European Tenure Games

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Abstract

We study a variant of the tenure game introduced by J. Spencer (Theor. Comput. Sci. 131 (1994), 415–429). In this version, faculty is not fired, but downgraded to the lowest rung instead.

For the upper bound we give a potential function argument showing that the value v_d of the game starting with d faculty on the first rung satisfies $v_d \leq \lfloor \log_2 d + \log_2 \log_2 d + 1.98 \rfloor$. We prove a nearly matching lower bound of $\lfloor \log_2 d + \log_2 \log_2 d \rfloor$ using a strategy that can be interpreted as an antirandomization of Spencer's original game. For d tending to infinity, these bounds improve to

$$\lfloor \log_2 d + \log_2 \log_2 d + 1 + o(1) \rfloor \leq v_d \leq \lfloor \log_2 d + \log_2 \log_2 d + 1.73 + o(1) \rfloor .$$

In particular, the set of all $d \in \mathbb{N}$ such that the value of the game is precisely $\lfloor \log_2 d + \log_2 \log_2 d + 1 \rfloor$ has lower density greater than $\frac{1}{5}$.

1 Introduction

In this paper we study a variant of the Tenure Game introduced by Spencer in [11]. Tenure games are so-called Pusher-Chooser games. These are two player perfect information games where each round the player called 'Pusher' splits the position into two alternatives and 'Chooser' selects one thereof. Hence the theme of these games is on-line balancing: Pusher has to find a balanced split (in the sense that neither alternative is too favorable to Chooser),

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whereas Chooser tries to detect and exploit such imbalances. Prominent examples of Pusher-Chooser games include vector balancing games ([2,3,6,7,9,10]) and liar games. Concerning the latter, we refer to the survey [8] and its extensive bibliography of 120 references. Internet routing problems gave rise to the related "guessing secrets" problem, that attracted attention recently ([1,4,5]).

In his marvelous paper "Randomization, Derandomization and Antirandomization: Three Games", Spencer [11] shows a generic method to convert a random strategy for Chooser in such a game into a deterministic algorithm. Moreover, he also provides a method coined 'antirandomization' that produces a matching lower bound constructively, i.e., including a strategy for Pusher.

The game Spencer demonstrated these methods most easily is the *Tenure Game*. We cite the rules from [11]:

> The tenure game is a perfect information game between two players, Paul — chairman of the department — and Carole — dean of the school. An initial position is given in which various faculty are at various pre-tenured positions. Paul will win if some faculty member receives tenure — Carole wins if no faculty member receives tenure. Each year Chair Paul creates a promotion list L of the faculty and gives it to Dean Carole who has two options: (1) Carole may promote all faculty on list L one rung and simultaneously fire all other faculty. (2) Carole may promote all faculty *not* on list L one rung and simultaneously fire all faculty on list L.

In this paper we study a slight variant of this tenure game. We will assume that not-promoted faculty is not fired, but downgraded to the first rung instead. Apart from an intrinsic interest, there are two further reasons to investigate this game. In [6] we showed that good strategies for this game yield good strategies for the on-line vector balancing problem with aging, i.e., where decisions in the past become less important compared to the actual one. A second motivation is that this variant seemingly does not admit the randomization–derandomization–antirandomization approach.

Let us state the rules precisely. Whether Carole can win or not of course depends on the number of rungs we have. To remove this parameter without losing information about the game, we assume to have infinitely many rungs and play an optimization version of the game: The highest rung reached by some faculty member is called the pay-off for Paul. Naturally, he tries to maximize this pay-off, whereas Carole tries to minimize it. To further simplify the setting we assume that all faculty is on the first rung at the beginning of the game. Exchanging people by innocent chips and baptizing this version 'European Tenure Game', we have:

> **Rules of the European Tenure Game:** The game is played with a fixed number d of chips which lie on levels numbered with the positive integers. At the start of the game, all d chips are on level one. The game is a two-player

perfect information game. The first player, called 'Pusher', tries to get a chip to a possibly high level. The maximum level ever reached by a chip during the game is called pay-off to Pusher. Each round Pusher chooses a subset of chips he proposes to be promoted. If the second player ('Chooser') accepts, then these chips each move up one level, and the remaining chips are moved down to the first level. If Chooser rejects, then the remaining chips move up one level, and Pusher's choice is downgraded to level one.

From the rules it is clear that Pusher has some advantage in the European Tenure Game compared to Spencer's original game, which we shall call 'American Tenure Game'. The American Tenure Game played with d chips has a value (the maximum level Pusher can reach) of $\lfloor \log d \rfloor + 1$, where we write $\log(\cdot)$ to denote the dyadic logarithm. In the European version Pusher roughly gains an extra $\log \log d$ levels.

