Proof of Berge's Strong Path Partition Conjecture for $k = 2$

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Abstract. Berge's strong path partition conjecture from 1982 generalizes and extends Dilworth's Theorem and the Greene-Kleitman Theorem which are well known for partially ordered sets. The conjecture is known to be true for all digraphs only for $k = 1$ (by the Gallai - Milgram Theorem) and for $k \geq \lambda$ (where λ is the cardinality of the longest path in the graph). The attempts made, so far, to prove the conjecture for other values of k have yielded proofs for acyclic digraphs, but not for general digraphs. In this paper, we prove the conjecture for $k = 2$ for all digraphs. The proof is constructive and it extends the proof for $k = 1$.

1. Introduction and Terminology

Let $G = (V, E)$ be a directed graph containing no loops or multiple edges, defined by a set V of vertices and a set $E \subseteq V \times V$ of directed edges. A *path* P in G is a sequence of distinct vertices $(v_1, v_2, \ldots, v_l)$ such that $(v_i, v_{i+1}) \in E$, for $i = 1, 2, \ldots, l-1$. We write $v_i <_P v_j$ if $1 \leq i < j \leq l$. The set of vertices $\{v_1, v_2, \ldots, v_l\}$ of a path $P = (v_1, v_2, \ldots, v_l)$ is denoted by $V(P)$, and the set of edges by $E(P)$. The *cardinality* of P, denoted by $|P|$, is $|V(P)|$. A path of cardinality one is called a *trivial* path. For any path in G, (directed or undirected) we denote the first vertex by $in(P)$ and the last vertex by $ter(P)$.

For a set of vertex disjoint paths $\mathcal{P}$, we write $V[\mathcal{P}] = \bigcup\{V(P); P \in \mathcal{P}\}$, $E[\mathcal{P}] = \bigcup\{E(P); P \in \mathcal{P}\}$, $in[\mathcal{P}] = \{in(P); P \in \mathcal{P}\}$, and $ter[\mathcal{P}] = \{ter(P); P \in \mathcal{P}\}$. We write $u >_{\mathcal{P}} v$ if $u >_P v$ for some $P \in \mathcal{P}$.

A family $\mathcal{P}$ of paths is called a *path partition* of G if its members are vertex disjoint and $\cup\{V(P); P \in \mathcal{P}\} = V$. A directed graph, or for brevity, *digraph*, may have many path partitions. The *trivial* path partition, where every path is a trivial path, is an example of a path partition. For each nonnegative integer k, the k-norm $|\mathcal{P}|_k$ of a path partition $\mathcal{P} = \{P_1, \ldots, P_m\}$ is defined by

$$|\mathcal{P}|_k = \sum_{i=1}^{m} min\{|P_i|, k\}. \tag{1}$$

A partition which minimizes $|\mathcal{P}|_k$ is called *k-optimal*. Note that a 1-optimal path partition is a partition that contains a minimum number of paths and $|\mathcal{P}|_1 = |\mathcal{P}|$. Denote by $\mathcal{P}^{\geq k}$ the set of paths in $\mathcal{P}$ of cardinality at least k, (which we also call *long* paths), and by $\mathcal{P}^{<k}$ the set of paths in $\mathcal{P}$ of cardinality less than k, (called *short* paths). Then equation (1) can be alternatively written as

$$|\mathcal{P}|_k = \sum_{i=1}^{m} min\{|P_i|, k\} = k|\mathcal{P}^{\geq k}| + |V[\mathcal{P}^{<k}]|$$

Given a path partition $\mathcal{P}$, denote by $x^+_{\mathcal{P}}$, or for short, x^+ the vertex (if exists) which succeeds x on $\mathcal{P}$. Similarly, $x^-_{\mathcal{P}}$ (or briefly, x^-) is the vertex (if exists) which precedes x on $\mathcal{P}$. For a subset $X \subseteq V$, denote by $X^+ = \{x^+; x \in X\}$ and $X^- = \{x^-; x \in X\}$.

For convenience, we draw all paths in a path partition vertically downwards, i.e. x^- is drawn above x (see Figure 1).

A *k-colouring* is a family $\mathcal{C}^k = \{C_1, C_2, \ldots, C_k\}$ of k disjoint independent sets called *colour classes*. (Some of the colour classes may be empty). The *cardinality* of a k-colouring is the sum of the sizes of the colour classes, i.e., $|\mathcal{C}^k| = \sum_{i=1}^{k} |C_i|$ and $\mathcal{C}^k$ is said to be *optimal* if $|\mathcal{C}^k|$ is as large as possible. A path partition $\mathcal{P}$ and a k-colouring $\mathcal{C}^k$ are *orthogonal* if every path P_i in $\mathcal{P}$ meets $min\{|P_i|, k\}$ different colour classes of $\mathcal{C}^k$. Note that this is the maximum number of different colour classes that a path can intersect a k-colouring.

Conjecture 1.1 (Berge's strong path partition conjecture [3]**).** *Let G be a digraph and let k be a positive integer. Then for every k-optimal path partition $\mathcal{P}$ there exists a k-colouring orthogonal to it.*

The conjecture holds for $k = 1$ for all digraphs by the Gallai-Milgram Theorem [7]. Berge's strong path partition conjecture has also been proved for acyclic digraphs (see [5], [12], [2], and [1]). It is not difficult to see, as was shown in [3], that the conjecture is true when $k \geq \lambda$, where λ is the cardinality of the longest path in G, and when the k-optimal path partition contains only short paths, (i.e., paths of cardinality less than k). In [1] it was proved that the conjecture holds also in the case that the given k-optimal path partition contains only long paths, i.e., $\mathcal{P} = \mathcal{P}^{\geq k}$. For a survey of Berge's conjecture and related problems see [9]. See also [6], [13], and [14] for related results.

Denote by $\alpha_k(G)$ the cardinality of an optimal k-colouring in G, and by $\pi_k(G)$ the k-norm of a k-optimal path partition in G. Conjecture 1.1 implies the following:

Conjecture 1.2 (Weak path partition conjecture -Linial [10]**).** *For any digraph G and positive integer k, $\alpha_k(G) \geq \pi_k(G)$.*

If G is transitive and acyclic (i.e., the graph of a partially ordered set), then Green-Kleitman's theorem [8] states that $\alpha_k(G) = \pi_k(G)$, implying Conjecture 1.2.

