mathcal{C}$ by adding the rest of P as singeltons and $\mathcal{C}$ is orthogonal to a k-antichain family $\mathcal{A}$, then

$$\sum_{A \in \mathcal{A}} |A| = \sum_{C \in \mathcal{C}^+} \sum_{A \in \mathcal{A}} |A \cap C| = \sum_{C \in \mathcal{C}^+} \min(|C|, k).$$

Thus $\mathcal{C}^+$ is k-saturated. Similarly a ℓ-saturated antichain partition can be obtained from an orthogonal pair $\mathcal{A}, \mathcal{C}$ where $\mathcal{C}$ is a ℓ chain family.

Using the minimal cost flow algorithm of Ford and Fulkerson [4], Frank could prove the existence of a sequence of orthogonal chain and antichain families. This sequence is rich enough to allow the derivation of the whole theory (i.e. theorems 1 to 5).

The purpose of this paper is to show how the concept of 1–weightings and the technique used by Frank can be used to obtain a similar theory for directed graphs.

In the next section we develop the network flow method and show how this method can be used to obtain a sequence of orthogonal pairs in directed graphs. The role of k antichain families is taken by k weighting families (i.e k 'disjoint' 1–weightings), a family of disjoint paths and cycles including ℓ paths takes care of the role of ℓ chain families.

In section 3 we show how the orthogonal structures of section 2 allow generalizations of theorems 1-5 to acyclic directed graphs and partly even to arbitrary directed graphs.

2 The Network Flow Method

The proof of theorems 1-5 given by Frank is based on the construction of a well-behaved sequence of orthogonal pairs. Given a directed graph $D = (V, E)$ (partially ordered sets have $E = \{(x, y)|\ x < y\}$), we associate the *Frank network* $N = (V_N, E_N)$ as follows.

$$V_N = \{s, t\} \cup \{x'|\ x \in V\} \cup \{x''|\ x \in V\}$$
$$E_N = \{(s, x')|\ x \in V\} \cup \{(x', x'')|\ x \in V\} \cup \{(x'', t)|\ x \in V\} \cup \{(x', y'')|\ (x, y) \in E\}$$

We set all arc capacities $c(e)$ for $e \in E_N$ to one and take costs as

$$a(e) = \begin{cases} 1 \text{ if } e = (x', x'') \text{ for some } x \in V \\ 0 \text{ otherwise.} \end{cases}$$

A maximal flow that can be sent from s to t through this network has flow value $n = |V|$ and costs $\leq n$. The Ford Fulkerson algorithm solves the problem of finding a minimal cost flow for all flow values v with $0 \leq v \leq n$. It invokes dual variables $\pi(x)$ ('potentials') assigned to the vertices of N. The fundamental feature of the algorithm is given by the theorem.

Theorem. *Let f be a flow with value v. If there exists a* potential π *such that*

$$\begin{array}{llcl} (i) & \pi(s) = 0; \quad \pi(t) = k & & (k \textit{ is called the } \text{potential value}) \\ (ii) & \pi(y) - \pi(x) < a(x, y) & \Rightarrow & f(x, y) = 0 \\ (iii) & \pi(y) - \pi(x) > a(x, y) & \Rightarrow & f(x, y) = c(x, y) \end{array}$$

then f has minimal cost among the flows of value v.

The algorithm begins with zero potential and zero flow. It iteratively increases either the flow or the potential, always maintaining the optimality criteria given in the theorem. The decision, which step is carried out to reach the next stage, depends on an auxiliary network $N^* = (V_N, E^*)$. With an edge $(x, y) \in E_N$ we have

$$\begin{array}{llcl} (*) & (x, y) \in E^* & \Longleftrightarrow & \pi(y) - \pi(x) = a(x, y) \text{ and } f(x, y) < c(x, y) \\ (**) & (y, x) \in E^* & \Longleftrightarrow & \pi(y) - \pi(x) = a(x, y) \text{ and } f(x, y) > 0 \end{array}$$

The steps of the algorithm then are:

Step (F)

If a path leading from s to t exists in N^*.