More precisely, a bound of $\lfloor \log d \rfloor + \lfloor \log \log d \rfloor \leq v_d \leq \log d + \log \log d + 4$ for the value v_d of the European Tenure Game played with d chips was shown in [6]. In this paper we make some progress towards a full understanding of the game. We reduce the gap between lower and upper bound, so that there at most three possibilities for each d. For larger d, the gap further reduces to at most two values, though we are able to determine a precise value for a set having positive lower density:

Theorem 1 *Let v_d denote the value of the European Tenure Game played with d chips. Then*

$$\lfloor \log d + \log \log d \rfloor \leq v_d \leq \lfloor \log d + \log \log d + 1.98 \rfloor$$

holds for all d. For d tending to infinity, these bounds improve to

$$\lfloor \log d + \log \log d + 1 + o(1) \rfloor \leq v_d \leq \lfloor \log d + \log \log d + 1.73 + o(1) \rfloor .$$

In particular, the set $S = \{d \in \mathbb{N} \mid v_d = \lfloor \log d + \log \log d + 1 \rfloor\}$ has lower density greater than $\frac{1}{5}$. [1]

2 Upper Bound: Chooser's Strategy

Let us assume $d \geq 3$ since smaller cases are trivial. We describe a position of the game by a function $P : \mathbb{N} \to \mathbb{N}_0$ such that $\sum_{i \in \mathbb{N}} P(i) = d$. Hence $P(i)$ denotes the number of chips on level i.

[1] The lower density $\underline{d}(S)$ of a set $S \subseteq \mathbb{N}$ is $\underline{d}(S) := \liminf_{n \to \infty} \frac{1}{n} |\{s \in S \mid s \leq n\}|$. Roughly speaking, the last paragraph of the theorem states that we know the precise value of the game for more than a fifth of the values for d.

For the upper bound we use a potential function argument. Let $\lambda := \frac{2\log d-1}{\log d}$. Define a potential function v by $v(P) := \sum_{i\in\mathbb{N}} P(i)\lambda^{i-1}$ for all positions $P : \mathbb{N} \to \mathbb{N}_0$. We analyze the strategy for Chooser to choose that one of the alternatives which minimizes $v(P)$ for the resulting position. This yields:

Lemma 1 *The value of the game played with $d \geq 3$ chips is at most*

$$\begin{aligned}\frac{\log(d\log d - d + 1)}{\log\left(2 - \frac{1}{\log d}\right)} + 1 &\leq \log d + \log\log d + 1 + \tfrac{1}{2\ln 2} + o(1)\\ &\approx \log d + \log\log d + 1.73 + o(1).\end{aligned}$$

PROOF. Clearly we have $v(P) \leq d\log d$ for the starting position. Now let P be an arbitrary position of the game such that $v(P) \leq d\log d$. Denote by P_1, P_2 the two positions resulting from either accepting or rejecting a given Pusher move. Then $v(P_1) + v(P_2) = d + \lambda v(P)$, since each chip is promoted in either P_1 or P_2 (increasing its potential by a factor of λ) and downgraded (leading to a potential of one) in the other alternative. Thus

$$\begin{aligned}\min\{v(P_1), v(P_2)\} &\leq \tfrac{1}{2}(v(P_1) + v(P_2))\\ &= \tfrac{1}{2}(d + \lambda v(P))\\ &\leq \tfrac{1}{2}(d + \tfrac{2\log d-1}{\log d} d\log d)\\ &= d\log d.\end{aligned}$$

Hence Chooser's strategy of minimizing $v(P)$ ensures that $v(P) \leq d\log d$ holds throughout the game.

Let P be any position such that $v(P) \leq d\log d$. Let l denote the level of the highest-ranking chip. Since the remaining $d-1$ chips at least are on level one [2], we have $\lambda^{l-1} \leq d\log d - d + 1$. Hence

$$l \leq \log_\lambda(d\log d - d + 1) + 1 = \frac{\log(d\log d - d + 1)}{\log\left(2 - \frac{1}{\log d}\right)} + 1.$$

For $d \geq 3$, the latter term is strictly less than $\log d + \log\log d + 1.98$, as can be easily shown. Put $l = \log d$. Then $\log(2 - \frac{1}{l}) = 1 + \log(l - \frac{1}{2}) - \log l \geq 1 - \frac{1}{2}\frac{1}{(l-1/2)\ln 2}$, as the logarithm is concave. Thus

[2] This seems to be a negligible advantage. For large d in fact it is, but for smaller values this is enough to reduce the upper bound from $l + \log l + 4$ to $l + \log l + 1.98$.