In this paper we will prove Conjecture 1.1 for $k = 2$ for all graphs. In Section 1 we give a proof of the conjecture for $k = 1$ which will set the foundations for the proof for $k = 2$. From here on, for a path partition $\mathcal{P}$, we denote the set of non-trivial paths as $\mathcal{P}^{>1}$ and the set of trivial paths as $\mathcal{P}^1$.

2. Proof for $k = 1$, the Gallai-Milgram Theorem

An *undirected trail* Q in G is a sequence $Q = (v_0, e_1, v_1, e_2, v_2, \ldots, e_l, v_l)$ such that, for each $1 \leq i \leq l$, either $e_i = (v_{i-1}, v_i) \in E(G)$ or $e_i = (v_i, v_{i-1}) \in E(G)$, and all the edges are distinct. We assign a direction to Q from v_0 to v_l, so that if $e_i = (v_{i-1}, v_i)$ then e_i is considered a *forward* edge, and if $e_i = (v_i, v_{i-1})$ then e_i is considered a *backward* edge. For trails $Q_1 = (v_0, \ldots, v_s)$ and $Q_2 = (v_s, \ldots, v_l)$ we denote by $Q_1 * Q_2 = (v_0, \ldots, v_s, \ldots, v_l)$ the concatenation of Q_1 and Q_2.

Definition 2.1 (alternating trail, proper alternating trail). *Given a path partition $\mathcal{P}$, an undirected trail $Q = (v_0, e_1, v_1, e_2, v_2, \ldots, e_l, v_l)$ is* alternating relative to $\mathcal{P}$ *if the following two conditions hold:*

- *All forward edges of Q, denoted by F_Q, are in $E(G) - E[\mathcal{P}]$, all backward edges, denoted by B_Q are in $E[\mathcal{P}]$, and every forward edge (u, v), where $v \in V[\mathcal{P}^{>1}]$, is followed by a backward edge, unless $v \in in[\mathcal{P}^{>1}]$ and $v = v_l$.*

- *For every vertex $v \in V(Q)$ there exists at most one forward edge $(u,v) \in E(Q)$, entering v, and at most one forward edge $(v,w) \in E(Q)$, leaving v.*

An alternating trail is proper *if $v_0 \in ter[\mathcal{P}]$. An alternating cycle is an alternating trail $Q = (v_0, e_1, v_1, e_2, v_2, \ldots, e_l, v_l)$ where $v_0 = v_l$ and if $v_l \in in[\mathcal{P}^{>1}]$ then necessarily e_l is a backward edge.*

We denote by $\mathcal{P}\Delta Q$ the spanning subgraph of G containing edges in the symmetric difference $E[\mathcal{P}]\Delta E(Q) = E[\mathcal{P}] \cup E(Q) \setminus (E[\mathcal{P}] \cap E(Q))$. The following claim is self evident from the definition above:

Claim 2.2. *Let Q be either a proper alternating trail or an alternating cycle relative to $\mathcal{P}$. Then*

1. *$\mathcal{P}' = \mathcal{P}\Delta Q$ is a collection of disjoint paths and cycles in G.*
2. *If $\mathcal{P}'$ contains no cycles then $|\mathcal{P}'| - |\mathcal{P}| = (n - |E(\mathcal{P}')|) - (n - |E(\mathcal{P})|) = |E(\mathcal{P})| - |E(\mathcal{P}')| = |B_Q| - |F_Q|$.*

Definition 2.3 (1-Transversal). *Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a path partition in G. A set of vertices $Y \subseteq V$ is a* 1-transversal, *(or a* transversal*), of $\mathcal{P}$ if for each $1 \leq i \leq m$, $|Y \cap V(P_i)| \geq 1$. Denote by $first_{\mathcal{P}}(Y)$ the set of vertices in Y which appear first on each path in $\mathcal{P}$.*

To prove the Gallai-Milgram theorem we shall show that every optimal path partition $\mathcal{P}$ contains a 1-transversal Y such that $X = first_{\mathcal{P}}(Y)$ is an independent set. The proof is constructive: Given any path partition $\mathcal{P}$, then either we find a transversal Y where $X = first_{\mathcal{P}}(Y)$ is an independent set, or we find an alternating trail Q such that $\mathcal{P}' = \mathcal{P}\Delta Q$ is a path partition with $|\mathcal{P}'| < |\mathcal{P}|$.

For ease of notation, if $Y \subseteq S$, and $x \in S$, then we use the short notation $Y + x$ (or $Y - x$) to denote $Y \cup \{x\}$ (or $Y \setminus \{x\}$).

Definition 2.4 (1-snapshot). *Let $\mathcal{P}$ be path partition in G. A* 1-snapshot *in $\mathcal{P}$ is the quadruple (Y, A, p, t) where Y is a transversal of $\mathcal{P}$, $p : Y \to Y$, is a function, called the* predecessor function *of a 1-transversal Y, $A \subseteq E$ is a subset called* admissible edges, *and $t : Y \cup A \to \mathbb{N}$ is a function called* time function. *A 1-snapshot is initialized and updated as defined below:*

2.1. Initialize snapshot

1. Transversal: $Y \leftarrow ter[\mathcal{P}]$
2. Admissible edges: $A \leftarrow \emptyset$
3. Predecessor: $p(y) \leftarrow y$, *for all* $y \in Y$
4. Time function: $t(y) = 0$ *for all* $y \in Y$

2.2. Update snapshot

Let $X = first_{\mathcal{P}}(Y)$. For $e = (u,v) \in E, u, v \in X$ and $v \notin in[\mathcal{P}]$ let:
1. Transversal: $Y \leftarrow Y + v^-$
2. Admissible edges: $A \leftarrow A + e$
3. Predecessor: $p(v^-) \leftarrow u$
4. Time function: $t(v^-) = t(e) \leftarrow t_{max} + 1$ *(where t_{max} denotes the maximum value of the function t.)*

Lemma 2.5 (Properties of a Snapshot). *A 1-snapshot (Y, A, p, t) has the following properties:*

1. *Y is a transversal of $\mathcal{P}$.*
2. *If $e = (u,v) \in A$ then $u, v \in Y$ and $t(e) > t(u), t(e) > t(v)$.*
3. *If $u, v \in Y$ and $v >_{\mathcal{P}} u$ then $t(u) > t(v)$.*

