Semidefinite programming

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2024. Fall

Semidefinite Programming, SDP

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General Formulation of Semidefinite Programming

Minimize	$c^{\mathsf{T}}x$-t
subject to	$\sum_{i=1}^{n} x_i A_i \preceq B$
	$Dx = e,$

where $c, x \in \mathbb{R}^n$, $A_i, B \in \mathcal{S}^k = \{M \in \mathbb{R}^{k\times k} : M^{\mathsf{T}} = M\} \subset \mathbb{R}^{k\times k}$, $D \in \mathbb{R}^{\ell\times n}$, and $e \in \mathbb{R}^{\ell}$.

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Notation

$A \preceq B$ if and only if $A, B \in \mathcal{S}^n$ and $0 \preceq B - A$, i.e., $B - A$ is positive semidefinite, denoted as $B - A \in \mathcal{S}^n_+$.

$\sum_{i=1}^{n} x_i A_i$

$$\begin{pmatrix} A_{1,1}^{(1)}x_1 + \ldots + A_{1,1}^{(n)}x_n & A_{1,2}^{(1)}x_1 + \ldots + A_{1,2}^{(n)}x_n & \ldots & A_{1,k}^{(1)}x_1 + \ldots + A_{1,k}^{(n)}x_n \\ A_{2,1}^{(1)}x_1 + \ldots + A_{2,1}^{(n)}x_n & A_{2,2}^{(1)}x_1 + \ldots + A_{2,2}^{(n)}x_n & \ldots & A_{2,k}^{(1)}x_1 + \ldots + A_{2,k}^{(n)}x_n \\ \vdots & \vdots & & \vdots \\ A_{k,1}^{(1)}x_1 + \ldots + A_{k,1}^{(n)}x_n & A_{k,2}^{(1)}x_1 + \ldots + A_{k,2}^{(n)}x_n & \ldots & A_{k,k}^{(1)}x_1 + \ldots + A_{k,k}^{(n)}x_n \end{pmatrix}$$

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Example

$$\begin{pmatrix} 2x_1 - 3x_2 + x_3 & 5x_1 + 2x_2 - x_3 & x_1 - x_2 + 8x_3 & 6x_1 + 5x_2 + x_3 \\ 5x_1 + 2x_2 - x_3 & -x_1 + 7x_2 - 2x_3 & 9x_1 - 3x_2 + x_3 & -2x_1 - x_2 + 4x_3 \\ x_1 - x_2 + 8x_3 & 9x_1 - 3x_2 + x_3 & 10x_1 - 2x_2 + 2x_3 & 8x_1 + x_2 + x_3 \\ 6x_1 + 5x_2 + x_3 & -2x_1 - x_2 + 4x_3 & 8x_1 + x_2 + x_3 & -11x_1 + 2x_2 - 3x_3 \end{pmatrix}$$

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Observation

The LP problem is a special case of an SDP problem.

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$\mathcal{S}^n$ is a linear subspace of $\mathbb{R}^{n\times n}/\mathbb{R}^{[n]\times[n]}/\mathbb{R}^{H\times H}$. Its dimension is $\left(\!\!\binom{n}{2}\!\!\right)$.

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Notation

$\langle .,. \rangle : \mathcal{S}^n \times \mathcal{S}^n \to \mathbb{R}$ is the following inner product: For $M, N \in \mathcal{S}^n$,

$$\langle M, N\rangle = \mathrm{Tr}(M^{\mathrm{T}} N) = \sum_j (M^{\mathrm{T}} N)_{jj} = \sum_{i,j} M^{\mathrm{T}}_{ji} N_{ij} = \sum_{i,j} M_{ij} N_{ij}.$$

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An alternative: Let $vec : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$ be the vectorization of a matrix (i.e., stacking the columns of a table into a vector). Then $\langle M, N\rangle = vec(M)^{\mathsf{T}} vec(N)$.

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Below we summarize the properties of the introduced inner product. Interested students can verify these properties themselves.

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Lemma

Let $M, N, P \in \mathcal{S}^n$

(i) $\langle M, N \rangle = \langle N, M \rangle$

(ii) $\langle MN, P \rangle = \langle M, PN \rangle$

(iii) $\langle M, M \rangle = \mathrm{Tr} M^2 = \sum_{i=1}^{n} \lambda_i^2 \geq 0$

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$\|M\|_F = \sqrt{\langle M, M \rangle}$, the Frobenius norm of a symmetric matrix M.

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(iv) Every symmetric submatrix of M (obtained by deleting s rows and the corresponding s columns) has a nonnegative determinant.

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$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \qquad \Lambda^{\frac{1}{2}} = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}.$$

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- Thus,

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- Choosing $M^{\frac{1}{2}} = Q \Lambda^{\frac{1}{2}} Q^{\mathsf{T}}$ proves the Lemma.

