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### Independent sets, perfect graphs

Péter Hajnal

2025 Spring

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#### Reminder

We call a set  $F \subseteq V(G)$  an independent vertex set if every edge of G has at most one endpoint in F, i.e., if the vertices of F are not connected by an edge.

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Using it, we define the following polytope.

#### Definition, vertex packing polytope

$$\mathcal{PP}(G) := \operatorname{conv} \{\chi_F : F \subseteq V(G) \text{ independent vertex set}\} \subseteq \mathbb{R}^V$$
,

where  $\chi_F \in \{0,1\}^V$  is the characteristic vector of the vertex set *F*.

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Perfect Graphs

Fulkerson—Chvatal Theorem

## Simple inequalities

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# Simple inequalities

#### Observation

We can write a simple containment for this set:

$$\mathcal{PP}(G) \subseteq \{x \in \mathbb{R}^V : x \succeq 0, \text{ and for every } e = uv \text{ edge}, \ x_u + x_v \leq 1\} =: \mathcal{PP}_0(G).$$

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The set on the right-hand side consists of non-negative vectors for which the sum of two components is less than or equal to 1 whenever there is an edge between the corresponding vertices in G.

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The set on the right-hand side consists of non-negative vectors for which the sum of two components is less than or equal to 1 whenever there is an edge between the corresponding vertices in G.

To establish the containment, we only need to verify that the characteristic vectors of independent vertex sets are contained in the intersection of the half-spaces described on the right-hand side (which obviously defines a convex vertex set).

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#### Theorem

Let G be a bipartite graph without isolated vertices. Then

 $\mathcal{PP}(G) = \mathcal{PP}_0(G).$ 

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In the general case, further conditions are necessary,

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Fulkerson—Chvatal Theorem

### **Clique Conditions**

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Among the missing conditions, perhaps the simplest ones are given below.

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# Clique Conditions

Among the missing conditions, perhaps the simplest ones are given below.

### Definition, clique conditions

$$\widehat{\mathcal{PP}}(G) := \{ x \in \mathbb{R}^V : x \succeq 0, \text{ and for every } K \text{ clique,} \ \sum_{u \in K} x_u \leq 1 \}.$$

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## Simple Containments

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• A pair of vertices connected by an edge actually forms a clique of size two. Thus, the edge conditions are special clique conditions.

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• A clique and an independent vertex set can have at most one common vertex (since if there were two vertices, they would have to be connected, which they are not).

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• Thus,

$$\mathcal{PP}(G)\subseteq\widehat{\mathcal{PP}}(G).$$

• Summarizing,

$$\mathcal{PP}(G) \subseteq \widehat{\mathcal{PP}}(G) \subseteq \mathcal{PP}_0(G)$$

Perfect Graphs



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### Example

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Note that  $\alpha(C_5) = 2$ , so the independent vertex sets have at most two elements.

Thus, in their characteristic vectors, the sum of components does not exceed 2. The same holds true for any element of their convex hull.

However, the vector  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  does not fit this condition.

Independent vertex sets and LP

Perfect Graphs

Fulkerson—Chvatal Theorem

### Another Example

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#### Perfect Graphs

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For  $C_3$ , not only the edge conditions appear, but also the inequality  $x_1 + x_2 + x_3 \le 1$ .

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• It's easy to verify that

$$\{x \in \mathbb{R}^3 : x \succeq 0, x_1 + x_2 + x_3 \le 1\} = \operatorname{conv}(0, e_1, e_2, e_3),$$

where  $e_i$  are the standard basis, i.e., one-element subsets (which are exactly the nonempty matchings in  $K_3$ ) characteristic vectors.

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$$\widehat{\mathcal{PP}}(\mathcal{C}_3) \subsetneq \mathcal{PP}_0(\mathcal{C}_3) \ni (1/2, 1/2, 1/2)$$

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• Under what conditions on G will  $\mathcal{PP}(G) = \widehat{\mathcal{PP}}(G)$  hold? We explore this question further, but it requires a small digression into graph theory.

### Break



Perfect Graphs

## Colorings, Cliques

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# Colorings, Cliques

#### Reminder

Let *c* be a proper coloring, and *K* be a clique of *G*. We know that a clique can only be colored by assigning different colors to each of its vertices. Therefore, the number of required colors is at least |K|.