- Increase the flow along this path by one.
- Actualize the auxiliary network.

Step (P)

If there is no $s \to t$ path in N^*.
Let $I(s) = \{x|$ there is *no* $s \to x$ path in $N^*\}$

- Increase the potential of all $x \in I(s)$ by one.
- Actualize the auxiliary network.

If this algorithm is applied to a Frank network we can state an additional invariant.

Lemma 1 *For the actual potential π at any stage of the algorithm and all edges $(x', y'') \in E_N$*

$$\pi(y'') - \pi(x') \leq a(x', y'').$$

Proof. To get $\pi(y'') - \pi(x') > a(x', y'')$ we would have to come across a situation with $\pi(y'') - \pi(x') = a(x', y'')$ and $x' \notin I(s)$, $y'' \in I(s)$.

Let $x' \notin I(s)$ and suppose $f(x', y'') = 0$ then $(x', y'') \in E^*$. By definition of $I(s)$ there exists a $s \to x'$ path in N^*. This path can be enlarged to a $s \to y''$ path, thus $y'' \notin I(s)$.

On the other hand if $x' \notin I(s)$ and $f(x', y'') = 1$, then the flow in this edge comes from (s, x'), thus $(s, x') \notin E^*$. This forces the last edge of the $s \to x'$ path in N^* to be a backward edge. The only choice for this edge is (x', y''), thus revealing a $s \to y''$ path, again $y'' \notin I(s)$. □

This Lemma together with property (ii) from the theorem gives

$(iv)\quad f(x', y'') = 1 \quad \Rightarrow \quad \pi(y'') - \pi(x') = a(x', y'')$

Now let $D = (V, E)$ be a directed graph on n vertices.

Definition. A family $\mathcal{W} = \{W_1, \ldots, W_t\}$ of disjoint subsets of V is called a ℓ *path/cycle family* of D if each W_i is either the support of a simple path or the support of a simple cycle and at most ℓ of the W_i are the support of a path.

Definition. A function χ is called a k*–weighting* (k a positive integer) of D iff

1. $\chi : V \to \{-k, \ldots, -1, 0, 1\}$
2. $\chi(P) \leq k$ if P is the support of a path in D
3. $\chi(C) \leq 0$ if C is the support of a cycle in D.

Two 1–weightings χ_1, χ_2 are *disjoint* if $\chi_1^{-1}(1) \cap \chi_2^{-1}(1) = \emptyset$. A family of k pairwise disjoint 1–weightings is called a k *weighting family.*

Remark. If $\chi_1, \ldots, \chi_k$ are disjoint 1–weightings, then $\sum_1^k \chi_i$ is a k–weighting. The opposite is true too, any k–weighting admits a decomposition into disjoint 1–weightings. The second fact is nontrivial but remains unproved here.

Consider a stage of the minimum cost flow computation in the Frank network of D. Let (f, π) be the current flow-potential pair, with flow value v and potential value k.

Associate with f a $(n - v)$ path/cycle family $\mathcal{W}$:

Let $H_f = \{x \in V |\ f(x', x'') = 1\}$ and note that $|H_f|$ is just the cost of f. Start from the partition of $V \setminus H_f$ into 1-element paths. Use the edges (x, y) with $x \neq y$ and $f(x', y'') = 1$ to hook the tail of the path ending with x with the beginning of the path starting with y, thus reducing the number of paths by one. In graphs which are not acyclic this procedure may connect the ends of a single path to produce a cycle. In the path/cycle family corresponding this way to the flow f we find $(n - |H_f|) - (v - |H_f|) = n - v$ as the number of paths.