$$\frac{\log\left(d\log d-d+1\right)}{\log\left(2-\frac{1}{\log d}\right)}\leq\frac{l+\log l}{\log\left(2-\frac{1}{l}\right)}$$
$$\leq\frac{l+\log l}{1-1/(2(l-\frac{1}{2})\ln 2)}$$
$$=l+\log l+\tfrac{1}{2\ln 2}+\tfrac{\log l}{2l\ln 2-1-\ln 2}+\tfrac{1+\ln 2}{2\ln 2(2l\ln 2-1-\ln 2)},$$

proving the asymptotics. □

3 Lower Bound: Pusher's Strategy

For a position $P:\mathbb{N}\to\mathbb{N}_0$ put $f(P)=\sum_{i\in\mathbb{N}}P(i)2^i$. Note that this function was used as a potential function for the American Tenure Game. We propose a strategy for Pusher that consists of first maximizing the f–value of the position and then using the strategy for the American Tenure Game to convert the f–potential into a high-ranking chip.

Let us sketch this last phase first. For a more detailed discussion we refer to [11]. Assume Pusher got to a position having f–potential at least 2^ℓ for some $\ell\in\mathbb{N}$. Then either there is a chip on level ℓ, or Pusher can split the chips into two subsets each having potential at least $2^{\ell-1}$. Regardless of Chooser's move, the promoted part already has an f–potential of at least 2^ℓ. Therefore, Pusher may ignore the downgraded chips and continue like this on the promoted group. Since the number of non-ignored chips decreases but the potential does not drop below 2^ℓ, Pusher ends up with a chip on level ℓ. Thus using this strategy Pusher can enforce a chip to level $\lfloor\log(f(P))\rfloor$ from a position having potential $f(P)$.

3.1 First Phase

The case $d=2$ is solved by a moment's thought, so let us assume $d\geq 3$. We assume first that $d=2^l$ is a power of 2 and deal with the general case at the end of this section. The first part of our strategy is identical with the one presented in [6].

It is clear that Pusher can change any position P such that all $P(i)$ are even, to the position P' defined by $P'(1)=\frac{1}{2}d$ and $P'(i+1)=\frac{1}{2}P(i)$ for all $i\in\mathbb{N}$. All pusher has to do is to select half of the chips of each level. Then, regardless of Chooser's choice, he ends up with position P'. We call this procedure an *easy split*.

From the starting position with $d=2^l$ chips on the first level, Pusher can play l easy splits and reach a position P with $P(i)=2^{l-i}$ for all $i=1,\dots,l$, with

$P(l+1) = 1$ and $P(i) = 0$ for $i \geq l+2$. Playing up to this position is the first part of Pusher's strategy.

3.2 Second Phase

We continue our strategy to maximize f. In the remainder of this paper, we will call $f(P)$ simply the potential of P omitting the f. Since in Phase 1 a greedy strategy of maximizing the function f was successful, one might be tempted to continue this. As each level has potential d (except level $l+1$, which has potential $2d$), it is not too difficult to split the levels into two parts having equal potential.[3] Thus the surviving part carries the whole potential (recall that moving up doubles the potential of a chip), and we gain a potential of 2 for each chip that is downgraded. We can continue this roughly $\log l$ times. If, while doing so, we partition the downgraded chips evenly, we can gain an extra potential of roughly $d \log l$. Since we need roughly dl extra to prove our main result, we are not done yet.

The problem is that having played this way, we might end up with one chip on level $l + \log l$ holding most of the potential of the whole position. Hence Chooser will downgrade this chip in his next move, and all our clever gains are gone.

The solution is modesty. Of course, we cannot prevent the chip on level $l+1$ to move up to $l + \log l$ in $\log l - 1$ moves. Chooser can enforce this by simply downgrading that part of the chips that does not contain this highest-ranking one. Therefore, we partition the chips into classes having different potentials: The one containing the highest-ranking chips has so large a potential, that we are immediately satisfied if it survives (ending with a potential of at least $2dl$). On the other hand, if the 'lower class' chips survive, we gain only little potential (an additional d), but end up with a flexible position (in particular having no too high-ranking chips, and allowing a similar step again). Here are the details:

We call the position

$$P_k : \mathbb{N} \to \mathbb{N}_0 ; i \mapsto \begin{cases} 2^{l-i} & \text{if } i < k \\ 1 & \text{if } i = k \\ 2^{l+1-i} & \text{if } k < i \leq l+1 \\ 0 & \text{otherwise.} \end{cases}$$

[3] From the rules of the tenure games it is clear that it makes no difference whether Pusher proposes some set of chips or its complement. Therefore we may view any Pusher move simply as partition of the set of chips into two classes.

