Geometry of linear programming

Péter Hajnal

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Minimize	$c^{\mathsf{T}}x$-t
subject to	$Ax \preceq b$

where $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{k\times n}$, $b \in \mathbb{R}^k$.

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- In this normal form, only linear inequalities are allowed among the constraints.

- Another common normal form is:

Minimize	$c^{\mathsf{T}}x$-t
subject to	$Ax = b,$
	$x \succeq 0.$

LP Duality

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(i) $p^* = d^*$, i.e., strong duality holds,

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- If $p^* = -\infty$, weak duality guarantees strong duality.

- The only loophole for an LP problem to evade strong duality is to have $p^* = \infty$ and $d^* = -\infty$. That is, both primal and dual problems are infeasible. This possibility is not theoretical; it can occur in concrete examples.

Solution Set of a Linear Equation

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Solution Set of Linear Inequalities

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$\nu \in \mathbb{R}^n - \{0\}$. The solution set of the non-trivial linear homogeneous inequality $\nu^\mathsf{T} x \leq 0/\nu^\mathsf{T} x \geq 0$ is not trivial.	$\nu \in \mathbb{R}^n - \{0\}$ is a normal vector. The inequality $\nu^\mathsf{T} x \leq 0/\nu^\mathsf{T} x \geq 0$ defines a CLOSED half-space bounded by a hyperplane passing through the origin and perpendicular to ν.
$\nu \in \mathbb{R}^n - \{0\}$, $b \in \mathbb{R}$. The solution set of the non-trivial linear inequality $\nu^\mathsf{T} x \leq b/\nu^\mathsf{T} x \geq b$ is not trivial.	$\nu \in \mathbb{R}^n - \{0\}$ is a normal vector. The inequality $\nu^\mathsf{T} x \leq b = \nu^\mathsf{T} v_0/\nu^\mathsf{T} x \geq b$ defines a CLOSED half-space bounded by a hyperplane passing through v_0 and perpendicular to ν.

Formal Definitions

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Definition

Let $\nu \in \mathbb{R}^n$ be a nonzero vector, τ any real number. Then the set $\{x \in \mathbb{R}^n : \nu^\mathsf{T} x = \tau\}$ is called a hyperplane in $\mathbb{R}^n$. The sets of the form $\{x \in \mathbb{R}^n : \nu^\mathsf{T} x \leq \tau\}$ are called (closed) half-spaces.

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Every hyperplane defines two closed half-spaces, which share the same boundary.

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Lemma

Half-spaces and hyperplanes are convex.

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$A \in \mathbb{R}^{k\times n}$. Solution set of the homogeneous linear inequality system $Ax \preceq 0$.	Intersection of finitely many closed half-spaces passing through the origin $\equiv$ *polyhedral (closed, convex) cone*.

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$A \in \mathbb{R}^{k\times n}, b \in \mathbb{R}^k$. Solution set of the linear inequality system $Ax \preceq b$.	Intersection of finitely many closed half-spaces $\equiv$ *(convex, closed) polyhedron*.

Formal Definitions

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Definition: Linear Combination of Vectors

Let $v_1, v_2, \ldots, v_N \in \mathbb{R}^n$ be vectors in a finite system and $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}$ be a system of real numbers. Then

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N$$

is called the linear combination of the v_i vectors.

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$\mathcal{L} \subset \mathbb{R}^n$ is a linear subspace if $0 \in \mathcal{L}$ and closed under linear combination.

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Example

Example: Finitely Generated Linear Subspace

$$\langle v_1, v_2, \ldots, v_N \rangle_{\text{lin}} = \{\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N : \lambda_i \in \mathbb{R}\}.$$

Formal Definitions (continued)

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Definition: Affine Combination of Vectors

Let $v_1, v_2, \ldots, v_N \in \mathbb{R}^n$ be vectors in a finite system and $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}$ be a system of real numbers such that $\lambda_1 + \lambda_2 + \ldots + \lambda_N = 1$. Then

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N$$

is called the affine combination of the v_i vectors.