Associate with π a k weighting family Θ:

Define the function χ_j for $1 \leq j \leq k$ by:

$$\chi_j(x) = \begin{cases} 1 & \text{iff} \quad \pi(x') < j - \frac{1}{2} < \pi(x''), \\ -1 & \text{iff} \quad \pi(x') > j - \frac{1}{2} > \pi(x''), \\ 0 & \text{otherwise.} \end{cases}$$

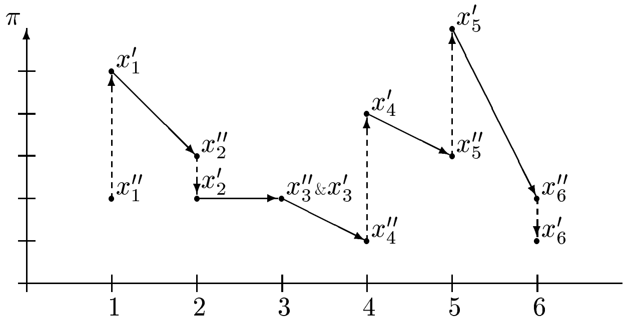

Figure 1: Potential diagram of a path $x_1, \ldots, x_6$

Lemma 2 *The set $\{\chi_j | \, 1 \leq j \leq k\}$ is a k weighting family for D.*

Proof. By definition $\chi_j : V \to \{-1, 0, 1\}$ for all j. Now let $P = x_1, \ldots, x_r$ be a path in D. Consider the potential diagram of this path (cf. Figure 1).

By definition the value $\chi_j(P)$ is just the number of dashed arrows crossing the potential $j - 1/2$ downwards minus the number of dashed arrows crossing $j - 1/2$ upwards.

From the continuity of the arrow sequence we get that the difference of downward crossing and upward crossing (arbitrary) arrows is $1, 0$ or -1. Lemma 1 spezializes to $\pi(x''_{i+1}) - \pi(x'_i) \leq 0$, thus forcing the slope of the nondashed arrows to be non positive. The restriction of the difference to the dashed arrows can therefore only deminish the result and we get the second property of 1–weightings, namely $\chi_j(P) \leq 1$ for all j between 0 and k.

If $C = x_1, \ldots, x_r$ is a cycle in D, then we get an additional nondashed arrow from $\pi(x'_r)$ to $\pi(x''_1)$ and the difference over all crossings is exactly 0. Again the restriction to dashed arrows can only deminish the result of the difference and we get $\chi_j(C) \leq 0$.

The disjointness of χ_{j_1} and χ_{j_2} with $j_1 \neq j_2$ again follows from lemma 1. This time we conclude $\pi(x''_i) - \pi(x'_i) \leq 1$, therefore the dashed downarrows have length ≤ 1 and can only contribute to a single χ_j. □

Definition. Call a path/cycle family $\mathcal{W}$ and a weighting family Θ an *orthogonal pair* iff

1. $$V = \Big(\sum_{\chi \in \Theta} \chi\Big)^{-1}(1) \;\cup\; \Big(\bigcup_{W \in \mathcal{W}} W\Big),$$

2. $\chi(P) = 1$ for all paths $P \in \mathcal{W}$ and all $\chi \in \Theta$.

Theorem 6 *The families $\mathcal{W}$ and Θ associated with the current flow-potential pair (f, π) of any stage of the algorithm are orthogonal.*

Proof. 1) Let $x \notin \bigcup_{W \in \mathcal{W}} W$, then by definition of $\mathcal{W}$, we have $f(x', x'') = 1$ and by (iv) also $\pi(x'') - \pi(x') = 1$. We conclude $\chi_j(x) = 1$ for $j = \pi(x'')$.

2) Let $P = x_1, \ldots, x_r$ be a path in $\mathcal{W}$. Consider the potential diagram of $\mathcal{W}$ and recall the arguments used in the proof of lemma 2. This time we have flow in all the edges (x'_i, x''_{i+1}). With (iv) we conclude $\pi(x''_{i+1}) - \pi(x'_i) = 0$ and the slope of the nondashed arrows must be 0. Therefore $\chi_j(P) = 1$ iff $\pi(x''_1) > j$ and $\pi(x'_r) < j$. We complete the proof with two claims.