a *logarithmic ladder with gap at level* k. Further on, we define for all $0 \leq j < k$

$$Q_{k,j} : \mathbb{N} \to \mathbb{N}_0 ; i \mapsto \begin{cases} d(1-2^{-j}) & \text{if } i = 1 \\ 2^{l+1-i} & \text{if } j+2 \leq i \leq k \\ 1 & \text{if } i = k+1 \\ 2^{l+2-i} & \text{if } k+2 \leq i \leq l+2 \\ 0 & \text{otherwise.} \end{cases}$$

We compute the potentials of these positions:

Lemma 2 *For all* $0 \leq j < k \leq l+1$*, we have*

$$\begin{aligned} f(P_k) &= d(2l - k + 1) + 2^k, \\ f(Q_{k,j}) &= d(4l - 2(k+j) + 4) + 2^{k+1} - 2^{l-j+1}). \end{aligned}$$

PROOF. The proof is a simple calculation. Note that the levels of P_k below the gap each contribute d to the potential, whereas those above contribute $2d$. □

Lemma 3 *From a logarithmic ladder with gap* P_k*, Pusher can enforce for any* $j < k$ *one of the positions* P_{l+1-j} *and* $Q_{k,j}$*. In particular, he can advance from* P_k *to either* P_{k-1} *and* $Q_{k,l+2-k}$*, if* $k > \frac{l}{2} + 1$*.*

PROOF. In position P_k, Pusher chooses those chips that have level at most j. If Chooser rejects, these $d(1-2^{-j})$ chips move down to level one, the remaining move up one level and position $Q_{k,j}$ is reached. Hence suppose that Chooser accepts. Then $d2^{-j} = 2^{l-j}$ chips move to level one, and the other chips move up one level. Now the number of chips on each level is a multiple of 2^{l-j}. Thus Pusher can play $l - j$ easy splits and reach position P_{l+1-j}. The second claim follows from the first by choosing $j = l + 2 - k$. □

From what we showed so far we already get a first lower bound:

Lemma 4 *For any* $\lceil (l+1)/2 \rceil \leq s \leq l+1$*, Pusher has a strategy enforcing one of the positions* $Q_{k,l+2-k}$ *for* $k = s+1, \ldots, l$*, and* P_s*. For* $s = \lceil (l+1)/2 \rceil$*, this strategy yields a potential of at least* $1.5d \log d$*, and thus a lower bound for the value of the game of* $\lfloor \log d + \log \log d + 0.58 \rfloor$*.*

PROOF. From the starting position with 2^l chips on level one, Pusher does l easy splits and reaches position $P_l = P_{l+1}$ (this is Phase 1). Once in Position P_i

for some $l \geq i \geq s+1$, he applies Lemma 3 with $j = l+2-i$ and reaches $Q_{i,j}$ or P_{i-1}. This proves the statement concerning the possible positions. With $s = \lceil (l+1)/2 \rceil$, the bound for the value of the game follows directly from Lemma 2 and the discussion of the American Tenure Game. $\square$

Since the positions $Q_{k,l+2-k}$ all have a potential of more than $2dl$, the lower bounds of Lemma 4 just depend on the potential of $P_{\lceil (l+1)/2 \rceil}$ of about $\frac{3}{2}dl$. We therefore continue Pusher's strategy on this position.

3.3 Third Phase

The reason why we could not continue applying Lemma 3 is that the gap k and the position j where Pusher splits the levels would meet. Splitting the levels above the gap leads to slightly more complicated positions having two gaps. For $0 < r < s \leq l+2$, we define

$$P_{r,s} : \mathbb{N} \to \mathbb{N}_0 ; i \mapsto \begin{cases} 2^{l-i} & \text{if } i < r \\ 1 & \text{if } i = r \\ 2^{l+1-i} & \text{if } r < i < s \\ 0 & \text{if } i = s \\ 2^{l+2-i} & \text{if } s < i \leq l+2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence P_{rs} is again a logarithmic ladder, this time having one gap on level r (holding one chip) and a second one on level s, which is empty. Note that $P_k = P_{k,l+2}$.

We also need for $0 < r < j < s \leq l+2$

$$Q_{r,s,j} : \mathbb{N} \to \mathbb{N}_0 ; i \mapsto \begin{cases} d(1-2^{-j+1}) & \text{if } i = 1 \\ 1 & \text{if } i = r+1 \\ 2^{l+2-i} & \text{if } j+2 \leq i < s+1 \\ 0 & \text{if } i = s+1 \\ 2^{l+3-i} & \text{if } s+1 < i \leq l+3 \\ 0 & \text{otherwise.} \end{cases}$$

Again we compute their potentials:

Lemma 5

$$f(P_{r,s}) = d(4l - r - 2s + 5) + 2^r,$$
$$f(Q_{r,s,j}) = d(8l - 4s - 4j + 14) + 2^{r+1} - 2^{l-j+2}.$$

The following lemma shows that also logarithmic ladders with two gaps allow comprehensible strategies.

