Semidefinite programming and eigenvalues

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2024. Fall

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The eigenvalues (with multiplicities) form the spectrum of the matrix.

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What are its eigenvalues, eigenvectors?

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- The all-1 vector: $\underline{1} = j = (1, 1, \cdots, 1)^{\mathsf{T}} \in \mathbb{R}^k$ is an eigenvector of the matrix: $J\underline{1} = k\underline{1}$. So, the eigenvalue associated with the found eigenvector is k.

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Specifically, $\lambda_{\max}(J_{k\times k}) = k$.

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It is easy to see and well-known that

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- This is an SDP formulation of determining $\lambda_{\max}$.

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Let $X = \sum_{1=1}^{n} x_i A_i$, where $A_i \in \mathcal{S}^k$ (so $X \in \mathcal{S}^k$ holds).

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That is, X has the following form

$$\begin{pmatrix} \alpha_{11}^{(1)} x_1 + \alpha_{11}^{(2)} x_2 + \dots & \alpha_{12}^{(1)} x_1 + \alpha_{12}^{(2)} x_2 + \dots & \cdots & \alpha_{1n}^{(1)} x_1 + \alpha_{1n}^{(2)} x_2 + \dots \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1}^{(1)} x_1 + \alpha_{11}^{(2)} x_2 + \dots & \alpha_{n2}^{(1)} x_1 + \alpha_{n2}^{(2)} x_2 + \dots & \cdots & \alpha_{nn}^{(1)} x_1 + \alpha_{nn}^{(2)} x_2 + \dots \end{pmatrix}_{k \times}$$

i.e., an $k \times k$ matrix, where each element is a linear function.

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Task

Find $x \in \mathbb{R}^n$ such that $\lambda_{\max}(X)$ is minimized.

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Minimize	μ-t
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That is, our eigenvalue question can again be formulated as an SDP problem.

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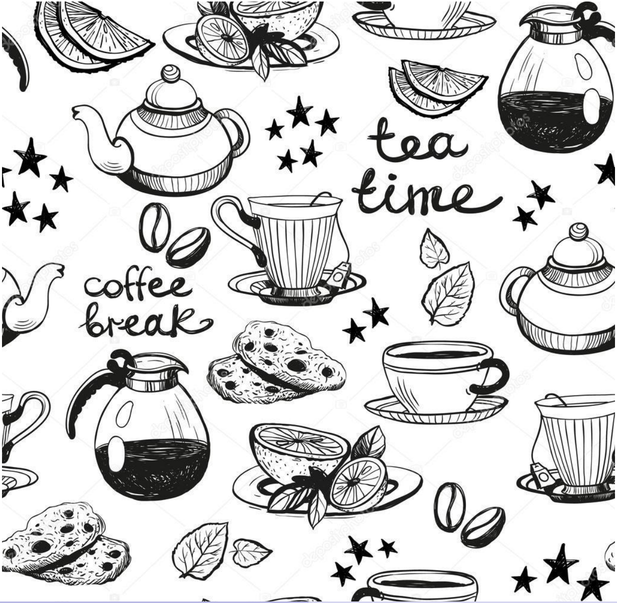
- We provide an estimation for its optimum using eigenvalues.

- Then, we formulate the assertion of the best estimate as an SDP problem.

Break



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The adjacency matrix A_G of a simple graph G

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- To represent A_G, we need to fix an ordering of the vertices.
- In an ordering-independent view of the matrix, we must interpret it as a function $V \times V \to \{0, 1\}$.

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The standard notation for the optimal value is $\alpha(G)$.

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That is,

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F is an independent set if and only if $\overline{A_G}|_{F \times F} = J$, the constant matrix of ones.