Formal Definitions (continued)

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Definition: Affine Subspace of $\mathbb{R}^n$

$\mathcal{A} \subset \mathbb{R}^n$ is an affine subspace if closed under affine combination.

Formal Definitions (continued)

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Definition: Affine Subspace of $\mathbb{R}^n$

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Example

Example: Finitely Generated Affine Subspace

$$\langle v_1, v_2, \ldots, v_N \rangle_{\text{affine}} = \{\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N : \lambda_i \in \mathbb{R}, \sum_i \lambda_i = 1\}.$$

Formal Definitions (continued)

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Definition: Cone Combination of Vectors

Let $v_1, v_2, \ldots, v_N \in \mathbb{R}^n$ be vectors in a finite system and $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}_+$ be nonnegative real numbers. Then

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$\mathcal{C} \subset \mathbb{R}^n$ is a (convex) cone if closed under cone combination.

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Formal Definitions (continued)

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$\mathcal{K} \subset \mathbb{R}^n$ is a convex point set if closed under convex combination.

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Let $0 \in \mathcal{L} \subset \mathbb{R}^n$. Then the following are equivalent:

(i) Closed under line joining.

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(iii) Solution set of $Ax = 0$ for some $A \in \mathbb{R}^{k \times n}$.

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Theorem

Let $\mathcal{A} \subset \mathbb{R}^n$. Then the following are equivalent:

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Theorems (continued)

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Minkowski-Weyl Theorem

Let $\mathcal{C} \subset \mathbb{R}^n$. Then the following are equivalent:

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Fundamental Theorem of Polytopes

Let $\mathcal{T} \subset \mathbb{R}^n$. Then the following are equivalent:

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Let $\mathcal{P} \subset \mathbb{R}^n$. Then the following are equivalent:

(i) Polyhedron, i.e., solution set of $Ax \preceq b$ for some $A \in \mathbb{R}^{k\times n}$, $b \in \mathbb{R}^k$.

(ii) $\mathcal{T} + \mathcal{C}$, where $\mathcal{T}$ is a polytope/finitely generated convex set and $\mathcal{C}$ is a polyhedral/finitely generated cone.

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(iv) $\operatorname{ext} \mathcal{P} = \emptyset$.

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- These are exactly those cones for which there exists a hyperplane passing through the origin, such that all nonzero vectors of the cone lie strictly on one side of it. (This needs to be proved!)

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Theorem

Let $\mathcal{P}$ be an arbitrary polyhedron. Then

$$\mathcal{P} = \mathcal{T} + \mathcal{C}_{\text{pointed}} + \mathcal{L},$$

where $\mathcal{T}$ is polytope, $\mathcal{C}_{\text{pointed}}$ is a pointed cone, and $\mathcal{L}$ is a linear subspace.

Break

Vertices of Polyhedra

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LINEAR ALGEBRA	GEOMETRY

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A solution m of a linear inequality system $Ax \preceq b$ (assuming A has no zero rows) is exactly an interior point of m (and any neighborhood of m contains only solutions) if every condition is satisfied with strict inequalities. That is, every condition is tight.	The boundary points of a polyhedron $\mathcal{P}$ are those points that have both $\mathcal{P}$-interior and $\mathcal{P}$-exterior points in every neighborhood. The set of boundary points, or the boundary itself, is denoted by $\partial\mathcal{P}$. The polyhedron $\mathcal{P}$ is closed, thus $\partial\mathcal{P} \subseteq \mathcal{P}$.

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Theorem

Let $K \subseteq \mathbb{R}^n$ be a closed convex set. The following are equivalent:

(i) $p \in \partial K$,

(ii) $p \in K$ and a supporting hyperplane can be placed on it.