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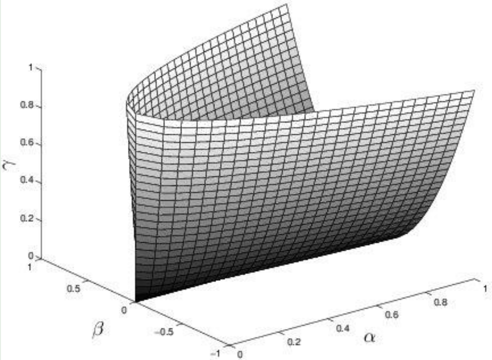
The Set of Positive Semidefinite Matrices $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$



Normal Forms of SDP

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SDP: I. Normal Form

$$\begin{array}{|lll|}\hline \text{Minimize} & \langle C, X\rangle\text{-t} & \\ \text{subject to} & \langle A_i, X\rangle = b_i, & i = 1, \dots k \\ & X \succeq 0, & \\ \hline \end{array}$$

where $C, X, A_i \in \mathcal{S}^n$ and $b \in \mathbb{R}^k$.

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SDP: II. Normal Form

$$\begin{array}{|ll|}\hline \text{Minimize} & c^{\mathsf{T}}x\text{-t} \\ \text{subject to} & \sum_{i=1}^n x_i A_i \preceq B, \\ \hline \end{array}$$

where $c, x \in \mathbb{R}^n$ and $A_i, B \in \mathcal{S}^k$.

Break

Duality (Lagrange Method)

We examine the primal problem in the first normal form:

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However, we cannot incorporate the positive semidefiniteness constraint into L, so we "constrain" the domain of L with this condition.

$$L(X, \mu) = \langle C, X\rangle + \sum \mu_i(\langle A_i, X\rangle - b_i), \quad \mathrm{dom}\, L = \{X : X \succeq 0\}.$$

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Minimizing this function leads to the objective function of the dual problem.

$$\widehat{c}(\mu) = \inf_{X:\ X \succeq 0} L(X, y) = -\sum b_i \mu_i + \inf \left\langle C + \sum \mu_i A_i, X \right\rangle.$$

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We need to solve the optimization problem

$$\inf_{X \succeq 0} \langle M, X \rangle = ?.$$

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If $M, N \succeq 0$, then $\langle M, N\rangle \geq 0$.

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We need a simple equality:

Auxiliary Lemma

Let $A \in \mathbb{R}^{k\times\ell}$ and $B \in \mathbb{R}^{\ell\times k}$. Then

$$\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA).$$

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$$\begin{aligned}
\langle M, X\rangle &= \operatorname{Tr}(MX) = \operatorname{Tr}(M^{1/2}M^{1/2}X^{1/2}X^{1/2}) \\
&= \operatorname{Tr}((M^{1/2}M^{1/2}X^{1/2})X^{1/2}) = \operatorname{Tr}(X^{1/2}(M^{1/2}M^{1/2}X^{1/2})) \\
&= \operatorname{Tr}((X^{1/2}M^{1/2})(M^{1/2}X^{1/2})) \\
&= \operatorname{Tr}((M^{1/2}X^{1/2})^{\mathsf{T}}(M^{1/2}X^{1/2})) \geq 0,
\end{aligned}$$

since $M^{1/2}$ and $X^{1/2}$ are symmetric.

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- Conversely, assume that $\langle M, X\rangle \geq 0$ for all $X \succeq 0$ matrices. Let $x \in \mathbb{R}^n$ be arbitrary. Apply our assumption to the positive semidefinite matrix $X = xx^{\mathsf{T}}$:

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$$\begin{aligned}0 &\leq \langle M, X\rangle = \operatorname{Tr}(M(xx^{\mathsf T})) = \operatorname{Tr}((Mx)x^{\mathsf T}) = \operatorname{Tr}(x^{\mathsf T}(Mx))\\ &= \operatorname{Tr}(x^{\mathsf T}Mx) = x^{\mathsf T}Mx.\end{aligned}$$

Thus, M is positive semidefinite.

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• Indeed, if $M \succeq 0$, then by the lemma $\langle M, X\rangle$ cannot be negative, but $M = 0$ shows it can be 0.

• Similarly, by the lemma, if $M \not\succeq 0$, then $\langle M, X\rangle$ can be negative. Multiplying X by a positive scalar, we can make it arbitrarily large in absolute value and negative.

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Lemma

(i) $\mathcal{S}_+^n$ is a cone.

(ii) $\mathcal{S}_+^n$ is self-dual, $(\mathcal{S}_+^n)^* = \mathcal{S}_+^n$.