Lemma 6 *Let $0 < r < s \leq l + 2$. For any j such that $r < j < s$, Pusher can advance position $P_{r,s}$ to either $P_{l+2-j,l+r+2-j}$ and $Q_{r,s,j}$.*

PROOF. Pusher chooses all chips on level at most j except the single chip on level r. If Chooser rejects, we are immediately in position $Q_{r,s,j}$. Otherwise, 2^{l-j+1} chips move to level one and Pusher's choice moves up one level. As all levels hold a multiple of 2^{l-j+1} chips, Pusher can play $l - j + 1$ easy splits and reach position $P_{l+2-j,l+r+2-j}$. □

Here is an outline of the third phase: Using Lemma 6, we apply a modesty strategy again. By Lemma 4 (and one extra step if l is odd), we reach $P_{(l+1)/2,l+1}$ or $P_{(l+2)/2,l+2}$. Once in position $P_{x,2x}$ for some $x \in [\lfloor (x+7)/3 \rfloor, \lfloor (l+2)/2 \rfloor]$, Pusher slowly increases the potential through the position $P_{x-1,2x-1}$ to $P_{x-1,2(x-1)}$. The first step increases the potential by roughly $3d$, the second by $2d$. If Chooser tries to foil this strategy, he immediately ends up with a Q–position having a potential of roughly $2dl$ (and leading to a potential of at least $2dl$ within a few moves).

Lemma 7 *In the European Tenure Game played with $d = 2^l$ chips, Pusher can reach one of the positions*

- $Q_{k,l+2-k}$ *for* $k = \lceil (l+1)/2 \rceil + 1, \ldots, l$,
- $Q_{x,2x,l+3-x}$ *for* $x = \lfloor (l+7)/3 \rfloor, \ldots, \lfloor (l+2)/2 \rfloor$,
- $Q_{x-1,2x-1,l+3-x}$ *for* $x = \lfloor (l+7)/3 \rfloor, \ldots, \lfloor (l+3)/2 \rfloor$,
- $P_{\lfloor (l+4)/3 \rfloor, 2\lfloor (l+4)/3 \rfloor}$.

In consequence, Pusher can reach a potential of at least $2dl$, and thus a pay-off of at least $\lfloor \log d + \log\log d + 1 \rfloor$.

PROOF. Applying Lemma 4 with $s = \lceil (l+1)/2 \rceil$, Pusher can get one of the positions $Q_{k,l+2-k}$ for $k = \lceil (l+3)/2 \rceil, \ldots, l$, or $P_{\lceil (l+1)/2 \rceil}$. Note that $P_{\lceil (l+1)/2 \rceil} = P_{\lceil (l+1)/2 \rceil, l+2}$. In particular, we have $P_{\lceil (l+1)/2 \rceil} = P_{\lfloor (l+4)/3 \rfloor, 2\lfloor (l+4)/3 \rfloor}$ for $l = 2$. Thus we may assume $l \geq 3$ from now on.

If l is odd, we apply Lemma 6 with $j = \frac{l+1}{2} + 1$ and end up with either $Q_{(l+1)/2,l+2,(l+3)/2}$ (which is $Q_{x-1,2x-1,l+3-x}$ for $x = \lfloor (l+3)/2 \rfloor$) or $P_{(l+1)/2,l+1}$.

For $l = 3$, the latter equals $P_{\lfloor (l+4)/3 \rfloor, 2\lfloor (l+4)/3 \rfloor}$. Hence assume $l \geq 4$ in the remainder of this proof. If l is even, our actual position is $P_{(l+2)/2, l+2}$.

The rest is an easy induction: Assume that we are in position $P_{x,2x}$ for some $x = \lfloor (l+7)/3 \rfloor, \ldots, \lfloor (l+2)/2 \rfloor$. Note that this implies $l \geq 4$. Applying Lemma 6 with $j = l + 3 - x$ on this position, we get $Q_{x,2x,l+3-x}$ or $P_{x-1,2x-1}$. Applying Lemma 6 on the latter with $j = l + 3 - x$ again, we reach position $Q_{x-1,2x-1,l+3-x}$ or $P_{x-1,2(x-1)}$. This proves the claim concerning the reachable positions.