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Definition

Let K be a convex set and F be a face. Let $\text{aff}(F)$ be the affine hull of the set F, i.e., the smallest affine subspace containing F. The dimension of F is $\dim(\text{aff}(F))$.

Special Faces: Vertices

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Theorem

Let $\mathcal{P} : \{x : Ax \preceq b\} \subset \mathbb{R}^n$ be a polyhedron, $e \in \mathcal{P}$. Then the following are equivalent:

(i) There exists a supporting hyperplane that intersects $\mathcal{P}$ only at e.

(ii) There is no line segment in $\mathcal{P}$ that contains e as an interior point.

(iii) Let $I = \{i : a_i^T e = b_i\}$. Then I is such that $\{a_i : i \in I\}$ spans $\mathbb{R}^n$.

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Let $\mathcal{P}$ be a polyhedron, $p \in \partial\mathcal{P}$

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Lemma

C_p is a convex cone.

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Theorem

Let $\mathcal{P}$ be a polyhedron, $\mathcal{P} = \{x \colon Ax \preceq b\}$, $p \in \partial \mathcal{P}$. The following are equivalent:

(i) $p \in \mathrm{ext}(\mathcal{P})$,

(ii) C_p has an interior point (in $\mathbb{R}^n$),

(iii) there exist row vectors $a_{i_1}^{\mathsf{T}}, a_{i_2}^{\mathsf{T}}, \ldots, a_{i_n}^{\mathsf{T}}$ in A such that

  (1) they are linearly independent,

  (2) $a_{i_j}^{\mathsf{T}} p = b_{i_j}$ for every $j = 1, 2, \ldots, n$.

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• That is, C_p is full-dimensional if and only if p is a vertex. Generally, the dimension of C_p determines the dimension of the interior point of the boundary p point.

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- Then $\mathcal{P}$ supports the hyperplane. The supporting points are the optimal points.

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• If $c^\mathsf{T} k \geq 0$, we can assume $k = 0$, i.e., o falls into the *polytope part* of our polyhedron.

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- Obviously, $c = \nu$ is a good choice.

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- Specifically, we can write a system of n equations, whose matrix is a submatrix of A, constants are the components of b, and e is the unique solution.

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Break Time

Farkas' Lemma: First Alternative Form

Farkas' Lemma, First Alternative Form

Let $Ax \preceq b$ be a system of equations, where $A \in \mathbb{R}^{k\times n}$,

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and $b \in \mathbb{R}^k$. Then exactly one of the following two statements holds:

(i) The system of equations is solvable, i.e., there exists $x_0 \in \mathbb{R}^n$ such that $Ax_0 \preceq b$.

(ii) There exists $0 \preceq \lambda \in \mathbb{R}^k$ such that $\lambda^\mathsf{T} A = 0^\mathsf{T}$ and $\lambda^\mathsf{T} b = -1$.

Second Alternative Form

Farkas' Lemma, Second Alternative Form

Consider the system of equations $\begin{cases} Ax = b \\ x \succeq 0 \end{cases}$, where $A \in \mathbb{R}^{\ell \times n}$,

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and $b \in \mathbb{R}^{\ell}$. Then exactly one of the following two statements holds:

(i) The system of equations is solvable, i.e., there exists $0 \preceq x_0 \in \mathbb{R}^n$ such that $Ax_0 = b$.

(ii) There exists $\lambda \in \mathbb{R}^{\ell}$ such that $\lambda^{\mathsf{T}} A \succeq 0^{\mathsf{T}}$ and $\lambda^{\mathsf{T}} b = -1$.