$\pi(x''_1) = \pi(t)$. There is no flow entering the vertex x''_1, so, there can be no flow leaving the vertex and $f(x''_1, t) = 0$. By $(**)$ we get $\pi(t) - \pi(x''_1) \leq a(x''_1, t) = 0$, i.e. $\pi(t) \geq \pi(x''_1)$, but $\pi(t)$ equals the number of potential increasing steps (i.e. Step(P))accomplished by now. Thus $\pi(y) \leq \pi(t)$ for all y.

$\pi(x'_r) = 0$. There is no flow leaving x'_r, so $f(s, x'_r) = 0$. Apply $(**)$ to get $\pi(x'_r) \leq \pi(s) = 0$ and compare with $\pi(y) \geq 0$ for all y. □

Remark. If $C = x_1, \ldots, x_r$ is a cycle in $\mathcal{W}$ then, since $f(x'_r, x''_1) = 1$, we have $\pi(x''_r) = \pi(x'_1)$. This proves $\chi(C) = 0$ for all $\chi \in \Theta$.

3 Duality Theorems for Directed Graphs

In the first part of this section we show how the orthogonal pairs arising from a run of the Ford Fulkerson algorithm on the Frank network of an acyclic directed graph give raise to generalizations of theorems 1 to 5. In the second part we analyse which part of the theory can be adapted to arbitrary directed graphs.

3.1 The Acyclic Case

Since the graphs in question here never contain cycles we will replace the fussy 'path/cycle' simply by 'path' throughout this part.

Let P be a path and Θ a k weighting family, then $\sum_{\chi \in \Theta} \chi(P) \leq \min(|P|, k)$, since the 1–weightings constituing Θ are disjoint and $\chi(P) \leq 1$ for $\chi \in \Theta$. Summing up over the paths of a path partition $\mathcal{P}$ we get $\sum_{\chi \in \Theta} \chi(V) \leq \sum_{P \in \mathcal{P}} \min(|P|, k)$ for all k weighting families Θ and path partitions $\mathcal{P}$.

Now consider the families $\mathcal{P}$ and Θ associated with the current flow-potential pair after the algorithm did climb up to the potential value k. Let $\mathcal{P}^+$ be obtained from $\mathcal{P}$ by adding the rest of V as 1-element paths. Since the k–weighting family Θ is orthogonal to $\mathcal{P}$ we obtain

$$\sum_{\chi \in \Theta} \chi(V) = \sum_{P \in \mathcal{P}^+} \Big(\sum_{\chi \in \Theta} \chi(P) \Big) = \sum_{P \in \mathcal{P}} k \; + \sum_{P \in \mathcal{P}^+ \setminus \mathcal{P}} 1 = \sum_{P \in \mathcal{P}^+} \min(|P|, k).$$

Thus we have proved the theorem:

Theorem 7 *For any acyclic directed graph D and any positive integer k*

$$\max \sum_{\chi \in \Theta} \chi(V) = \min \sum_{P \in \mathcal{P}} \min(|P|, k)$$

where the maximum is taken over all k weighting families Θ and the minimum over all path partitions $\mathcal{P}$ of D.

A path partition $\mathcal{P}$ which minimizes the right hand side is called *k-saturated.*

Theorem 8 *For any $k \geq 1$ there is a path partition which is simultaneously k-saturated and $(k+1)$-saturated.*

Proof. To obtain a simultaneously k and $(k+1)$-saturated path partition consider the step increasing the potential from k to $k+1$. Let f be the current flow in this step. Our construction gives a path family $\mathcal{P}$ being orthogonal to a k weighting family Θ_k and a $(k+1)$ weighting family Θ_{k+1}. The partition $\mathcal{P}^+$ thus has the desired properties. □

To state the duals of theorem 7 and 8 we first have to define the term 'weighting partition'.