For the potentials we look up in Lemma 2 and 5 and compute:

$$\begin{aligned} f(Q_{k,l+2-k}) &= 2dl + 3 \cdot 2^{k-1}, \\ f(Q_{x,2x,l+3-x}) &= 4d(l-x) + 2d + 3 \cdot 2^{x-1}, \\ f(Q_{x-1,2x-1,l+3-x}) &= 4d(l-x) + 6d + 2^{x-1}, \\ f(P_{\lfloor (l+4)/3 \rfloor, 2\lfloor (l+4)/3 \rfloor}) &= d(4l - 5\lfloor (l+4)/3 \rfloor + 5) + 2^{\lfloor (l+4)/3 \rfloor}. \end{aligned}$$

All potentials except the one of $Q_{x,2x,l+3-x}$ for $x = \frac{l}{2} + 1$ and even $l \geq 4$ are at least $2dl$. For $l \geq 28$, we briefly sketch how to obtain the missing potential of slightly less than $2d$ for this position also. The remaining small cases can be solved easily by hand or computer. A nicer proof though will be given at the end of this section.

Assume $l \geq 28$. Note that the $r = \frac{l}{2} - 1$ levels $\frac{l}{2} + 4$ to $l + 2$ of $Q_{l/2+1,l+2,l/2+2}$ each contain chips of total potential $4d$, whereas the remaining 'low' chips have a potential of slightly more than $2d$ only. Pusher may choose the chips on the $\left\lceil \frac{r}{2} \right\rceil \geq 7$ highest levels together with the low chips. This choice alone has a potential of more than dl, thus if Chooser accepts, Pusher clearly reached a potential of more than $2dl$. If Chooser rejects, a potential of more than $8d \left\lfloor \frac{r}{2} \right\rfloor + 1d \geq 2dl - 7d$ results. As all levels hold a multiple of 2^7 chips, Pusher plays seven easy splits and finally gets the desired potential of $2dl$. $\square$

3.4 If d is not a Power of 2

So far we assumed that d is a power of two. Since we may always ignore some of the chips in our play, this immediately yields bounds for the general case as well. As we ignore less than half the chips, our loss is not very big. For the value of the game, we just lose an additive term of $1 + o(1)$. Unfortunately, our upper and lower bounds are already that close that such a loss is significant.

A first idea would be to partition the chips into subsets of cardinalities of powers of two, and then play the above strategies on each separately. It is a problem though to synchronize the strategies. It might happen (and Pusher cannot prevent this) that one subset already reached a Q–position ending the strategy, while another set is in the middle of a series of easy splits. To make this approach work, we would need a way to conserve the potential of a favorable position like a Q–position for several moves. This seems to be a difficult task.

Fortunately, an easy trick solves the problem and shows that the general case is not far away from the special case of powers of 2.

Lemma 8 *Let $i \in \mathbb{N}$ such that $2^i \le d$. Then Pusher can earn a potential of $2 \cdot 2^i i \lfloor d/2^i \rfloor$. In consequence, Pusher has a strategy ensuring him a potential of at least $2d \log d(1 + o(1))$.*

PROOF. Let $d_0 = \lfloor d/2^i \rfloor 2^i$, the largest multiple of 2^i not exceeding d. This is Pusher's strategy: He plays with d_0 chips only, ignoring the rest. The set of d_0 non-ignored chips is partitioned into $\lfloor d/2^i \rfloor$ groups of 2^i chips each. These groups will never be split in the course of the game, so we may assume these chips to be glued together forming each a big chip. As there are 2^i big chips, Pusher follows his strategy for powers of 2 and ends up with a position of potential $2 \cdot 2^i i$ in terms of big chips. Since each big chip consists of $\lfloor d/2^i \rfloor$ ordinary ones ('un-gluing the big chips'), this position has a potential of $2 \cdot 2^i i \lfloor d/2^i \rfloor$.

Put $l := \log d$ and $i = \lfloor l - \log l \rfloor$. Then

$$\begin{aligned} 2 \cdot 2^i i \left\lfloor d/2^i \right\rfloor &\ge 2(d - 2^i)(l - \log l - 1) \\ &= 2dl(1 - \tfrac{1}{l})(1 - (\log l + 1)/l) \\ &= 2dl(1 + o(1)). \end{aligned}$$

□

From Lemma 1 and 8 we deduce that we know the precise value of the game, namely $\lfloor \log d + \log \log d + 1 \rfloor$, for all sufficiently large d such that the fractional part of $\log d + \log \log d$ is contained in $[0, 0.27[$. Some elementary calculus leads to the conclusion that the set of all d such that we know the precise value of the game has lower density greater than $\frac{1}{5}$.

3.5 Small Cases

In this last subsection on lower bounds we deal with the small cases solved by hand or computer in the proof of Lemma 7. Recall that only the cases

$4 \leq l < 28$, l even, where not proven completely. We apply the trick of Lemma 8 to reduce the seemingly more difficult case of l even to odd l. We will show that for all $l \geq 3$ odd, Pusher can gain a potential of $2ld + d$.