Farkas' Lemma: Geometric Form

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Let $\mathcal{C} \subset \mathbb{R}^n$ be a finitely generated cone. That is, there exists a matrix $G \in \mathbb{R}^{n \times k}$ such that

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Farkas' Lemma: Geometric Form (continued)

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- According to Farkas' Lemma, the infeasibility of $\begin{cases} Gx = b, \\ 0 \preceq x \end{cases}$ is equivalent to the existence of a vector $\lambda \in \mathbb{R}^n$ such that

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- In other words, the hyperplane $\mathcal{H} : \lambda^{\mathsf{T}} x = 0$ passing through the origin separates the cone and the point b, where one side $\mathcal{F}^{\geq} : \lambda^{\mathsf{T}} x \geq 0$ contains the cone $\mathcal{C}$, while the other side $\mathcal{F}^{\leq} : \lambda^{\mathsf{T}} x \leq 0$ contains b.

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Farkas' Lemma: Geometric Form

Let $\mathcal{C} \subset \mathbb{R}^n$ be a finitely generated cone, $b \notin \mathcal{C}$. Then there exists a hyperplane $\mathcal{H} : \lambda^\mathsf{T} x = 0$ that separates the cone and b.

Proof of Weyl's Theorem: If a cone is finitely generated, then it's polyhedral

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Lemma

We know that $\mathcal{C}$ is both a polyhedron and a cone. Then $\mathcal{C}$ is a polyhedral cone.

Minkowski's Lemma

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- This is equivalent to saying that

the elements of AG are all non-positive.

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- These are equivalent to the proposition to be proven.

Polytopes

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A polyhedron $\mathcal{P} \subset \mathbb{R}^n$ is called a polytope if it is bounded.

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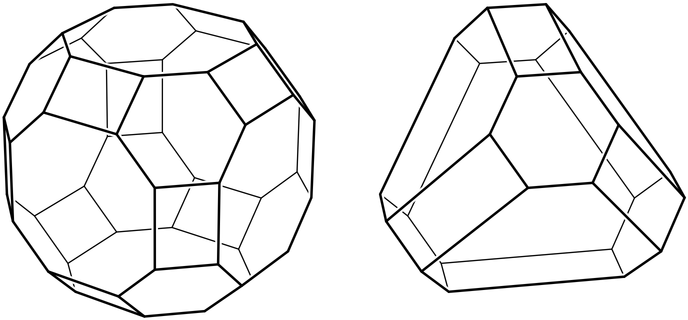

Fundamental Theorem of Convex Polytopes

Fundamental Theorem of Convex Polytopes

Theorem

Let $\mathcal{P} \subset \mathbb{R}^d$. Then the following are equivalent:

(i) $\mathcal{P}$ is a bounded polyhedron.

(ii) $\mathcal{P}$ is the convex hull of finitely many points in $\mathbb{R}^d$.

Polyhedra: Coning, Homogenization

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Let $\mathcal{P}$ be a polyhedron, i.e.,

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Define

$$\widehat{\mathcal{P}} = \left\{ \binom{x}{\lambda} : x \in \mathbb{R}^d, \lambda \in \mathbb{R}, Ax \preceq \lambda b, 0 \leq \lambda \right\} \subset \mathbb{R}^d \times \mathbb{R}_+ \subset \mathbb{R}^{d+1}.$$

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Example

$$\mathcal{P} = \{(x, y)^\mathsf{T} : x \leq 0, y \leq 0\} \subset \mathbb{R}^2.$$

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Coning of Polyhedra: The Observation

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Observation

(i) $x \in \mathcal{P}$ if and only if $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \widehat{\mathcal{P}}$.

(ii) $\widehat{\mathcal{P}}$ is a polyhedral cone.

Fundamental Theorem of Convex Polytopes: Proof (i)⇒(ii)

Fundamental Theorem of Convex Polytopes: Proof (i)$\Rightarrow$(ii)

- Since $\mathcal{P}$ is bounded, the polyhedral cone $\widehat{\mathcal{P}}$ contains only 0 from the hyperplane $\lambda = 0$.