4. *If* $e = (u, v), f = (w, x) \in A$*,* $u, v, w, x \in Y$ *and* $w \geq_{\mathcal{P}} v$ *then* $t(e) > t(f)$*.*
5. *If* $e = (u, v) \in A$ *then* $p(v^-) = u$
6. *Every* $u \in Y$ *satisfies one of the following:*
   - $p(u) = u \in ter[\mathcal{P}]$
   - $p(u) \in Y$*,* $e = (p(u), u^+) \in A$ *and* $t(u) = t(e)$
7. *The graph induced by* $A \cup E[\mathcal{P}]$ *is acyclic; furthermore, for* $u, v \in first_{\mathcal{P}}(Y)$*, the graph induced by* $A \cup E[\mathcal{P}] \cup \{(u, v)\}$ *is acyclic.*

Proof:

1. By definition of a snapshot.
2. Follows from the fact that the time counter is increased every time a new edge is added to A.
3. Follows from Section 2.2 (update snapshot), and the fact that in each update v^- is added to Y only when $v \in first_{\mathcal{P}}(Y)$.
4. Assume, by contradiction, that $t(f) > t(e)$. By update snapshot, once e is added to A, v^- enters Y, and clearly $w^- >_{\mathcal{P}} v$. Hence $w \notin first_{\mathcal{P}}(Y)$, and it is impossible that at a later stage the edge $f = (w, x)$ is chosen to be added to A.
5. Follows directly from line 3 in Section 2.2 (update snapshot).
6. Follows directly from lines 2-4 in 2.2 and line 1 in 2.1
7. Follows easily from 2.2
8. Since u is the first vertex in some path $P \in \mathcal{P}$ that belongs to Y, no edge of A can enter P in u or in a vertex that precedes u in P. Therefore, no cycle in $A \cup E[\mathcal{P}] \cup \{(u, v)\}$ can contain u. Assume, by contradiction, that $A \cup E[\mathcal{P}]$ contains a cycle C. Then C is of the form $C = (e_1 \overline{P_2} e_2 \overline{P_3} \ldots e_l \overline{P_1})$, where $t \geq 2$, $\overline{P_i}$ is a subpath of $P_i \in \mathcal{P}$ ($\overline{P_i}$ may be either a single vertex, or a non-trivial subpath of P_i or equal to P_i), and $e_i \in A$ for all $1 \leq i \leq t$. (See Figure 1). By (4), it follows that $t(e_1) > t(e_2) > \ldots > t(e_l) > t(e_1)$, a contradiction.

■

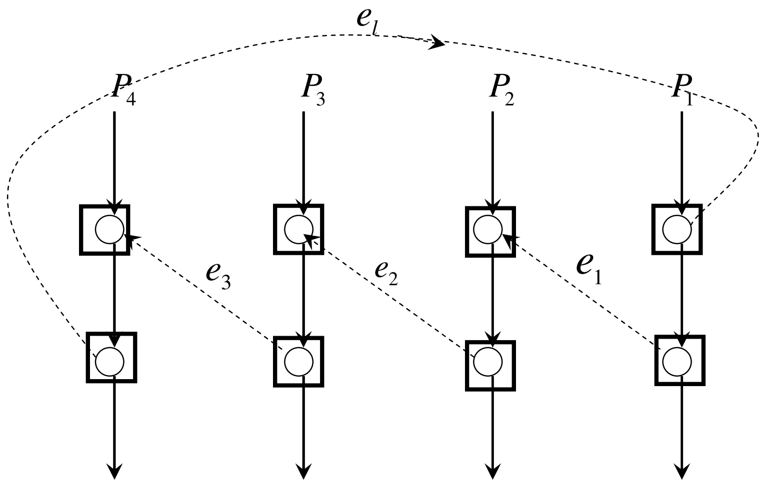

FIGURE 1. a cycle

Since every vertex $u \in Y$ has a unique predecessor $p(u)$, it is possible to retrieve a trail from u back to $p(u)$, back to $p(p(u))$, etc. until we end up with a vertex in $ter[\mathcal{P}]$. The next definition formalizes this process:

Definition 2.6 (Path Segment in a snapshot). *Let (Y, A, p, t) be a snapshot. For a vertex $u \in Y$, an* alternating path segment *$Q_1(u)$ of u is defined as follows:*

1. *If $p(u) = u$ then $Q_1(u)$ is the trivial alternating path (u).*
2. *If $u \neq p(u)$ then $Q_1(u)$ is the alternating trail (in fact, a path) $(p(u), u^+, u)$ with one forward edge (the edge $(p(u), u^+) \in A$) and one backward edge (the edge $(u, u^+) \in E(\mathcal{P})$).*

We write $Q_k(u) = Q_1(p^{k-1}(u) * Q_1(p^{k-2}(u)) \ldots * Q_1(p(u)) * Q_1(u)$. Note that $|B_{Q_1(u)}| = |F_{Q_1(u)}|$ and hence $|B_{Q_k(u)}| = |F_{Q_k(u)}|$. It is easy to check the following:

Claim 2.7. *$Q_k(u)$ is a subgraph of $A \cup E[\mathcal{P}]$ and is an alternating trail relative to $\mathcal{P}$, as defined in Definition* 2.1.

Lemma 2.8. *Let $\mathcal{P}$ be a path partition and (Y, A, p, t) a 1-snapshot in $\mathcal{P}$. If there exists an edge $e = (u, v)$ with $u, v \in first_{\mathcal{P}}(Y)$ and $v \in in[\mathcal{P}]$ then $\mathcal{P}$ is not optimal.*

Proof: Let k be such that $p^k(u) = p^{k+1}(u)$ and consider the proper alternating trail $Q = Q_k(u) * e$.

Let $\mathcal{P}' = \mathcal{P} \Delta Q$. Since $\mathcal{P}'$ is a subgraph of $A \cup E[\mathcal{P}] \cup \{e\}$ which is acyclic (as was proved in Lemma 2.5-(7)), it follows that $\mathcal{P}'$ is a path partition. By Claim 2.2, $|\mathcal{P}'| - |\mathcal{P}| = |B_Q| - |F_Q| = |B_{Q_k(u)}| - |F_{Q_k(u)}| - 1 = -1$.

■

Theorem 2.9 (Gallai-Milgram [7]**).** *Let $\mathcal{P}$ be a path partition in a graph G. Then either there exists an independent transversal of $\mathcal{P}$, or there exists a path partition $\mathcal{P}'$ satisfying $|\mathcal{P}'| < |\mathcal{P}|$.*

Proof: We give an algorithmic proof based on the lemmas above.