Assume this shown for the moment. In the case $d = 2^l$, l even, Pusher splits the chips into pairs, glues each one together and plays his strategy for l odd. Un-gluing them yields a position having potential at least $2(l-1)d + d$. Since the number of chips on each level is even, Pusher can play an easy split, gaining him an extra potential of d. Thus Pusher can gain a potential of at least $2ld$ for all $d = 2^l$. We end this section by proving the remaining claim for l odd.

Lemma 9 *In the game played with $d = 2^l$, l odd, chips, Pusher can enforce a potential of at least $2ld + d$.*

PROOF. Pusher follows the strategy of Lemma 7 and reaches one of the positions described there. We continue play from there:

From $Q_{k,l+2-k}$ for some $k \in [(l+3)/2, l]$: If $k = l$, then $f(Q_{k,l+2-k}) = 2dl + 3 \cdot 2^{k-1} = 2dl + \frac{3}{2}d$, hence there is nothing to show. Assume $k < l$. Then level $l+2$ holds one chip and $l+1$ holds two chips. Thus both levels hold the same potential of $4d$. Further on, there is one single chip on level $k+1$, which has the same potential as 2^k chips on the first level (there are much more on this level). Here is Pusher's split: The first partition class shall contain the two single chips on level $l+2$ and $k+1$. The second class shall contain the two chips on level $l+1$ and 2^k of the chips on the first level. The remaining chips (each level contains an even number thereof) shall be split evenly. Thus both partition classes contain chips having equal potential in total. The first partition class contains less chips, namely $2 + \frac{1}{2}(d - 4 - 2^k) = \frac{d}{2} - 2^{k-1}$. Hence even if this class does not survive (resulting in fewer chips downgraded and adding a potential of 2 each), Pusher ends up with a potential of $2dl + 3 \cdot 2^{k-1} + 2(\frac{d}{2} - 2^{k-1}) > 2dl + d$.

From $Q_{x-1,2x-1,l+3-x}$ for some $x \leq (l+3)/2$: Since $f(Q_{x-1,2x-1,l+3-x}) = 4d(l-x) + 6d + 2^{x-1}$, only the case $x = (l+3)/2$ has to be regarded. We treat the case $l = 3$ separately first. Though not difficult, it shows the problems arising in small cases. $Q_{x-1,2x-1,l+3-x}$ for $l = 3$ and $x = (l+3)/2$ has six chips on the first level and one each on level three and five. We shall denote this position by $(6, 0, 1, 0, 1, 0, \dots)$. Pusher selects the single chip on level five, which will not survive (or we are done with a potential of over $8d$). Thus we are in position $(1, 6, 0, 1, 0, 0, \dots)$. Pusher selects four chips on level two. If they do not survive, we are in position $(4, 1, 2, 0, 1, 0, \dots)$ having potential $60 = 2ld + d + 4$. Otherwise we are in position $(4, 0, 4, 0, 0, 0, \dots)$ and two easy splits get us to $(4, 2, 1, 0, 1, 0, \dots)$ having potential $56 = 2ld + d$.

We may thus assume $l \geq 5$ (recall that we only considered odd l). Then a split similar as in the first case solves the problem: The first partition class shall contain the single chips on level $l+2$ and $(l+3)/2$, the second the two chips on level $l+1$ and $2^{(l+1)/2}$ of the chips on level one. The remaining chips shall be split evenly. As above we compute a potential of $4d(l-x)+6d+2^{x-1}+2(2+\frac{1}{2}(d-4-2^{(l+1)/2})=2ld+d$ if the second class survives, which solves this case.

From $Q_{x,2x,l+3-x}$ for some $x \leq (l+1)/2$: Since $f(Q_{x-1,2x-1,l+3-x})=4d(l-x)+2d+3\cdot 2^{x-1}$, only the case $x=(l+1)/2$ has to be regarded. Again very small values of l, namely $l=5$ and $l=7$ need special attention.

Let $l=5$, thus $x=3$. We deviate from the strategy described in Lemma 7. Once in position $P_{3,6}=(16,8,1,4,2,0,1,0,\dots)$ (which precedes $Q_{3,6,5}$ in Lemma 7), Pusher chooses the two single chips on level 3 and 7 and 10 of the chips on level one. Regardless of Chooser's move the resulting position has potential at least $11d$.

Let $l=7$. Again Pusher avoids position $Q_{4,8,6}$. Once in Position $P_{4,8}=(64,32,16,1,8,4,2,0,1,0,\dots)$, he chooses the two single chips on level 4 and 9, the two chips on level 7 and 52 of the chips on level one. If Chooser accepts these chips to be promoted, as potential of $15d$ results. Otherwise, Pusher continues from $(56,12,32,16,0,8,4,0,\dots)$ with two easy splits and obtains a potential greater than $15d$.