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- It's easy to verify that

$$|F| = \frac{\chi_F^\mathsf{T} \overline{A_G} \chi_F}{\chi_F^\mathsf{T} \chi_F} \leq \max_{x : x \in \mathbb{R}^V - \{0\}} \frac{x^\mathsf{T} \overline{A_G} x}{x^\mathsf{T} x} = \lambda_{\max}(\overline{A_G}).$$

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- We arrive at the following result:

Theorem

Let G be a simple graph, F an independent set in our graph. Then

$$\lambda_{\max}\left(\overline{A_G}\right) \geqslant |F|.$$

Specifically,

$$\lambda_{\max}\left(\overline{A_G}\right) \geqslant \alpha(G).$$

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- By calculating $\lambda_{\max}\left(\overline{A_G}\right)$, we obtain an estimate for an $\mathcal{NP}$-hard function.

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Then

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- Thus, it's straightforward to formulate the sharpest version of the theorem.

Maximally Utilizing the Consequence

- The consequence has two „participants".
  (i) the matrix with property $\mathcal{T}_G$,
  (ii) the independent set F.
- One is on the left, the other on the right side.
- Thus, it's straightforward to formulate the sharpest version of the theorem.

Theorem

Let G be a simple graph. Then

$$\min\{\lambda_{\max}(M) : \ M \in \mathcal{S}^V \text{ satisfies } \mathcal{T}_G\} \geqslant \max\{|F| : \ F \text{ is an independent set}\}.$$

The Left-hand Side of the Inequality

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Minimize	$\lambda_{\max}(M)$-t
subject to	$M_{uu} = 1$, for all $u \in V$,
	$M_{uv} = 1$, for all $uv \notin E$,
	$M \in \mathcal{S}^n$.

Let's reformulate the problem and see that its determination is an SDP problem.

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That is,

$$S_e(u, v) = \begin{cases} 1, & u = x, v = y \text{ or } u = y, v = x \\ 0 & \text{otherwise.} \end{cases}$$

The Reformulated Problem

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After this, a symmetric matrix M's satisfying the property $\mathcal{T}_G$ is equivalent to $M = J - \sum_{e \in E} x_e S_e$ for some $x_e \in \mathbb{R}^E$ vector: The main diagonal and the non-edges of the J matrix are 1, and everywhere else (at the position of edge e) we modify it by x_e (to any value).

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Minimize	μ-t
subject to	$-\mu I - \sum_{e\in E} x_e S_e \preceq -J.$

Summary

Summarizing:

Definition/Notation

Let G be a simple graph. Then

$$\vartheta(G) = \text{ the optimal value of the above SDP problem.}$$

We obtained that

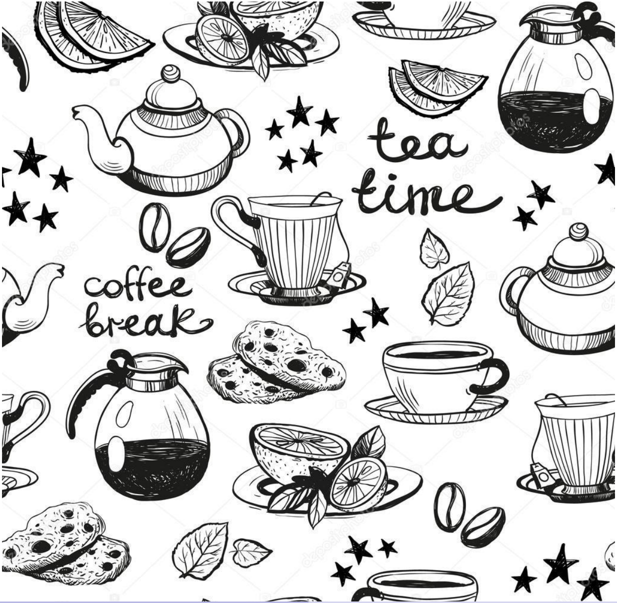
Theorem

Let G be a simple graph. Then

$$\vartheta(G) \geq \alpha(G).$$

Knowing that the optimum of an SDP optimization problem can be efficiently determined, the left-hand side of the theorem's inequality is $\mathcal{NP}$-hard, while the right-hand side is tractable.