2.3. Algorithm

0. **Input:** G, $\mathcal{P}$
1. **Initialize snapshot:** as in Section 2.1. Let $X = first_{\mathcal{P}}(Y)$.
2. **while** $(\exists e = (u, v), u, v \in X, v \notin in[\mathcal{P}])$ **do**
3. Update snapshot as in Section 2.2
4. **if** $(\exists e = (u, v), u, v, \in X, v \in in[\mathcal{P}])$
5. find $Q \leftarrow Q_k(u) * e$
6. $\mathcal{P}' \leftarrow \mathcal{P} \Delta Q$
7. **stop**
8. **else** X is an independent transversal

In line 6 $|\mathcal{P}'| < |\mathcal{P}|$ by Lemma 2.8, and in line 8, X is independent since any edge $e = (u, v), u, v \in X, v \notin in[\mathcal{P}]$ is used to update and increase the snapshot. Once the snapshot is maximal with respect to inclusion, and no edge $e = (u, v), u, v, \in X, v \in in[\mathcal{P}]$ exists, X is an independent transversal. ■

3. Proof for $k = 2$

3.1. Outline of the proof:

We begin by defining a 2-transversal and a 2-snapshot. Unlike the definition of a 1-snapshot, we do not define a 2-snapshot constructively, but by its properties. We continue with an example of an initial 2-snapshot (Example 3.3). If this 2-snapshot contains a 2-colouring orthogonal to $\mathcal{P}$, then we are done. Otherwise, as for $k = 1$, we either show how to update the 2-snapshot, or find a *'breakthrough'*, an alternating trail Q in $A \cup E[\mathcal{P}]$ that produces a path partition with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$. In Lemma 3.7 we show how a 2-snapshot can be updated. It is similar to the update in Section 2.2, and it is a *'direct'* update. We then make two observations (3.5, and 3.6) and establish in Lemma 3.7 that $A \cup E[\mathcal{P}]$ is acyclic in order to prove that $\mathcal{P}' = \mathcal{P}\Delta Q$ is a path partition. We continue by setting up the machinery for defining an alternating trail Q (Definitions 3.8 and 3.10) and for finding one (Lemma 3.11). A final step is to define an *'update with regret'* of a 2-snapshot - Definition 3.12, which is applied in results 3.13 and 3.14. Theorems 3.15 and 3.16 complete the proof.

Definition 3.1 (2-Transversal). *Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a path partition in G. The sets $Y_1, Y_2 \subseteq V$, $Y_1 \cap Y_2 = \emptyset$ form a* 2-transversal *of $\mathcal{P}$ if for each $1 \le i \le 2$, and for each $P \in \mathcal{P}^{>1}$, $|Y_i \cap V(P)| \ge 1$, and $V[\mathcal{P}^1] \subseteq Y_1 \cup Y_2$.*

Denote by $X_i = first_{\mathcal{P}}(Y_i), i = 1, 2$ the sets of vertices in Y_i which appear first on each path in $\mathcal{P}$.

For a vertex u write $c(u) = i$ if $u \in Y_i$ and $c(u) = 0$ if $u \notin Y_1 \cup Y_2$.

Note that since every path in $\mathcal{P}^1$ is trivial, then, naturally, $V[\mathcal{P}^1] \subseteq X_1 \cup X_2$. Therefore X_1, X_2 is also a 2-transversal.

We shall give a constructive proof for the case $k = 2$ in Conjecture 1.1 similar to the case $k = 1$, though the algorithm is more complicated: Given a path partition $\mathcal{P}$, either we find a 2-transversal $Y_1 \cup Y_2$ such that $X_1 \cup X_2$ is a 2-colouring orthogonal to $\mathcal{P}$, or we find an alternating trail Q such that $\mathcal{P}' = \mathcal{P}\Delta Q$ is a path partition with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$. Unlike the case $k = 1$, we prefer to define a 2-snapshot by its properties. Its properties naturally extend the properties proven in Lemma 2.5.

Definition 3.2 (2-snapshot). *Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a path partition in G. Let Y_1, Y_2 be subsets of $V(G)$, let A be a subset of $E(G)$, let $p : Y_1 \cup Y_2 \to Y_1 \cup Y_2$ and $t : Y_1 \cup Y_2 \cup A \to \mathbb{N}$ be functions. We say that $\mathcal{S} = (Y_1, Y_2, A, p, t)$ is a* 2-snapshot in $\mathcal{P}$ *if the following properties hold:*

1. *Y_1, Y_2 is a 2-transversal of $\mathcal{P}$.*
2. *If $e = (u, v) \in A$ then $u \in Y_1 \cup Y_2$ and $t(e) > t(u)$.*
3. *If $c(u) = c(v) \neq 0$ and $v >_{\mathcal{P}} u$ then $t(u) > t(v)$.*
4. *If $e = (u, v), f = (w, x) \in A$, $c(u) = c(w)$ and $w \ge_{\mathcal{P}} v$ then $t(e) > t(f)$.*
5. *If $(u, v) \in A$ then one of the following holds:*
   - *$u \in Y_i$, $v \in V[\mathcal{P}^{>1}] \setminus in[\mathcal{P}]$, $v^- \in Y_i$ for some $i = 1, 2$ and $p(v^-) = u$*
   - *$u \in Y_2$, $v \in Y_1 \cap V[\mathcal{P}^1]$ and $p(v) = u$*
6. *Every $u \in Y_2$ satisfies one of the following:*
   - *$p(u) \in Y_2$, $e = (p(u), u^+) \in A$ and $t(u) = t(e)$*
   - *$p(u) = u \in V[\mathcal{P}^1]$*
   - *$p(u) = u^+ \in Y_1$ and $t(u) = t(u^+)$*
7. *Every $u \in Y_1$ satisfies one of the following:*
   - *$p(u) \in Y_1$ and $e = (p(u), u^+) \in A$ and $t(u) = t(e)$*
   - *$p(u) = u \in ter[\mathcal{P}^{>1}]$*
   - *$u \in V[\mathcal{P}^1]$ and $(p(u), u) \in A$*
8. *If $P \in \mathcal{P}^{>1}$ and $x \in first_{\mathcal{P}}(Y_1) \cap V(P)$, then $x^- \in Y_2$ and $p(x^-) = x$.*
9. *For each $P \in \mathcal{P}^{>1}$, all the vertices of Y_2 come before all the vertices of Y_1 on P.*

Example 3.3. *Let $Y_1 = ter[\mathcal{P}^{>1}]$, $Y_2 = Y_1^- \cup \mathcal{P}^1$, $A = \emptyset$, $t \equiv 0$ and define $p(u) = u$ for $u \in Y_1 \cup V[\mathcal{P}^1]$ and $p(u) = u^+$ for $u \in Y_2 \cap V[\mathcal{P}^{>1}]$. Then (Y_1, Y_2, A, p, t) is a 2-snapshot in $\mathcal{P}$.*

The example above describes an initial 2-snapshot $\mathcal{S}$. If $X_i = first_{\mathcal{P}}(Y_i)$ for $i = 1, 2$ is an independent set, then it is a 2-colouring orthogonal to $\mathcal{P}$, and we are done. Otherwise, we shall update $\mathcal{S}$. In the next lemma we show how $\mathcal{S}$ can be updated in an 'easy' way.