Dual Problem

Dual Problem

As a result of the detour, we obtain that

$$\inf_{X \succeq 0} L(X, \mu) = \begin{cases} -\sum b_i \mu_i, & C + \sum \mu_i A_i \succeq 0 \\ -\infty, & \text{otherwise} \end{cases}.$$

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Thus, we find the dual of the starting SDP (primal) problem.

Maximize	$-\sum b_i \mu_i$-t
subject to	$-\sum \mu_i A_i \preceq C$

Dual Problem

As a result of the detour, we obtain that

$$\inf_{X \succeq 0} L(X, \mu) = \begin{cases} -\sum b_i \mu_i, & C + \sum \mu_i A_i \succeq 0 \\ -\infty, & \text{otherwise} \end{cases}.$$

Thus, we find the dual of the starting SDP (primal) problem.

Maximize	$-\sum b_i \mu_i$-t
subject to	$-\sum \mu_i A_i \preceq C$

Equivalently,

Minimize	$\sum b_i \mu_i$-t
subject to	$C + \sum \mu_i A_i = S,$
	$S \succeq 0.$

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Claim

$\langle C, X \rangle \geq -b^{\mathsf{T}}\mu$.

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Claim

$\langle C, X \rangle \geq -b^{\mathsf{T}} \mu$.

The claim states that any primal objective value is at least as large as any dual objective value. From this, the theorem trivially follows.

Proof of Weak Duality

We have seen that the product of two positive semidefinite matrices is nonnegative. Hence,

$$\begin{aligned}0 \leq& \langle X, S\rangle = \langle X, C + \sum \mu_i A_i\rangle = \langle X, C\rangle + \sum \mu_i \langle A_i, X\rangle \\ =& \langle C, X\rangle + \sum \mu_i b_i = \langle C, X\rangle + b_i^{\mathsf{T}} \mu_i.\end{aligned}$$

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From this, the claim follows by rearrangement.

Example

Minimize	x_{12}-t
subject to	$\begin{pmatrix} 0 & x_{12} & 0 \\ x_{12} & x_{22} & 0 \\ 0 & 0 & 1+x_{12} \end{pmatrix} \succeq 0$

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This is equivalent to setting $x_{11} = 0$, $x_{13} = 0$, $x_{23} = 0$, $x_{33} = 1 + x_{12}$.

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- Setting $x_{13} = 0$: $\left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, X \right\rangle = 0$ (this equality constraint corresponds to the dual variable μ_2).

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- Setting $x_{23} = 0$: $\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, X \right\rangle = 0$ (this equality constraint corresponds to the dual variable μ_3).
- Setting $x_{33} = 1 + x_{12}$: $\left\langle \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X \right\rangle = 1$ (this equality constraint corresponds to the dual variable μ_4).

The Example in Normal Form

The new form of our problem:

Minimize	$\left\langle \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X \right\rangle$-t
subject to	$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X \right\rangle = 0, \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, X \right\rangle = 0,$
	$\left\langle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, X \right\rangle = 0, \left\langle \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, X \right\rangle = 1.$
	$X \succeq 0$

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Maximize $-\mu_4$-t

subject to

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} +$$

$$\mu_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \mu_4 \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \succeq 0.$$

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$$\mu_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \mu_4 \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \succeq 0.$$

So,

Maximize $-\mu_4$-t

subject to

$$\begin{pmatrix} \mu_1 & \frac{1-\mu_4}{2} & \mu_2 \\ \frac{1-\mu_4}{2} & 0 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix} \succeq 0.$$

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If there exists an $X \in \mathcal{L}$, then the upper-left 2×2 principal minor of the matrix in the example's condition must be positive semidefinite. Thus, the product of its eigenvalues (determinant) must be greater than zero.

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In our case, $-x_{12}^2 \geqslant 0$, so $x_{12} = 0$. Hence $p^* = 0$.

d^* with Elementary Methods

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Maximize	$-\mu_4$-t
subject to	$\begin{pmatrix} \mu_1 & \frac{1-\mu_4}{2} & \mu_2 \\ \frac{1-\mu_4}{2} & 0 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix} \succeq 0.$

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d^*: Based on the previous considerations, it must hold that

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This only holds for $\mu_4 = 1$. So, $d^* = -1$.

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Thus, indeed the weak duality theorem holds, so $p^* \geqslant d^*$ $(0 \geqslant -1)$.

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However, there is no strong duality.

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$X \in \mathcal{L}_P^+$ if and only if $\langle A_i, X\rangle = b_i, i = 1, \ldots, \ell$ and $X \succ 0$ (i.e., X is positive definite).

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We won't prove the theorems here.

This is the End!

Thank you for your attention!