Definition. A *weighting partition* Θ of an acyclic directed graph $D = (V, E)$ is a set of (not necessarily disjoint) 1–weightings with

$$\sum_{\chi \in \Theta} \chi(x) = 1 \qquad \text{for all } x \in V.$$

Now if χ is a 1–weighting and $\mathcal{P}$ a ℓ path family then we would like to have $\chi\big(\bigcup_{P \in \mathcal{P}} P\big)$ less or equal to $\min(\chi(V), \ell)$. This however will fail to be true in general. To overcome this difficulty we restrict our considerations to 1–weightings which are either *positive* i.e. $\chi^{-1}(-1)=\emptyset$ or ℓ *large* i.e. $\chi(V)\geq\ell$.

Definition. A weighting partition Θ is called ℓ *admissible* if Θ consisting entirely of positive and ℓ large 1–weightings.

If Θ is a ℓ admissible weighting partition and $\mathcal{P}$ a ℓ path family then we conclude

$$\sum_{P\in\mathcal{P}}|P|=\sum_{\chi\in\Theta}\Big(\sum_{P\in\mathcal{P}}\chi(P)\Big)\ \leq \sum_{\chi,\ \ell\text{ large}}\ell\ +\sum_{\chi,\text{ positive}}\chi(V)\ \leq\sum_{\chi\in\Theta}\min(\chi(V),\ell).$$

Now consider the families $\mathcal{P}$ and Θ associated with the current flow-potential pair after the algorithm did climb up to the flow value $n-\ell$. Let Θ^+ be obtained from Θ by adding $1-\sum_{\chi\in\Theta}\chi(x)$ copies of the singular 1–weighting for each $x\in V$. The *singular* 1–weighting for x is defined by $\chi(y)=1$ if $x=y$ and $\chi(y)=0$ otherwise. Since all 1–weightings $\chi\in\Theta$ are ℓ large and the singular 1–weightings are positive we conclude: Θ^+ is a ℓ admissible weighting partition. From the orthogonality between Θ and the ℓ path family $\mathcal{P}$ we obtain

$$\sum_{P\in\mathcal{P}}|P|=\sum_{\chi\in\Theta^+}\Big(\sum_{P\in\mathcal{P}}\chi(P)\Big)=\sum_{\chi\in\Theta}\ell\ +\sum_{\chi\in\Theta^+\setminus\Theta}1=\sum_{\chi\in\Theta^+}\min(\chi(V),\ell).$$

This gives the theorem:

Theorem 9 *For any acyclic directed graph D and any positive integer ℓ*

$$\max\sum_{P\in\mathcal{P}}|P|=\min\sum_{\chi\in\Theta}\min(\chi(V),\ell)$$

where the maximum is taken over all ℓ path families $\mathcal{P}$ and the minimum over all ℓ admissible weighting partitions Θ of D.

A ℓ admissible weighting partition which minimizes the right hand side is called *ℓ-saturated.*

Theorem 10 *For any $\ell\geq 1$ there is a $(\ell+1)$ admissible weighting partition which is simultaneously ℓ-saturated and $(\ell+1)$-saturated.*

Proof. A simultaneously ℓ and $(\ell+1)$-saturated weighting partition is obtained from the step increasing the flow value from $n-\ell-1$ to $n-\ell$. From the current potential we get a weighting family Θ being orthogonal to both, a ℓ path family $\mathcal{P}_\ell$ and a $(\ell+1)$ path family $\mathcal{P}_{\ell+1}$. The weighting partition Θ^+ then is $(\ell+1)$ admissible as well as ℓ-saturated and $(\ell+1)$-saturated. □

To an acyclic directed graph D with n elements a partition λ of n is associated, such that for the Ferrers diagramm $G(D)$ corresponding to λ we get:

Theorem 11 *The number of squares in the ℓ longest columns of $G(D)$ equals the maximal value $\sum_{P\in\mathcal{P}}|P|$ of a ℓ path family $\mathcal{P}$ of D and the number of squares in the k longest rows of $G(D)$ equals the maximal value $\sum_{\chi\in\Theta}\chi(V)$ of a k weighting family Θ of D.*

Proof. The proof of this theorem is best accomplished by means of a diagram. Applying the Ford-Fulkerson algorithm to the Frank network of D we get a sequence of flow-potential pairs and thereby a sequence of flow-value – potential-value pairs (v,k). Show the pairs (v,k) as points in the diagram and connect two subsequent pairs by a line segment (cf. Figure 2). The diagram, thus obtained, depends on D only, not on the concrete run of the algorithm.

