# Heuristics

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- In such cases, we try to reduce the complete search of cases using heuristics.

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- Good heuristics can result in significant acceleration in special cases.

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**Definition: Rectangle**

$$[a_1, b_1] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$$

shaped point sets — where $n$ is the dimension and $-\infty < a_i < b_i < \infty \quad i = 1, \ldots, n$ — are called rectangles.

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   (A) If we divide a suitable side of $T$ into two halves:

   $$T = T_1 \mathring{\cup} T_2, \text{ i.e., for suitable } i \text{ and } v$$
   $$T_1 = [a_1, b_1] \times \ldots \times [a_{i-1}, b_{i-1}] \times [a_i, v] \times [a_{i+1}, b_{i+1}] \times \ldots \times [a_n, b_n],$$
   $$T_2 = [a_1, b_1] \times \ldots \times [a_{i-1}, b_{i-1}] \times [v, b_i] \times [a_{i+1}, b_{i+1}] \times \ldots \times [a_n, b_n],$$

   then $a_T \leq a_{T_1}, a_{T_2}$ and $f_{T_1}, f_{T_2} \leq f_T$ hold.

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   then $a_T \leq a_{T_1}, a_{T_2}$ and $f_{T_1}, f_{T_2} \leq f_T$ hold.
   (B) Furthermore, for every $\varepsilon > 0$, there exists $\delta > 0$, such that for any $T$ rectangle, if $\forall_i : b_i - a_i \leq \delta$, then $0 \leq f_T - a_T \leq \varepsilon$, meaning if a rectangle's size approximates a point, then the upper and lower bounds will be close.

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The smallest lower bound rectangle will have every point as a good output.

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• We only slice one rectangle at a time.

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WHILE $\Delta > \varepsilon$, DO repeat steps (1)-(4).

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(OUTPUT): $a_{\mathcal{T}}, f_{\mathcal{T}}$, where $a_{\mathcal{T}} \leq p^* \leq f_{\mathcal{T}}$. Moreover, let $T : a_T = a_{\mathcal{T}}$ and output $x \in T$.

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• $\text{Torz}(T) \geq 1$ and equality holds if and only if our $n$-dimensional rectangle is a cube.

• The advantage of splitting the longest side is that we will never have overly *distorted* rectangles during our procedure.

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- From the claim, the observation follows by induction.
- To prove the claim, consider the following two cases.

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**Case 2:** *Suppose torz $(T) \leq 2$.*

In this case, after halving, the original longest edge becomes the new brick's shortest edge, at half the size.

The length of the longest edge cannot increase, so the distortion of the new $T$ is at most 2, by the initial condition.

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$$\text{vol } T = \prod_i (b_i - a_i) \geq \max_i (b_i - a_i) \left( \min_i (b_i - a_i) \right)^{n-1} =$$

$$= \frac{(\max_i (b_i - a_i))^n}{\left( \frac{\max_i (b_i - a_i)}{\min_i (b_i - a_i)} \right)^{n-1}} = \frac{|T|^n}{(\text{torz } T)^{n-1}}.$$

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- This can only happen if $T$ was obtained by bisecting $T^-$ such that $|T^-| \leq \delta$.
- However, in this case, the algorithm should have stopped (see condition (3) for the beauty of $c$).

# Break

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Even for small $\nu$ values, $2^\nu$ becomes too large for efficient handling.

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- A *good* feasible solution is obtained by rounding the $[0,1]$-valued components of the optimal position $x^*_{\mathbb{R}}$ of the continuous problem to integers (i.e., to $\{0,1\}$).
- Using this, for the $F$ problem, we can provide a lower $a_F$ and an upper $f_F$ estimate for the optimal value.

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- The two new problems can be appended to the previous tree: From $\ell$ with $d_i = 0$ and $\ell$ with $d_i = 1$, $\ell_0$ and $\ell_1$ descendants are created.

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- The original problems and the two subproblems can be represented in a rooted tree.

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(0) Let $T$ be a 1-node rooted tree, with its only node (and leaf) representing the initial problem.
Calculate the lower/upper estimates $a, f$ for this problem.

WHILE $f - a > \varepsilon$

(1) Select a leaf/problem $\ell$ from $T$.

(2) Choose one non-fixed component $d$ from the selected leaf/problem. Denote this component as $i$.

# The Branch-and-Bound Algorithm (Continued)

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(3a) Let $\ell_0$ be the problem obtained from $\ell$ with $d_i = 0$ selection.

($3b_0$) Relax the remaining components and compute the $a_0, f_0$ lower and upper estimates in the same manner as before.

($3b_1$) Let $\ell_1$ be the problem obtained from $\ell$ with $d_i = 1$ selection. Relax the remaining components and compute the $a_1, f_1$ lower and upper estimates in the same manner as before.

(4) Let $T$ be the tree obtained from $\ell$ by branching into the $\ell_0$ and $\ell_1$ leaves. Set $a = \min\{a, a_0, a_1\}$ and $f = \min\{f, f_0, f_1\}$.

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- Our tree cannot grow beyond the full depth binary tree with $\nu$ levels.