We obtain the graph parameter  $\chi(G)$  by considering the minimal number of colors needed, and  $\omega(G)$  by considering the maximal clique size. However, the inequality holds true even between these optima, so  $\chi(G) \ge \omega(G)$ .

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#### Definition

G graph is nice if 
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Perfect Graphs

Fulkerson—Chvatal Theorem

### The Main Notion

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Perfect Graphs

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#### Definition

G graph is perfect if every induced subgraph R of G is nice.

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Perfect Graphs

### The Main Notion

#### Definition

G graph is perfect if every induced subgraph R of G is nice.

Usually, we only mention the authors of major theorems and proofs, but in some cases, it's worth mentioning the inventors of important definitions as well. Perfect graphs were first defined by Claude Berge, a French mathematician, in 1962.

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#### Example

The complete graph with *n* vertices,  $K_n$ , and the empty graph with *n* vertices,  $\overline{K_n} = E_n$ , are both perfect.

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- In empty graphs,  $\chi(E_n) = \omega(E_n) = 1$  for all n.

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- But also note that any induced subgraph of a complete graph or an empty graph remains complete or empty, respectively.

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- This is only a small part of the definition.
- But also note that any induced subgraph of a complete graph or an empty graph remains complete or empty, respectively.
- Thus, both are perfect.

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Perfect Graphs

### Examples II.

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#### Example

If *G* is bipartite, then it's also perfect. Here we provide an entire class of examples. As we know, any induced subgraph of a bipartite graph is also bipartite, so it suffices to prove the niceness of bipartite graphs. Suppose *G* has an edge. Then it can be colored with at most two colors, and its maximal cliques are the edges, so  $\chi(G) = \omega(G) = 2$ .

#### Example

If *G* is bipartite, then it's also perfect. Here we provide an entire class of examples. As we know, any induced subgraph of a bipartite graph is also bipartite, so it suffices to prove the niceness of bipartite graphs. Suppose *G* has an edge. Then it can be colored with at most two colors, and its maximal cliques are the edges, so  $\chi(G) = \omega(G) = 2$ .

#### Example

### If G is bipartite, then $\overline{G}$ is perfect.

In the complement of bipartite graphs, the lower and upper vertex sets form cliques, and the remaining edges connect vertices between these sets. The niceness of this is nontrivial, equivalent to Kőnig's theorem. We leave the verification to the interested student.

Perfect Graphs

### Examples III.

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#### Example

Consider an arbitrary partially ordered set (P, <). The partial ordering means that < is a relation, but we don't require that any two elements of P be comparable.

From this set, we can define a graph by taking the vertex set as P and connecting two points only if they are comparable as elements of the partially ordered set. This is called the comparability graph of (P, <).

Analogously, we can define the incomparability graph, and we can observe that these two concepts are graph theoretically related by complementation.

The interested student can also verify that both are perfect graphs.

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Perfect Graphs

### Examples IV.

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#### Example

 $C_{2k+1}$   $(k \ge 2)$ , an odd cycle of length 2k + 1, is not perfect and not even nice, as  $\chi(C_{2k+1}) = 3 \ne 2 = \omega(C_{2k+1})$ . This does not apply to the triangle graph, so we cannot generally speak of odd cycles, only those with more than five vertices. We leave it to the interested student to verify that  $\overline{C_{2k+1}}$   $(k \ge 2)$ is also not nice.

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### A Note

### Notation

 $G \supseteq R$  means R is an induced subgraph of G.

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### Observation

After the definition of perfection and the examples, the following are trivial:

- (i) If G is perfect and  $G \supseteq R$ , then R is also perfect.
- (ii) If G is not perfect and  $S \supseteq G$ , then S is also not perfect.
- (iii) If  $G \sqsupseteq C_{2k+1}$  or  $G \sqsupseteq \overline{C_{2k+1}}$  for some  $k \ge 2$ , then G is not perfect.