The claim is, that the shape of the 'squared' area in the diagram gives the Ferrer's diagram of a partition of n, meeting the properties stated by the theorem. Consider a flow increasing

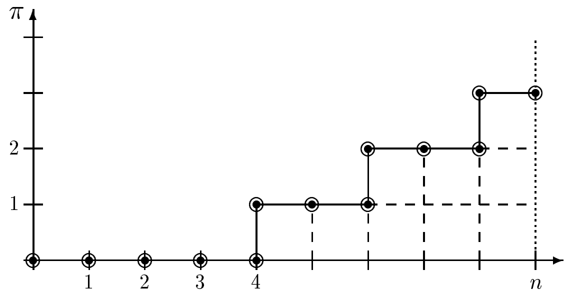

Figure 2: Kilter diagram of D.

step going from stage (v,k) to $(v+1,k)$. Associated we find a k weighting family Θ_k and $\ell = n-v$, respectively $\ell-1$ path families $\mathcal{P}_\ell$ and $\mathcal{P}_{\ell-1}$. Since Θ_k is orthogonal to both families we may use the construction, we used in proving theorem 9, to obtain

$$\sum_{P\in\mathcal{P}_\ell}|P| = \sum_{\chi\in\Theta_k}\ell \; + \sum_{\chi\in\Theta_k^+\setminus\Theta_k} 1 \quad\text{and}\quad \sum_{P\in\mathcal{P}_{\ell-1}}|P| = \sum_{\chi\in\Theta_k}(\ell-1) \; + \sum_{\chi\in\Theta_k^+\setminus\Theta_k} 1$$

The values $\sum_{P\in\mathcal{P}_\ell}|P|$ and $\sum_{P\in\mathcal{P}_{\ell-1}}|P|$ are maximal for ℓ and $(\ell-1)$ path families respectively and differ by k; but this is just the number of squares below the line segment $(v,k)\to(v+1,k)$. Starting from the empty path family we inductively obtain that the number of squares in the ℓ longest columns equals the maximal value $\sum_{P\in\mathcal{P}}|P|$ of a ℓ path family $\mathcal{P}$.

A n path family certainly will contain all of D, hence, there are exactly n 'squares' in the kilter diagram of D

Now consider a potential increasing step going from (v,k) to $(v,k+1)$. Associated we have a $\ell = n-v$ path family $\mathcal{P}_\ell$ and k, respectively $(k+1)$ weighting families Θ_k and Θ_{k+1}. Since $\mathcal{P}_\ell$ is orthogonal to both Θ_k and Θ_{k+1} we may use theorem 7 to obtain

$$\sum_{\chi\in\Theta_k}\chi(V) = \sum_{P\in\mathcal{P}_\ell}k \; + \sum_{P\in\mathcal{P}_\ell^+\setminus\mathcal{P}_\ell} 1 \quad\text{and}\quad \sum_{\chi\in\Theta_{k+1}}\chi(V) = \sum_{P\in\mathcal{P}_\ell}(k+1) \; + \sum_{P\in\mathcal{P}_\ell^+\setminus\mathcal{P}_\ell} 1.$$

Therefore the weights of Θ_k and Θ_{k+1} differ by ℓ. The number of squares lying to the right of the line segment $(v,k)\to(v,k+1)$ is ℓ as well. We get that the number of squares in the k longest rows of the diagram equals the maximal value $\sum_{\chi\in\Theta}\chi(V)$ of a k weighting family Θ. □

Remark. In the introduction we have sketched how a duality theorem for 1-weightings and chains can be used to derive Dilworth's theorem. We can use a similar observation to recognize theorems 1 - 5 as instances of 7 - 11:

In a partially ordered set $P = (V,<)$ every 1-weighting contained in a k weighting family Θ maximizing $\sum_{\chi\in\Theta}\chi(V)$ is positive, hence is the characteristic function of an antichain. Therefore maximal k weighting families in P are k antichain families.