Lemma 3.4 (Direct update). *Let $\mathcal{S} = (Y_1, Y_2, A, p, t)$ be a 2-snapshot. Let $X_i = first_{\mathcal{P}}(Y_i)$ for $i = 1, 2$. In the following cases $\mathcal{S}$ can be updated, remaining a 2-snapshot, where either Y_1 is increased, or Y_1 remains the same and Y_2 is increased.*

1. *If there exists an edge $e = (u, v), u, v \in X_2, v \in V[\mathcal{P}^{>1}], v \notin in[\mathcal{P}^{>1}]$ then the following update defines another snapshot where Y_1 is unchanged and Y_2 is increased:*
   - $Y_2 \leftarrow Y_2 + v^-$
   - $p(v^-) \leftarrow u$
   - $A \leftarrow A + e$
   - $t(v^-) = t(e) \leftarrow t_{max} + 1$
2. *If there exists an edge $e = (u, v)$, $u, v \in X_1$, $v, v^- \in V[\mathcal{P}^{>1}] \setminus in[\mathcal{P}^{>1}]$, $v^- \in Y_2$ and no edges of A leave v^- then in the following update Y_1 is increased.*
   - $Y_1 \leftarrow Y_1 + v^-$
   - $Y_2 \leftarrow Y_2 - v^- + v^{--}$
   - $p(v^-) \leftarrow u$; $p(v^{--}) \leftarrow v^-$
   - $A \leftarrow A + e$
   - $t(v^-) = t(v^{--}) = t(e) \leftarrow t_{max} + 1$
3. *If there exists an edge $e = (u, v)$, $u, v \in X_2$, $v \in V[\mathcal{P}^1]$ and no edges of A leave v then in the following update Y_1 is increased.*
   - $Y_1 \leftarrow Y_1 + v$
   - $Y_2 \leftarrow Y_2 - v$
   - $p(v) \leftarrow u$
   - $A \leftarrow A + e$
   - $t(v) = t(e) \leftarrow t_{max} + 1$

Proof: It is routine to check that properties 1-9 in Definition 3.2 are maintained after the update. ■

Observation 3.5. *Let (Y_1, Y_2, A, p, t) be a 2-snapshot on $\mathcal{P}$, then there is no path beginning in a vertex of Y_1 and ending in a vertex of Y_2 with all edges in $A \cup E[\mathcal{P}]$.*

Proof: Assume, by contradiction, that $P = (v_1, \ldots, v_l)$ is some path with all edges in $A \cup E[\mathcal{P}]$, where $v_1 \in Y_1$ and $v_l \in Y_2$. By taking the shortest such path, we may assume $v_2, \ldots, v_{l-1} \notin Y_1 \cup Y_2$. By part 9 of Definition 3.2 $E(P) \not\subseteq E[\mathcal{P}]$, so P must contain at least one edge from A. By part 2 of Definition 3.2, the only edge in P that can be from A is (v_1, v_2), and the rest are in $E[\mathcal{P}]$. By part 5 of Definition 3.2 we have $v_2 \in V[\mathcal{P}^{>1}] \setminus in[\mathcal{P}]$ and $v_2^- \in Y_1$. But v_2^- comes before v_l in $\mathcal{P}$. This is a contradiction to part 9 of Definition 3.2. ■

Observation 3.6. *Let (Y_1, Y_2, A, p, t) be a 2-snapshot on $\mathcal{P}$ and let $x_1, x_2 \in first_{\mathcal{P}}(Y_j)$ for some $j = 1$ or 2. Then the graph induced by $E[\mathcal{P}] \cup \{e = (u, v) \in A;\ u \in Y_j\} \cup \{(x_1, x_2)\}$ is acyclic.*

Proof: The proof is similar to the proof of Lemma 2.5- (7). Since x_1 is the first vertex in its path of $\mathcal{P}$ that is in Y_j, no cycle in this graph can contain the vertex x_1. Therefore, no cycle contains the edge (x_1, x_2). Assume, by contradiction, that the graph induced by $E[\mathcal{P}] \cup \{e = (u, v) \in A;\ u \in Y_j\}$ contains a cycle and let $e_1, \ldots, e_s$ be the edges from A in it (ordered the way they appear in the cycle). Write $e_i = (u_i, v_i)$ and we have $\{u_1 \ldots, u_s\} \subseteq Y_j$. But then by part 4 of Definition 3.2 we have $t(e_1) > t(e_2) > \ldots > t(e_s) > t(e_1)$, a contradiction. ■

Lemma 3.7. *Let (Y_1, Y_2, A, p, t) be a 2-snapshot on $\mathcal{P}$. Then the graph induced by $A \cup E[\mathcal{P}]$ is acyclic. Furthermore, if $u, v \in first_{\mathcal{P}}(Y_j)$ for some $j = 1, 2$, then the graph H induced by $A \cup E[\mathcal{P}] \cup \{(u, v)\}$ is acyclic.*

Proof: Assume, by contradiction, that H contains a cycle and let $e_1, \ldots, e_s$ be the edges from A in it (ordered the way they appear in the cycle). Write $e_i = (u_i, v_i)$. By part 2 of Definition 3.2 we have $u_1, \ldots, u_s \in Y_1 \cup Y_2$. By the previous observation $\{u_1, \ldots, u_s\}$ intersects both Y_1 and Y_2. Let P be the shortest segment of this cycle starting at Y_1 and ending at Y_2. By the minimality of P, it cannot contain the edge (u, v). This is impossible by Observation 3.5. ■

Definition 3.8 (Path segment in a 2-snapshot). *Let (Y_1, Y_2, A, p, t) be a 2-snapshot. For a vertex $u \in Y_1 \cup Y_2$ we define the* alternating path segment *$Q_1(u)$ of u as follows:*

1. *If $p(u) = u$ then $Q_1(u)$ is the trivial alternating path (u).*
2. *If $c(u) = 1$ and $c(p(u)) = 2$ then $Q_1(u)$ is the alternating path $(p(u), u)$ with one forward edge (which is in A).*
3. *If $c(u) = 2$ and $c(p(u)) = 1$ then we have $p(u) = u^+$ and we define $Q_1(u)$ to be the alternating path (u^+, u) with one backward edge.*
4. *If $c(u) = c(p(u))$ but $u \neq p(u)$ then $Q_1(u)$ is the alternating path $(p(u), u^+, u)$ with one forward edge (the edge $(p(u), u^+) \in A$) and one backward edge (the edge $(u, u^+) \in E[\mathcal{P}^{>1}]$).*

We write $Q_k(u) = Q_1(p^{k-1}(u)) * Q_1(p^{k-2}(u)) \ldots * Q_1(p(u)) * Q_1(u)$.