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- (ii) If G is not perfect and  $S \supseteq G$ , then S is also not perfect.
- (iii) If  $G \sqsupseteq C_{2k+1}$  or  $G \sqsupseteq \overline{C_{2k+1}}$  for some  $k \ge 2$ , then G is not perfect.

For all known non-perfect graphs, non-perfection follows from observation (iii). Based on this, Berge first formulated in 1962 as a conjecture that observation (iii) can be reversed (strong perfect graph conjecture).

Perfect Graphs

Fulkerson—Chvatal Theorem

# Strong Perfect Graph Conjecture/Theorem

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# Strong Perfect Graph Conjecture/Theorem

It was only proven in 2006 by the quartet of Chudnovsky—Seymour—Robertson—Thomas. Their paper is over 100 pages long, and the proof of the theorem alone would fit into a semester-long PhD course.

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Theorem (Strong Perfect Graph Theorem, Chudnovsky—Seymour—Robertson—Thomas, 2006)

*G* is not perfect if and only if it does not contain  $C_{2k+1}$  and  $\overline{C_{2k+1}}$  graphs as induced subgraphs for  $k \ge 2$ .

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*G* is not perfect if and only if it does not contain  $C_{2k+1}$  and  $\overline{C_{2k+1}}$  graphs as induced subgraphs for  $k \ge 2$ .

Let *G* be perfect. We know it cannot contain any odd cycle of length at least five or its complement as induced subgraphs. This property naturally holds for its complement. Thus, if the strong perfect graph conjecture is true, then  $\overline{G}$  is also perfect. Consequently, the perfection of *G* and  $\overline{G}$  go hand in hand.

### Weak Perfect Graph Theorem

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# Weak Perfect Graph Theorem

Berge also saw this line of thought. He couldn't prove it (it might be possible with the proof of the strong conjecture absent). He referred to this assertion as the weak perfect graph conjecture. It remained unsolved *only* for ten years.

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#### Theorem (Weak Perfect Graph Theorem, Lovász, 1972)

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#### Theorem (Weak Perfect Graph Theorem, Lovász, 1972)

G is perfect if and only if  $\overline{G}$  is perfect.

Without the strong version, this theorem is not a trivial statement either, but it can be proven in a single lecture of an MSc course. In the rest of the lecture, our aim will be to prove this theorem.

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#### Break



#### Perfect Graphs and Polytopes

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# Perfect Graphs and Polytopes

#### Theorem (Fulkerson—Chvatal Theorem, 1973)

The following are equivalent:

- (i)  $\mathcal{PP}(G) = \widehat{\mathcal{PP}}(G)$
- (ii) The vertices of  $\widehat{\mathcal{PP}}(G)$  are integers
- (iii) G is perfect

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The equivalence  $(i) \iff (ii)$  is based on a simpler statement, which we do not formally describe. After the previous ideas and reasoning, the interested student can easily understand it.

The proof of  $(ii) \iff (iii)$  relies on the following assertions:

G is perfect  $\Rightarrow_{(A)}$  The vertices of  $\widehat{\mathcal{PP}}(G)$  are integers  $\Rightarrow_{(B)} \overline{G}$  is perfect.

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Repeating the above inferences for  $\overline{G}$ , we obtain the complete proof of the Fulkerson—Chvatal Theorem and, along with it, the weak perfect graph theorem.

It also follows that for non-perfect graphs, beyond the clique conditions, additional inequalities are necessary for describing  $\mathcal{PP}$ . This research direction remains active and important to this day.

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Perfect Graphs

### Graph Theory Preliminaries, 1st Lemma

Perfect Graphs

# Graph Theory Preliminaries, 1st Lemma

To prove (A), we will first establish two lemmas.

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# Graph Theory Preliminaries, 1st Lemma

To prove (A), we will first establish two lemmas.

#### Lemma

G is perfect if and only if every spanning subgraph satisfies

(\*) : there exists an independent set intersecting every optimal (maximum size) clique.

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Independent vertex sets and LP

Perfect Graphs

Fulkerson—Chvatal Theorem

### Proof of 1st Lemma ( $\Leftarrow$ )

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Thus, we have obtained k independent sets, a k-coloring.