3.2 The General Case

In the sequel let D be an arbitrary directed graph. As we have seen in theorem 6 the Ford-Fulkerson algorithm gives a sequence of orthogonal pairs in D. Hence we may trace through section 3.1 to see which statements made there can be adapted to the present case.

Let C be a cycle and Θ a k weighting family then $\sum_{\chi\in\Theta}\chi(C)\le 0$. For a path P we again obtain $\sum_{\chi\in\Theta}\chi(P)\le\min(|P|,k)$. Hence for a path/cycle partition $\mathcal{W}$, consisting of its path subfamily $\mathcal{P}$ and its cycle subfamily $\mathcal{C}$, we get $\sum_{\chi\in\Theta}\chi(V)\le\sum_{P\in\mathcal{P}}\min(|P|,k)$.

Now, fill up a family $\mathcal{W} = (\mathcal{P},\mathcal{C})$, being orthogonal to a k weighting family Θ, with 1-element paths. We obtain the path/cycle partition $\mathcal{W}^+ = (\mathcal{P}^+,\mathcal{C})$. Since, for $C\in\mathcal{C}$ we have

$\sum_{\chi\in\Theta}\chi(C) = 0$, we get $\sum_{\chi\in\Theta}\chi(V) = \sum_{P\in\mathcal{P}+}\min(|P|,k)$ and have proven the following theorem.

Theorem 12 *For any directed graph D and any positive integer k*

$$\max\sum_{\chi\in\Theta}\chi(V) = \min\sum_{P\in\mathcal{P}}\min(|P|,k)$$

where the maximum is taken over all k weighting families Θ and the minimum over all path/cycle partitions $\mathcal{W} = (\mathcal{P},\mathcal{C})$ of D.

A path/cycle partition $\mathcal{W}$ which minimizes the right hand side is called *k-saturated.* With a proof similar to the proof of theorem 8 we get:

Theorem 13 *For any $k \geq 1$ there is a path/cycle partition which is simultaneously k-saturated and $(k+1)$-saturated.*

A major distinction to the acyclic case is, that now a flow of value n will not have cost n. In fact, the cost will be $\alpha(D) = |V| - \max|\bigcup_{C\in\mathcal{C}} C|$, the maximum being taken over all families of disjoint cycles. The parameter $\alpha(D)$ is called the *acyclicity* of D.

Since 1-weightings have $\chi(C) \leq 0$ for all cycles C, a set Θ of 1-weightings with $\sum_{\chi\in\Theta}\chi(x) \leq 1$ will nessesarily fulfill $\sum_{\chi\in\Theta}\chi(V) \leq \alpha(D)$. A directed graph D, containing cycles, therefore will not admit weighting partitions in the sense of section 3.1. We may ask, however, for partitions of the 'acyclic part' of D.

Definition. A *weighting partition* Θ of a directed graph $D = (V,E)$ is a set of 1-weightings with

1) $\displaystyle\sum_{\chi\in\Theta}\chi(V) = \alpha(D)$ 2) $\displaystyle\sum_{\chi\in\Theta}\chi(x) \leq 1$ for all $x \in V$.

The existence of a weighting partition is established by the weighting family Θ, corresponding to the current potential while the flow increases from $n-1$ to n. It seems unlikely, however, that all weighting families, orthogonal to some ℓ path/cycle family, can be augmented to ℓ admissible weighting partitions. Therefore, the question whether generalizations of theorems 9-11 to arbitrary directed graphs exist, has to remain open.

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