Remark 3.9. *By the definition above, in all cases, a path segment satisfies $|B_{Q_1(u)}| - |F_{Q_1(u)}| = c(u) - c(p(u))$ and thus $|B_{Q_k(u)}| - |F_{Q_k(u)}| = c(u) - c(p^k(u))$.*

Definition 3.10 ($p^\infty(v)$, $p^{-\infty}(v)$). *Let $p^\infty(v) = \{p(v), p(p(v)), p^3(v), \ldots\} = \bigcup_{i=1}^{\infty}\{p^i(v)\}$.*

Define $p^{-1}(v) = \{u; p(u) = v\}$. Note that p is a function (not necessarily a bijection) and p^{-1} is the inverse relation.

Let $p^{-\infty}(v) = \bigcup_{i=1}^{\infty} p^{-i}(v)$.

The following lemma extends Lemma 2.8 to $k = 2$.

Lemma 3.11. *Let $\mathcal{P}$ be a path partition, and let $\mathcal{S} = (Y_1, Y_2, A, p, t)$ be a 2-snapshot. Let $e = (u, v)$ be an edge in G. Then $\mathcal{P}$ is not 2-optimal in the following cases: (see Figure 2)*

1. *$u, v \in X_2$ and $v \in in[\mathcal{P}^{>1}]$.*
2. *$u, v \in X_1$ and $v \in V[\mathcal{P}^1]$.*
3. *$u, v \in X_1$ and $v^- \in in[\mathcal{P}^{>1}]$.*
4. *$u, v \in X_1$, $v^- \in Y_2$ and $p^{-\infty}(v^-) \cap Y_1 \neq \emptyset$.*
5. *$u, v \in X_2$, $v \in V[\mathcal{P}^1]$ and $p^{-\infty}(v) \cap Y_1 \neq \emptyset$.*

Proof:

In each of the mentioned cases, by Lemma 3.7, the graph H induced by $A \cup E[\mathcal{P}] \cup \{e\}$ is acyclic. Thus, for any alternating trail Q in H, $\mathcal{P}' = \mathcal{P} \Delta Q$ is acyclic. We shall show in each case that Q can be found such that $|\mathcal{P}'|_2 < |\mathcal{P}|_2$

1. Let k be the smallest integer such that $p^k(u) = p^{k+1}(u)$ and consider the alternating trail $Q = Q_k(u) * e$. Then, by Remark 3.9,

$$|B_Q| - |F_Q| = |B_{Q_k(u)}| - |F_{Q_k(u)}| - 1 = c(u) - c(p^k(u)) - 1 = 1 - c(p^k(u)).$$

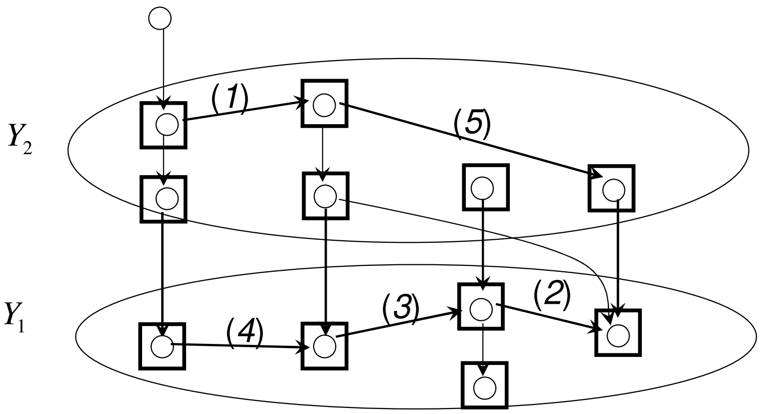

FIGURE 2. cases of a breakthrough

Let $\mathcal{P}' = \mathcal{P}\Delta Q$. Then

$$|\mathcal{P}'|_2 - |\mathcal{P}|_2 = |(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| + |\mathcal{P}'| - |\mathcal{P}| = |(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| + |B_Q| - |F_Q| =$$
$$= |(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| + 1 - c(p^k(u))$$

Hence we need to show that $|(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| + 1 - c(p^k(u)) < 0$. But this is indeed true. If $c(p^k(u)) = 1$ then Q is an alternating trail from $ter(\mathcal{P}^{>1})$ to $in(\mathcal{P}^{>1})$ and thus $|(\mathcal{P}')^{>1}| < |\mathcal{P}^{>1}|$. If $c(p^k(u)) = 2$ then Q is an alternating trail from $V(\mathcal{P}^1)$ to $in(\mathcal{P}^{>1})$, so we have $|(\mathcal{P}')^{>1}| = |\mathcal{P}^{>1}|$.

Note that the argument in this part is valid even if $p^j(u) = v$ for some $j \leq k$. In the following parts, however we have to be more careful.

2. Let k be minimal such that either $p^k(u) = p^{k+1}(u)$ or $p^k(u) = v$. Consider the alternating trail $Q = Q_k(u) * e$ and let $\mathcal{P}' = \mathcal{P}\Delta Q$. Similar to the previous part,

$$|\mathcal{P}'|_2 - |\mathcal{P}|_2 = |(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| + |B_Q| - |F_Q| = |(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| - c(p^k(u))$$

If $c(p^k(u)) = 1$ and Q is not a cycle, then Q is an alternating trail from $ter(\mathcal{P}^{>1})$ to $V[\mathcal{P}^1]$ and thus $|(\mathcal{P}')^{>1}| = |\mathcal{P}^{>1}|$. If $c(p^k(u)) = 2$, then Q is an alternating trail from $V[\mathcal{P}^1]$ to $V[\mathcal{P}^1]$, so we have $|(\mathcal{P}')^{>1}| = |\mathcal{P}^{>1}| + 1$. Finally, if $p^k(u) = v$, then Q is an alternating cycle with $c(p^k(u)) = 1$ and $|(\mathcal{P}')^{>1}| = |\mathcal{P}^{>1}|$. In all cases $|\mathcal{P}'|_2 - |\mathcal{P}|_2 < 0$.