Break



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• For a simple graph G, we can again describe it with a matrix. For us, the $A_{\overline{G}}$ matrix will be „convenient".
• Take a clique cover of G. Let ℓ be the number of cliques. That is, $V = K_1 \dot\cup K_2 \dot\cup \ldots \dot\cup K_\ell$, where K_i are disjoint cliques.

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• Similarly, a matrix of type $V \times V$ can be considered as a block matrix of type $\ell \times \ell$:

$$M = \begin{array}{c|c} & \overbrace{\quad}^{K_1} \quad \overbrace{\quad}^{K_2} \quad \ldots \quad \overbrace{\quad}^{K_\ell} \\ \begin{array}{c} K_1\{ \\ K_2\{ \\ \vdots \\ K_\ell\{ \end{array} & \left(\begin{array}{c|c|c|c} & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \end{array}\right) \end{array}.$$

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Let's specifically look at $A_{\overline{G}}$:

$$A_{\overline{G}} = \begin{array}{c c} & \begin{array}{cccc} \overbrace{\qquad}^{K_1} & \overbrace{\qquad}^{K_2} & \cdots & \overbrace{\qquad}^{K_\ell} \end{array} \\ \begin{array}{c} K_1\{ \\ K_2\{ \\ \vdots \\ K_\ell\{ \end{array} & \left(\begin{array}{c|c|c|c} \boxed{0} & & & \\ \hline & \boxed{0} & & \\ \hline & & \ddots & \\ \hline & & & \boxed{0} \end{array}\right) \end{array}.$$

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Let the eigenvalues of $A_{\overline{G}}$ be denoted by
$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n = \lambda_{\max}$ $(n = |V|)$.

v and $\widetilde{v}$

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$$\widetilde{v}^{\mathsf{T}} := \big(\|v_1\|, 0, \ldots, 0\big|\|v_2\|, 0, \ldots, 0\big|\ldots\big|\|v_\ell\|, 0, 0, \ldots, 0\big)$$

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where $\|w\| = \sqrt{\sum_{i=1}^{d}(w_i)^2}$, the L^2 norm of the d-dimensional vector w.

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• Thus, there exist orthogonal matrices Q_i such that $Q_i\widetilde{v_i} = v_i$, where $\widetilde{v_i} = (\|v_i\|_2, 0, 0, \ldots, 0)^{\mathsf{T}} \in \mathbb{R}^{K_i}$.

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• Then

$$
Q = \begin{array}{r c}
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$$Q = \begin{array}{r c} & \begin{array}{cccc} \overbrace{\hphantom{Q_1}}^{K_1} & \overbrace{\hphantom{Q_2}}^{K_2} & \ldots & \overbrace{\hphantom{Q_\ell}}^{K_\ell} \end{array} \\ \begin{array}{r} K_1\{ \\ K_2\{ \\ \vdots \\ K_\ell\{ \end{array} & \left(\begin{array}{c|c|c|c} Q_1 & \boxed{0} & \ldots & \boxed{0} \\ \hline \boxed{0} & Q_2 & \ldots & \boxed{0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \boxed{0} & \boxed{0} & \ldots & Q_\ell \end{array}\right) \end{array}, \qquad \text{and} \qquad Q\widetilde{v} = v.$$

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If u is an eigenvector corresponding to the eigenvalue λ of $A_{\overline{G}}$, then $Q^{-1}u = Q^{\mathsf{T}}u$ is an eigenvector of the matrix $Q^{-1}A_{\overline{G}}Q$ corresponding to the same eigenvalue λ.

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Indeed,

$$(Q^{-1}A_{\overline{G}}Q)(Q^{-1}u) = Q^{-1}A_{\overline{G}}(QQ^{-1}u) = Q^{-1}A_{\overline{G}}u = Q^{-1}\lambda u = \lambda Q^{-1}u.$$

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Note that the block structure and the zero blocks on the main diagonal of $Q^{-1}A_{\overline{G}}Q$ can also be recognized:

$$Q^{-1}A_{\overline{G}}Q = \begin{array}{c c} & \begin{array}{cccc} \overbrace{\quad}^{K_1} & \overbrace{\quad}^{K_2} & \cdots & \overbrace{\quad}^{K_\ell} \end{array} \\ \begin{array}{c} K_1\{ \\ K_2\{ \\ \vdots \\ K_\ell\{ \end{array} & \left(\begin{array}{c|c|c|c} \boxed{0} & & & \\ \hline & \boxed{0} & & \\ \hline & & \ddots & \\ \hline & & & \boxed{0} \end{array}\right) \end{array}.$$

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- We form a submatrix R of $A_{\overline{G}}$ by removing the rows and columns not in F. (This operation is called symmetric submatrix extraction.)