Independent vertex sets and LP

Perfect Graphs

Fulkerson—Chvatal Theorem

# Proof of 1st Lemma $(\Longrightarrow)$

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We will prove from the niceness of G that it satisfies  $(\star)$ .

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These classes are independent sets, so if we take an optimal clique, it can intersect each color class at most once, but due to the niceness of G, it is necessary that it also contains one vertex from each class.

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Thus, by choosing any color class, we exhibit a set satisfying  $(\star)$ .

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#### Perfect Graphs

#### A Note

At the end of the previous proof, we saw that we have great freedom in choosing the set satisfying  $(\star)$ , as any color class of an optimal coloring suffices. Based on this, we can replace  $(\star)$  with a stricter condition, the fulfillment of which remains equivalent to the perfection of *G* for every spanning subgraph.

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# Lemma<sup>+</sup> *G* is perfect if and only if every spanning subgraph satisfies $(\star)^+$ : for any $x \in V(G)$ , there exists an independent set containing *x*, intersecting every optimal clique.

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Perfect Graphs

### Graph Theory Preliminaries, Blow up

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# Graph Theory Preliminaries, Blow up

For the final lemma, we will need a graph-theoretical operation.

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#### Definition

Let G be a graph with an associated vector  $n = (n_v)_{v \in V} \in \mathbb{N}^V$ . Each component assigns a natural number to every vertex.

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The *blow up* of *G* (with the vector *n*) is the graph obtained by replacing each vertex with a clique of size  $n_v$ , and connecting the vertices of these resulting cliques (each with all others) if and only if the corresponding vertices in the original graph were connected.

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Perfect Graphs

Fulkerson—Chvatal Theorem

# Graph Theory Preliminaries, 2nd Lemma

#### 2nd Lemma

Let G be a perfect graph and  $n \in \mathbb{N}^V$ ,  $n = (n_v)_{v \in V}$  an arbitrary vector. Then G remains perfect when blown up with the vector n.

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#### Remarks

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#### 1st Remark

Let H be the graph obtained from G by blow up with n. According to the previous lemma, it suffices to show that  $(\star)$  holds for all spanning subgraphs of H. However, H's subgraphs are blown up graphs of G with different vectors, so once we prove that H satisfies  $(\star)$  for an arbitrary vector n, we need not concern ourselves with its spanning subgraphs.

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#### 2nd Remark

A blow up can be done step by step, only one vertex blown up at a time. Thus, we can assume that  $n = (1, ..., 1, n_v, 1, ..., 1)$ , as the ones in the vector represent cliques of a single vertex. The blown up vertex is denoted by v.

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Fulkerson—Chvatal Theorem

#### Proof of 2nd Lemma

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H is the graph obtained from the perfect graph G by blowing up the vertex v.

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Given *F*, we can construct an independent set *F'* in *H*, defined as  $\{v \text{ or one of the vertices blown up from } v\} \cup (F \setminus v)$ . We need to show that *F'* satisfies (\*) for *H*, i.e., intersects every optimal clique.

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Notice that the optimal cliques in H can be of two types:

- a) Contains the entire blown up vertex set of v,
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- a) Contains the entire blown up vertex set of v,
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Type a) cliques are intersected by F' because it includes one of the vertices blown up from v. Type b) cliques are intersected because of  $F \setminus \{v\}$ . This completes the proof of the lemma.

Fulkerson—Chvatal Theorem

# G is Perfect $\Rightarrow_{(A)}$ Vertices of $\widehat{\mathcal{PP}}(G)$ are Integers

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The lemma assures us that H is perfect.

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## G is Perfect $\Rightarrow_{(A)}$ Vertices of $\widehat{\mathcal{PP}}(G)$ are Integers

Let *G* be perfect,  $e \in \widehat{\mathcal{PP}(G)}$  a vertex of the polytope.

Our goal is to show that  $e \in \mathbb{Z}^V$ .

We know that  $e \in \mathbb{Q}^V$ . Let  $e = (\frac{n_V}{N})_{v \in V}$  be the common denominator form of the coordinates.

We use the previous lemma on the graph G and the vector  $n = (n_v)_{v \in V}$ , thus constructing the blown up graph H.

The lemma assures us that H is perfect.