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• We reduce our task to a mixed convex-integer type problem.

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- Our previous method can be applied.

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• If we obtain a new inequality in this manner, we say that we have made an L-inference.

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- The inequalities *derived* this way are consequences of the initial conditions. Adding them to our system of conditions does not change the set of possible solutions.
- If we obtain a new inequality in this manner, we say that we have made an L-inference. (L$\equiv$ linear, real.)

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- We can also make new logical inferences.

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- This proves the claim.

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- Consider the following problem:

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| Minimize | $-2x-5y$-t |
| subject to | $10x+3y\leq 45$ |
| | $4x+20y\leq 65$ |
| | $x,y\in\mathbb{N}$ |

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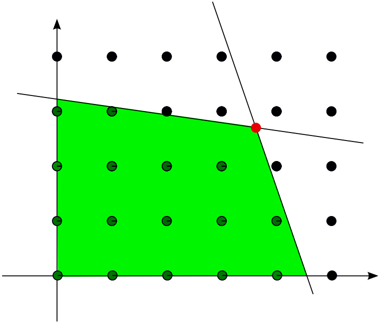

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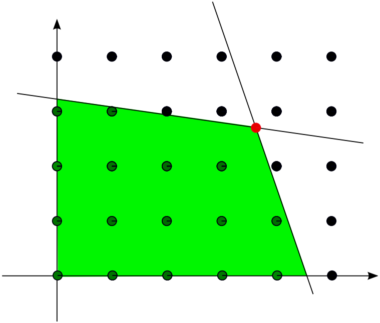

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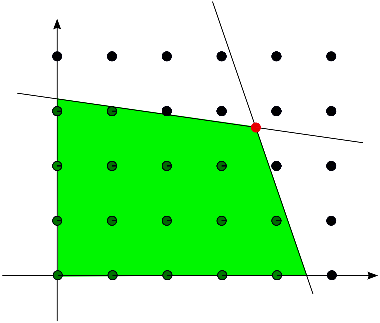

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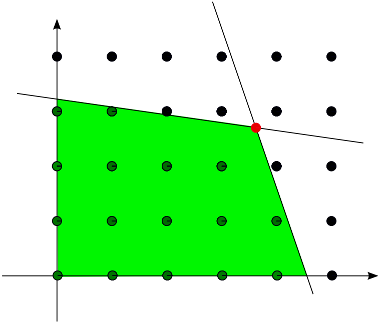

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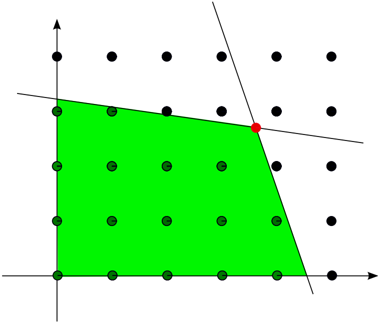

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(GEOMETRY: All three inferences describe a half-plane whose boundary passes through the red point (why?).)

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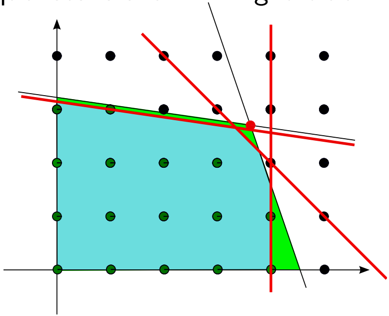

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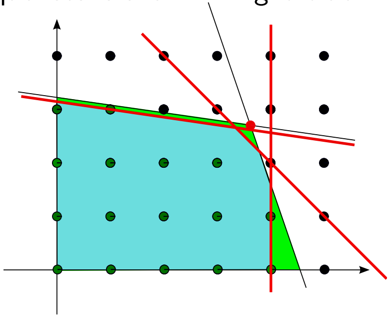

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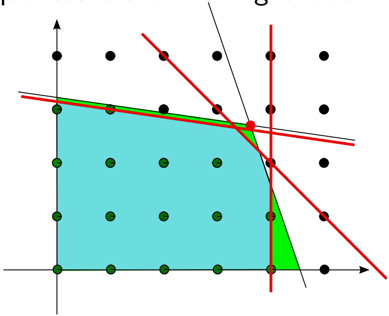

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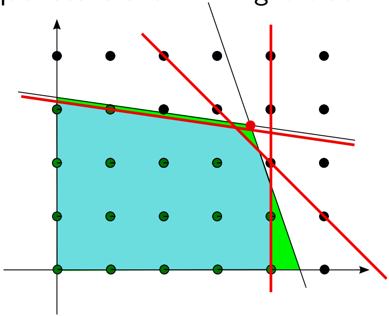

The decrease in the described polytope is obvious. During the decrease, the dark green points did not leave the solution set. The polytope described by the LP relaxation approached the convex hull of the dark green points.

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(L) // LOGIC
Make L- and I-inferences to add new linear inequalities to our problem. Return to the relaxation step.

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• The same situation occurs with the Gomory algorithm, it is generally not polynomial-time.

# This is the End!

Thank you for your attention!