Linear Algebraic Detour

Cauchy's Theorem

Let $M_{s\times s}$ be a symmetric matrix, and let's form a symmetric submatrix $R_{t\times t}$. Let $\lambda_1 \leq \ldots \leq \lambda_s$ be the eigenvalues of $M_{s\times s}$, and let $\mu_1 \leq \ldots \leq \mu_t$ be the eigenvalues of $R_{t\times t}$.

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(i) $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$, $\ldots$, $\lambda_t \leq \mu_t$,

(ii) $\mu_t \leq \lambda_s$, $\mu_{t-1} \leq \lambda_{s-1}$, $\ldots$, $\mu_1 \leq \lambda_{s-t+1}$.

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- Based on the above, the largest eigenvalue is $\lambda_{\max}$.
- The sum of eigenvalues is the trace of our matrix, which is 0.

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- If we assume that the known eigenvalues of $\lambda_{\max}$ are accompanied by $\ell - 1$ other eigenvalues estimated to be $\lambda_{\min}$, then we get: $0 = \text{trace } R = \sum_{i=1}^{\ell} \mu_i \geq (\ell - 1)\lambda_{\min} + \lambda_{\max}$.

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- If we assume that the known eigenvalues of λ_{max} are accompanied by $\ell - 1$ other eigenvalues estimated to be λ_{min}, then we get: $0 = \text{trace } R = \sum_{i=1}^{\ell} \mu_i \geq (\ell - 1)\lambda_{\mathsf{min}} + \lambda_{\mathsf{max}}$.
- Hence

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- That is,

$$\ell \geq 1 + \frac{\lambda_{\text{max}}}{-\lambda_{\text{min}}}.$$

Hoffman's Theorem, Complementary Form

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Then

$$\ell(\psi) \geq 1 + \frac{\lambda_{\max}(A_{\overline{G}})}{-\lambda_{\min}(A_{\overline{G}})}.$$

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Hoffman's Theorem, Original/Coloring Form

Let G be a simple graph, not empty. Let A_G be the adjacency matrix of the graph.

$$\chi(G) \geq 1 + \frac{\lambda_{\mathrm{max}}(A_G)}{-\lambda_{\mathrm{min}}(A_G)}.$$

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- The above proof can be repeated with $\widetilde{\mathcal{T}_G}$ instead of $A_{\overline{G}}$.
- This opens up the possibility to improve the Hoffman's estimation, even optimizing these improvements.

Hoffman's Theorem, Strong Form

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Let G be a simple graph that is not a complete graph.

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Let G be a simple graph that is not a complete graph.

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Then

$$\ell(f) \geq 1 + \frac{\lambda_{\max}(M)}{-\lambda_{\min}(M)}.$$

That is,

$$\min\{\ell(f) : \ f \text{ is a clique cover}\} \geq$$
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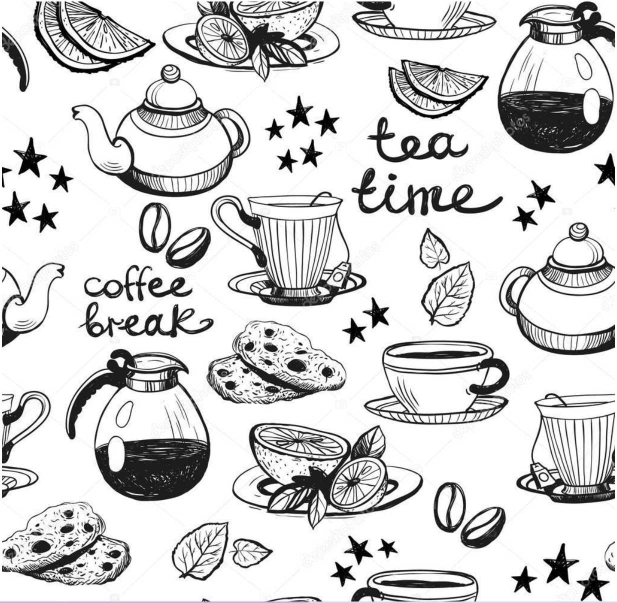
The value of the left-hand side of the final inequality is $\overline{\chi}(G)$.