Let's find the largest clique in H, or in other words, the optimal clique K of G for which  $\sum_{v \in K} n_v$  is maximal.

### Proof of (A) (continued)

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## Proof of (A) (continued)

Since  $(\frac{n_v}{N}) \in \widehat{\mathcal{PP}(G)}$ , we have

$$\sum_{v \in K} rac{n_v}{N} \leq 1$$
 and  $\sum_{v \in K} n_v \leq N,$ 

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From the perfection of H, we deduce that it can be colored with N colors.

Thus, there exist independent sets  $F_1, \ldots, F_N$  whose union covers V(H).

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### Proof of (A) (continued)

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## Proof of (A) (continued)

The sets  $F_i$  can be projected back to G: let  $F_i \subset V(H)$  correspond to  $\Phi_i \subset V(G)$ .

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The union of  $\Phi_i$  covers each vertex  $v n_v$  times. Thus,

$$\sum_{i=1}^N \chi_{\Phi_i} = (n_1, n_2, \ldots).$$

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Dividing by N, we obtain:

$$\sum_{i=1}^{N} \frac{\chi_{\Phi_i}}{N} = e.$$

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Independent vertex sets and LP

## Conclusion of (A)

Perfect Graphs

Fulkerson—Chvatal Theorem

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• The left-hand side represents an *average* of the independent set characteristic vectors (specifically,  $\widehat{\mathcal{PP}}$  elements).

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- Indeed: If e is a vertex of  $\widehat{\mathcal{PP}}$ , then a suitable half-space  $\{x \in \mathbb{R}^V : \nu^T x \leq b\}$  contains  $\widehat{\mathcal{PP}}$  such that only e satisfies the inequality as an equality.

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• This contradiction proves the claimed relationship,

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Perfect Graphs

Fulkerson—Chvatal Theorem

## Proving $\overline{G}$ Perfect if Vertices of $\widehat{\mathcal{PP}}(G)$ are Integers

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## Proving $\overline{G}$ Perfect if Vertices of $\mathcal{PP}(G)$ are Integers

The vertices of  $\widehat{\mathcal{PP}}(G)$  are integers, meaning that they represent characteristic vectors of independent sets.

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The boundaries of this half-space correspond to the characteristic vectors of maximal independent sets.

More precisely,

$$\widehat{\mathcal{PP}}(\mathcal{G}) \cap \{ x \in \mathbb{R}^{V} : 1^{\mathsf{T}} x \leq \alpha(\mathcal{G}) \} =$$
  
conv  $\{ \chi_{\mathcal{F}} : \mathcal{F} \text{ is a maximal independent set of size } \alpha(\mathcal{G}) \}.$ 

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This forms a facet of the polytope.

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## Proof of (B) (continued)

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# Proof of (B) (continued)

One of the supporting hyperplanes derived from the inequalities defining the polytope must contain it.

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One of the supporting hyperplanes derived from the inequalities defining the polytope must contain it.

Moreover, such a hyperplane must also exist among the clique constraints.

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One of the supporting hyperplanes derived from the inequalities defining the polytope must contain it.

Moreover, such a hyperplane must also exist among the clique constraints.

If the hyperplane  $\sum_{\nu:\nu\in K} x_{\nu} = 1$  associated with a clique K contains all characteristic vectors of maximal independent sets, then  $\alpha(G - K) < \alpha(G)$ .

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## Conclusion of Proof of (B)

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The conclusion of proving statement (B) is that if the vertices of  $\widehat{\mathcal{PP}}(G)$  are integers, then the vertices of  $\widehat{\mathcal{PP}}(R)$  are integers for every spanning subgraph  $R \sqsubseteq G$ .

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Seeing that we found the above clique K in G, we can find a similar clique K' in G - K.

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Continuing this process, we can cover V(G) with  $\alpha(G)$  cliques. This precisely means that  $\overline{G}$  is perfect.

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Continuing this process, we can cover V(G) with  $\alpha(G)$  cliques. This precisely means that  $\overline{G}$  is perfect.

The same line of reasoning holds for every spanning subgraph of  $\overline{G}$ , proving that  $\overline{G}$  is perfect.

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#### This is the end!

# Thank you for your attention!

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