Finally, let $l \geq 9$. Then $Q_{x,2x,l+3-x}$ contains four chips on level l and two on level $l+1$ to match the potential of the single chip on level $l+3$. Here is the partition of Chooser's move: The two single chips on level $l+3$ and $(l+3)/2$ shall be in the first partition class. The chips on levels l and $l+1$ as well as $2^{(l+1)/2}$ of the chips on level one shall be in the second. There is an even number of remaining chips on each level, which are split evenly. Thus both partition classes hold the same potential (of $\frac{1}{2}(2dl+3\cdot 2^{(l+1)/2})$) and the position resulting from Chooser's move has potential $2dl+3\cdot 2^{(l+1)/2}$ plus an extra 2 for each chip that is downgraded. Since both partition classes contain at least $2+\frac{1}{2}(d-6-2^{(l+1)/2})$ chips, this results in a potential of at least $2dl+d$.

From $P_{\lfloor (l+4)/3\rfloor, 2\lfloor (l+4)/3\rfloor}$: Apart from the case $l=5$, this position already has a potential of $2dl+d$. For $l=5$ note that $P_{\lfloor (l+4)/3\rfloor, 2\lfloor (l+4)/3\rfloor}=(16,8,1,4,2,0,1,0,\dots)$ can be partitioned into the two positions $(10,2,1,0,0,0,1,0,\dots)$ and $(6,6,0,4,2,0,0,0,\dots)$. Both contain the same potential and half the chips, hence not matter how Chooser plays, a position having potential $f(P_{\lfloor (l+4)/3\rfloor, 2\lfloor (l+4)/3\rfloor})+d>2dl+d$ results.

This ends the proof of Lemma 9. □

4 Remarks and Open Problems

The strategies we gave above have little in common with the generic approach of randomization, derandomization and antirandomization. A main problem seems to be that the game has no fixed end. Thus random play is not a good idea for Chooser: If the game lasts sufficiently long, a chip can reach arbitrary high levels with high probability.

An obvious problem left open in this paper is a precise determination of the value of the game for all or all but a few values of d. We only succeeded in doing so for a set of d having lower density $\frac{1}{5}$. For the remaining values apart from finitely many, two possibilities exist for the value of the game.

With quite some effort it is possible to continue Pusher's strategy from the Q–positions and thus gain a potential of γdl for some $\gamma > 2$. Unfortunately, these gains are not too big, in particular, they are not enough to determine the value of the game for asymptotically all numbers d.

More interesting than a slight increase of the set of numbers d such that the value of the game with d chips is determined might be the following: Assume that $d = 2^l$ is a power of two again. Then the proofs in Section 3 give a strategy for Pusher to obtain a potential of about $2dl$. A closer inspection of these proofs shows that Pusher might need more than l^2 moves to reach this aim. This is caused by the strategy which is quite unbalanced in the following sense: Whenever Chooser has two different alternatives, i.e., Pusher did not play an easy split, one of the alternatives immediately produces a potential of $2dl$, whereas the other only gains a modest additional potential of $\Theta(d)$ in up to $l-1$ easy splits.

We do not know whether a more balanced strategy exists. If Pusher could produce two alternatives gaining an $\Omega(d)$ potential increase in one move (like the easy splits do), this would result in a strategy that needs $O(l)$ moves only.

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References

[1] N. Alon, V. Guruswam, T. Kaufman, and M. Sudan. Guessing secrets efficiently via list decoding. In *Proceedings of the 13th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 254–262, 2002.

[2] I. Barany. On a class of balancing games. *J. Combin. Theory Ser. A*, 26:115–126, 1979.

[3] I. Barany and V. S. Grunberg. On some combinatorial questions in finite dimensional spaces. *Linear Algebra Appl.*, 41:1–9, 1981.

[4] F. Chung, R. Graham, and T. Leighton. Guessing secrets. *Electron. J. Comb.*, 8:Research Paper 13, approx. 25 pp. (electronic), 2001.

[5] F. Chung, R. Graham, and L. Lu. Guessing secrets with inner product questions. In *Proceedings of the 13th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 247–253, 2002.

[6] B. Doerr. Vector balancing games respecting aging. *J. Combin. Theory Ser. A*, 95:219–233, 2001.

[7] J. Olson. A balancing strategy. *J. Combin. Theory Ser. A*, 40:175–178, 1985.

[8] A. Pelc. Searching games with errors — fifty years of coping with liars. *Theor. Comput. Sci.*, 270:71–109, 2002.

[9] H. Peng and C. H. Yan. Balancing game with a buffer. *Adv. Appl. Math.*, 21:193–204, 1998.

[10] J. Spencer. Balancing games. *J. Combin. Theory Ser. B*, 23:68–74, 1977.

[11] J. Spencer. Randomization, derandomization and antirandomization: Three games. *Theor. Comput. Sci.*, 131:415–429, 1994.