Break



The Right-hand Side

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Let (L) denote the optimization problem on the right-hand side:

$$1 + \frac{\lambda_{\max}(M)}{-\lambda_{\min}(M)}$$

subject to:

$$M_{uu} = 0 \text{ for all } u \in V$$

and

$$\langle M, S_e \rangle = 0 \text{ for all } e \in E$$

with $M \in \mathcal{S}^n$.

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The journey towards our "beloved" SDP form will be long.

Intermediate Problem

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Let (K) be an intermediate problem:

$$\lambda_{\max}(N)$$

subject to:

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Theorem

The optimization problems describing the derivation of Hoffman's Theorem, (L), and the intermediate problem (K) are equivalent.

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Theorem

The optimization problems describing the derivation of Hoffman's Theorem, (L), and the intermediate problem (K) are equivalent.

The claim is that the optimal values of the two problems coincide. This is demonstrated by establishing both directions of the inequality between them.

From (L) Solution to (K) Solution with Non-worsening Objective

Maximize	$1 + \frac{\lambda_{\max}(M)}{-\lambda_{\min}(M)}$-t
subject to	$M_{uu} = 0$ for all $u \in V$
	$\langle M, S_e \rangle = 0$ for all $e \in E$
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	$M \in \mathcal{S}^n$.

We construct matrix $N = I + \frac{1}{-\lambda_{\min}(M)} M$ from a feasible M.

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It can be seen that the constructed N is a feasible solution to the intermediate problem.

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The smallest eigenvalue of $\frac{1}{-\lambda_{\min}(M)} M$ will be -1.

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It can be seen that the constructed N is a feasible solution to the intermediate problem.

The smallest eigenvalue of $\frac{1}{-\lambda_{\min}(M)} M$ will be -1. Thus, adding the identity matrix ensures all eigenvalues become non-negative.

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The smallest eigenvalue of $\frac{1}{-\lambda_{\min}(M)} M$ will be -1. Thus, adding the identity matrix ensures all eigenvalues become non-negative.

Moreover, its objective function value remains the same as the one defined on M in the original problem.

From (K) Solution to (L) Solution with Non-worsening Objective

Maximize	$\lambda_{\max}(N)$-t
subject to	$N_{uu} = 1$ for all $u \in V$
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This train of thought can be reversed. Let N be an optimal solution to the intermediate problem (K).

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This train of thought can be reversed. Let N be an optimal solution to the intermediate problem (K).

Observation

Suppose an off-diagonal element of a positive semidefinite matrix A is 0. Then, its corresponding row and column vectors are all 0 vectors.

From (K) Solution to (L) Solution (continued)

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- If N has a 0 off-diagonal element, then it has 0 row and column vectors on that part.

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From (K) Solution to (L) Solution (continued)

- If N has a 0 off-diagonal element, then it has 0 row and column vectors on that part. On this part, let N be 1 on the diagonal and 0 otherwise. The essential part of the construction lies in defining the other elements. We focus on this: essentially, we assume that the off-diagonal elements of Λ are non-zero.

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- Let u be a vector formed from square roots of the positive (assumed) elements on the diagonal of Λ. This will obviously be a unit vector.

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- Let N be the matrix for which $N \cdot_H U = \Lambda$ holds.

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- This is a feasible solution to problem (K) with a non-worsening objective function.