3. Note that in this case we must have $c(v^-) = 2$. Let k be minimal such that either $p^k(u) = p^{k+1}(u)$ or $p^k(u) = v^-$. Consider the alternating trail $Q = Q_k(u) * (u, v, v^-)$ and let $\mathcal{P}' = \mathcal{P}\Delta Q$. With arguments similar to those of the previous parts we show that $|\mathcal{P}'|_2 - |\mathcal{P}|_2 = |(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| + |B_Q| - |F_Q| = |(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| + 1 - c(p^k(u))$.

If $p^k(u) = v^-$ or $p^k(u) \in V[\mathcal{P}^1]$ we have $|(\mathcal{P}')^{>1}| = |\mathcal{P}^{>1}|$ and $c(p^k(u)) = 2$. If $p^k(u) \in ter[\mathcal{P}^{>1}]$ we have $|(\mathcal{P}')^{>1}| = |\mathcal{P}^{>1}| - 1$ and $c(p^k(u)) = 1$. So in all cases, the inequality $|(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| + 1 - c(p^k(u)) < 0$ holds as required.

4. As in the previous case, let k be minimal such that either $p^k(u) = p^{k+1}(u)$ or $p^k(u) = v^-$. Assume first that $p^k(u) = v^-$. Consider the alternating cycle $Q = Q_k(u) * (u, v, v^-)$ and let $\mathcal{P}' = \mathcal{P}\Delta Q$. With arguments similar to those of the previous parts we show that $|\mathcal{P}'|_2 - |\mathcal{P}|_2 = |(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| + |B_Q| - |F_Q| = |(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| + c(u) - c(v^-) = -1$. Thus $\mathcal{P}$ is not optimal.

Assume now $p^k(u) \neq v^-$. Let j be minimal such that $p^j(w) = v^-$ for some $w \in Y_1$. By the minimality of j and part 7 of Definition 3.2 we have $w \in V[\mathcal{P}^1]$. Let $Q = Q_k(u) * (u, v, v^-) * Q_j(w)$ and let $\mathcal{P}' = \mathcal{P}\Delta Q$. Here we have,

$$
\begin{aligned}
|B_Q| - |F_Q| = |B_{Q_k(u)}| - |F_{Q_k(u)}| + |B_{Q_j(w)}| - |F_{Q_j(w)}| = \\
= c(u) - c(p^k(u)) + c(w) - c(v^-) = \\
= 1 - c(p^k(u)) + 1 - 2 = -c(p^k(u))
\end{aligned}
$$

Thus,

$$|\mathcal{P}'|_2 - |\mathcal{P}|_2 = |(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| + |B_Q| - |F_Q| = |(\mathcal{P}')^{>1}| - |\mathcal{P}^{>1}| - c(p^k(u))$$

If $p^k(u) \in V[\mathcal{P}^1]$ we have $|(\mathcal{P}')^{>1}| = |\mathcal{P}^{>1}| + 1$ and $c(p^k(u)) = 2$. If $p^k(u) \in ter[\mathcal{P}^{>1}]$ we have $|(\mathcal{P}')^{>1}| = |\mathcal{P}^{>1}|$ and $c(p^k(u)) = 1$. In both case we get $|\mathcal{P}'|_2 < |\mathcal{P}|_2$.

5. Let k be minimal such that either $p^k(u) = p^{k+1}(u)$ or $p^k(u) = v$. If $p^k(u) = v$, consider the alternating cycle $Q = Q_k(u) * e$. Otherwise, like in Case 4, let j be minimal such that $p^j(w) = v$ for some $w \in Y_1$, and let $Q = Q_k(u) * (u, v) * Q_j(w)$. Let $\mathcal{P}' = \mathcal{P}\Delta Q$. As in previous parts, it is routine to check that in both cases $|\mathcal{P}'|_2 - |\mathcal{P}|_2 = -1$. Thus $\mathcal{P}$ is not 2-optimal.

■

Recall that in cases 2 and 3 of Lemma 3.4 we needed the condition that no edges of A leave a certain vertex x (where $x = v^-$ in case 2 and $x = v$ in case 3). If this condition is violated, i.e., there are edges in A leaving x, then after updating $\mathcal{S}$ as in the lemma we will violate part 2 of Definition 3.2 and possibly other parts of the definition. The most serious problem that may arise is the existence of paths in $A \cup E[\mathcal{P}]$ from x to Y_2, which may lead, if we continue the algorithm, to a cycle in $A \cup E[\mathcal{P}]$.

We will, therefore, suggest a more elaborate update where all edges leaving x (and others) are erased, unless $p^{-\infty}(x) \cap Y_1 \neq \emptyset$, as was in cases 4 and 5 of Lemma 3.11. Details will follow these definitions:

Definition 3.12 (Regret function). *Let $\mathcal{S} = (Y_1, Y_2, A, p, t)$ be a 2-snapshot, $v \in Y_2$. Assume that $p^{-\infty}(v) \subseteq Y_2$. The function $regret(v)$ updates the sets A and Y_2 in the snapshot as follows: (see Figure 3)*

- $A \leftarrow A \setminus \{(u, w) : \ u \in \{p^{-\infty}(v) \cup \{v\}\}\}$
- $Y_2 \leftarrow Y_2 \setminus p^{-\infty}(v)$

Note that if $p^{-\infty}(v) = \emptyset$ then $regret(v)$ does nothing to $\mathcal{S}$. In this case no regret is necessary (E. Piaf [11]).

Lemma 3.13. *Let (Y_1, Y_2, A, p, t) be a 2-snapshot to $\mathcal{P}$, and let $v \in Y_2$. If $p^{-\infty}(v) \subseteq Y_2$ then by applying $regret(v)$ we obtain a 2-snapshot.*

Proof: We shall show that parts 1-9 of Definition 3.2 are maintained after the regret function is applied. Let $P \in \mathcal{P}^{>1}$. The set Y_1 intersects P since the regret function does not change Y_1. By part 8, for $x \in first_{\mathcal{P}}(Y_1) \cap V(P)$, $x^- \in Y_2$ and $p(x^-) = x$. Hence x^- will not be removed from Y_2 with the function $regret(v)$, and P will meet both Y_2 and Y_1, in that order, after $regret(v)$ is applied. This proves properties 1 and 9 of Definition 3.2.