The Final Form, $(\widetilde{L})$

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After these, we show that the intermediate problem can be reformulated into the following SDP form $(\widetilde{L})$:

Maximize	$\langle J, \Lambda \rangle$-t
subject to	$\langle S_e, \Lambda \rangle = 0$
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The intermediate problem (K) and the SDP problem $(\widetilde{L})$ are equivalent (their optimal values coincide).

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Corollary

The optimal value of the SDP problem $(\widetilde{L})$ is an estimate of the twisted Hoffman bound for the clique cover problem.

From (K) Solution to ($\widetilde{L}$) Solution with Non-worsening Objective

From (K) Solution to $(\widetilde{L})$ Solution with Non-worsening Objective

The value of the objective function is

$$\langle J, \Lambda \rangle = \langle J, N \cdot_H (uu^\mathsf{T}) \rangle = u^\mathsf{T} N u = \lambda_{\max}(N).$$

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If we examine the positions in $\Lambda = N \cdot_H U$ where 0 appears in N, by definition, we see 0s.

$$\langle I, \Lambda \rangle = Tr(N \cdot_H (uu^\mathsf{T})) = Tr(uu^\mathsf{T}) = u^\mathsf{T} u = 1.$$

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- The positive semidefiniteness of Λ is left to be shown. This is obvious from the following lemma.

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• However, the positive semidefiniteness of these is obvious.

• With this lemma and one direction of the inequality, we have derived both directions.

From $(\widetilde{L})$ Solution to (K) Solution

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Let's reverse the above argument for the other direction of the inequality.

Maximize	$\langle J, \Lambda \rangle$-t
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- Suppose $\Lambda = \frac{1}{|V|} \cdot I$ is a feasible solution to the SDP.

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- Suppose $\Lambda = \frac{1}{|V|} \cdot I$ is a feasible solution to the SDP. The objective function value is at least 1.
- Let Λ be any feasible solution to the SDP.

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- Let Λ be any feasible solution to the SDP.

Note

Assume that one of the diagonal elements of a positive semidefinite matrix is 0. Then, the row and column vectors on this element are all 0 vectors.

From ($\widetilde{L}$) Solution to (K) Solution (continued)

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- If Λ has a 0 diagonal element, then it has 0 row and column vectors on that part.

From $(\widetilde{L})$ Solution to (K) Solution (continued)

- If Λ has a 0 diagonal element, then it has 0 row and column vectors on that part. On this part, let N be such that it is 1 on the diagonal and 0 otherwise.

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- If Λ has a 0 diagonal element, then it has 0 row and column vectors on that part. On this part, let N be such that it is 1 on the diagonal and 0 otherwise. The essential part of the construction lies in defining the other elements. We focus on this: essentially, we assume that the off-diagonal elements of Λ are non-zero.

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- Let u be a vector formed from square roots of the positive (assumed) elements on the diagonal of Λ. This will obviously be a unit vector.

- Let $U = uu^{\mathsf{T}}$. According to our assumption, this is a non-zero matrix.

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Theorem

$$\widetilde{\vartheta}(G) \leq \overline{\chi}(G).$$

Reminder

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For both $\alpha(G)$ and $\overline{\chi}(G)$, we provided an SDP problem, the optimal value of which estimates the corresponding graph parameter:

Minimize	μ-t
subject to	$-\mu I - \sum_{e\in E} x_e S_e \preceq -J.$

Maximize	$\langle J, \Lambda\rangle$-t
subject to	$\langle S_e, \Lambda\rangle = 0$
	$\langle I, \Lambda\rangle = 1$
	$\Lambda \succeq 0.$

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Lovász's Sandwich Theorem

$$\alpha(G) \leq \vartheta(G) \leq \overline{\chi}(G).$$

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• However, the two extreme graph optimization questions are complex. $\mathcal{NP}$-hard. We see no possibility for efficient calculation (according to the general belief).

• The value of the intermediate function, however, can be calculated/approximated efficiently (SDP problems are manageable).

This is the End!

Thank you for your attention!