Properties 2-5 are preserved because for each vertex u removed from Y_2, we also remove all edges (u, w) from A.

Part 6 is preserved because if a vertex u is kept after the regret then so is $p(u)$. Parts 7 and 8 follow from the condition $p^{-\infty}(v) \subseteq Y_2$, implying that no vertex of Y_1 is deleted. ■

Lemma 3.13 together with cases 2 and 3 of Lemma 3.4 imply

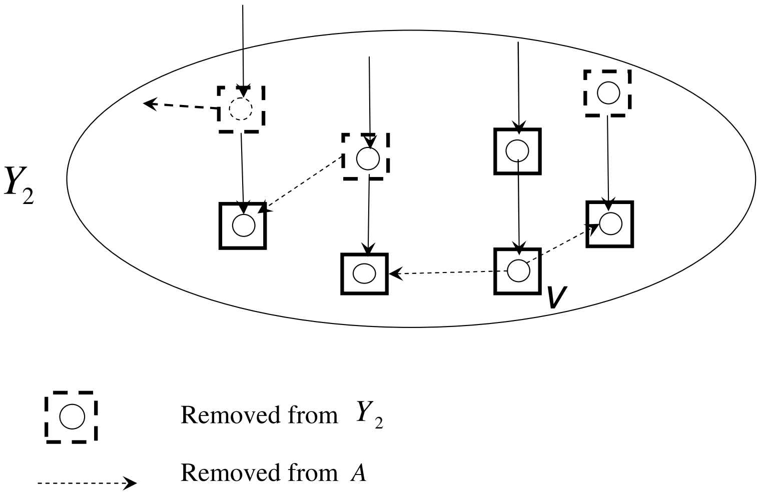

FIGURE 3. Regret(v)

Corollary 3.14 (Update 2-snapshot with regret). *Let $\mathcal{S} = (Y_1, Y_2, A, p, t)$ be a 2-snapshot to $\mathcal{P}$, and $X_i = first_{\mathcal{P}}(Y_i), i = 1, 2$. In the following cases $\mathcal{S}$ can be updated remaining a 2-snapshot, where Y_1 is increased.*

1. *If there exists an edge $(u, v), u, v \in X_1,\ v, v^- \in V[\mathcal{P}^{>1}] \setminus in[\mathcal{P}^{>1}],\ v^- \in Y_2$ and $p^{-\infty}(v^-) \subseteq Y_2$, then*
    - $regret(v^-)$
    - $Y_1 \leftarrow Y_1 + v^-$
    - $Y_2 \leftarrow Y_2 - v^- + v^{--}$
    - $p(v^-) \leftarrow u$; $p(v^{--}) \leftarrow v^-$
    - $A \leftarrow A + e$
    - $t(v^-) = t(v^{--}) = t(e) \leftarrow t_{max} + 1$
2. *If there exists an edge $(u, v), u, v \in X_2, v \in V[\mathcal{P}^1]$ and $p^{-\infty}(v) \subseteq Y_2$, then*
    - $regret(v)$
    - $Y_1 \leftarrow Y_1 + v$
    - $Y_2 \leftarrow Y_2 - v$
    - $p(v) \leftarrow u$
    - $A \leftarrow A + e$
    - $t(v) = t(e) \leftarrow t_{max} + 1$

Theorem 3.15. *Let $\mathcal{P}$ be a path partition. Let (Y_1, Y_2, A, p, t) be a 2-snapshot in $\mathcal{P}$ in which Y_1 is maximal with respect to inclusion among all possible snapshots. Then either $X_1 = first_{\mathcal{P}}(Y_1)$ is an independent set, or there exists a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$.*

Proof: Assume that $e = (u, v) \in E(G)$ and $u, v \in X_1$.

Case 1: $v \in V[\mathcal{P}^1]$. By case 2 of Lemma 3.11 there exists a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$.

Case 2: $v \in V[\mathcal{P}^{>1}]$ and $v^- \in in[\mathcal{P}^{>1}]$. By case 3 of Lemma 3.11 there exists a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$.

Case 3: $v, v^- \in V[\mathcal{P}^{>1}]$, $v^- \in Y_2$, $p^{-\infty}(v^-) \cap Y_1 \neq \emptyset$. By case 4 of Lemma 3.11 there exists a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$.

Case 4: $v, v^- \in V[\mathcal{P}^{>1}]$, $v^- \in Y_2$ and $p^{-\infty}(v^-) \subseteq Y_2$. By Corollary 3.14 case 1, Y_1 can be increased, contradicting the assumption of the theorem.

$\blacksquare$

Theorem 3.16. *Let $\mathcal{P}$ be a path partition. Let (Y_1, Y_2, A, p, t) be a 2-snapshot in $\mathcal{P}$ in which Y_2 is maximal to inclusion among all possible snapshots in which Y_1 is maximal to inclusion. Then either $X_2 = first_{\mathcal{P}}(Y_2)$ is an independent set or there exists a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$.*

Proof: Assume that $u, v \in X_2$ and $e = (u, v) \in E(G)$.

Case 1: $v \in in[\mathcal{P}^{>1}]$. By case 1 of Lemma 3.11 there exists a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$.

Case 2: $v \in V[\mathcal{P}^{>1}] \setminus in[\mathcal{P}^{>1}]$. By case 1 of Lemma 3.4, Y_2 can be increased, contradicting the assumption of the theorem.

Case 3: $v \in V[\mathcal{P}^1]$ and $p^{-\infty}(v) \subseteq Y_2$. By case 2 of Corollary 3.14, Y_1 can be increased, contradicting the assumption of the theorem.

Case 4: $v \in V[\mathcal{P}^1]$ and $p^{-\infty}(v) \cap Y_1 \neq \emptyset$. By case 5 of Lemma 3.11 there exists a path partition $\mathcal{P}'$ with $|\mathcal{P}'|_2 < |\mathcal{P}|_2$.

$\blacksquare$

Theorems 3.15 and 3.16 easily imply:

Corollary 3.17. *Let $\mathcal{P}$ be a 2-optimal path partition and let $\mathcal{S} = (Y_1, Y_2, A, p, t)$ be a 2-snapshot in $\mathcal{P}$ in which Y_2 is maximal to inclusion among all possible snapshots in which Y_1 is maximal to inclusion. Then the sets $X_i = first_{\mathcal{P}}(Y_i)$ for $i = 1, 2$ form a 2-colouring orthogonal to $\mathcal{P}$, proving Conjecture* 1.1 *for $k = 2$.*

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