Fundamental Theorem of Convex Polytopes: Proof (i)⇒(ii)

- Since $\mathcal{P}$ is bounded, the polyhedral cone $\widehat{\mathcal{P}}$ contains only 0 from the hyperplane $\lambda = 0$.
- By Weyl's theorem,

$$\widehat{\mathcal{P}} = \langle \widehat{g}_1, \widehat{g}_2, \ldots, \widehat{g}_k \rangle_{\text{cone}} = \left\langle \begin{pmatrix} g_1 \\ 1 \end{pmatrix}, \begin{pmatrix} g_2 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} g_k \\ 1 \end{pmatrix} \right\rangle_{\text{cone}}$$

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- Thus,

$$\begin{pmatrix} g \\ 1 \end{pmatrix} \in \widehat{\mathcal{P}}$$

if and only if

$$g \in \langle g_1, g_2, \dots, g_k \rangle_{\text{convex}}$$

Fundamental Theorem of Convex Polytopes: Proof (ii)$\Rightarrow$(i)

Fundamental Theorem of Convex Polytopes: Proof (ii)⇒(i)

Assume $\mathcal{P} = \langle g_1, g_2, \ldots, g_k \rangle_{\text{convex}}$.

Fundamental Theorem of Convex Polytopes: Proof (ii)⇒(i)

Assume $\mathcal{P} = \langle g_1, g_2, \dots, g_k \rangle_{\text{convex}}$. Clearly, $\mathcal{P}$ is bounded.
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a finitely generated polyhedral cone.

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a finitely generated polyhedral cone.

By Weyl's theorem, there exists a matrix $(A| - b)$ such that

$$\widehat{\mathcal{P}} = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : (A| - b) \begin{pmatrix} x \\ \lambda \end{pmatrix} \preceq 0 \right\}.$$

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Then

$$\mathcal{P} = \{x : Ax \preceq b\},$$

i.e., $\mathcal{P}$ is a polyhedron.

Combining Geometric Sets

Combining Geometric Sets

Definition

Let $A, B \subset \mathbb{R}^d$. Then

$$A + B = \{a + b : a \in A, b \in B\}$$

is called the direct or Minkowski sum of sets A and B.

Minkowski–Weyl Theorem

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(i) Let $\mathcal{P}$ be any polyhedron. Then there exist finitely generated convex sets/polytopes $\mathcal{T}$ and $\mathcal{C}$

$$\mathcal{P} = \mathcal{T} + \mathcal{C}.$$

Minkowski–Weyl Theorem

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(i) Let $\mathcal{P}$ be any polyhedron. Then there exist finitely generated convex sets/polytopes $\mathcal{T}$ and $\mathcal{C}$

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(ii) Let $\mathcal{T}$ be a finitely generated convex set/polytope and $\mathcal{C}$ be a finitely generated cone. Then $\mathcal{T} + \mathcal{C}$ is a polyhedron.

Minkowski–Weyl Theorem: Proof: (i)

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- Then

$$\mathcal{P} = \langle g_1, g_2, \ldots, g_k \rangle_{\text{convex}} + \langle h_1, h_2, \ldots, h_\ell \rangle_{\text{cone}},$$

Minkowski–Weyl Theorem: Proof: (ii)

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Assume $\mathcal{P} = \langle g_1, g_2, \ldots, g_k \rangle_{\text{convex}} + \langle h_1, h_2, \ldots, h_\ell \rangle_{\text{cone}}$.
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a finitely generated cone.

Minkowski–Weyl Theorem: Proof: (ii)

Assume $\mathcal{P} = \langle g_1, g_2, \ldots, g_k \rangle_{\text{convex}} + \langle h_1, h_2, \ldots, h_\ell \rangle_{\text{cone}}$.
Let

$$\widehat{\mathcal{P}} = \left\langle \begin{pmatrix} g_1 \\ 1 \end{pmatrix}, \begin{pmatrix} g_2 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} g_k \\ 1 \end{pmatrix}, \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \begin{pmatrix} h_2 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} h_\ell \\ 0 \end{pmatrix} \right\rangle_{\text{cone}},$$

a finitely generated cone.

By Weyl's theorem, there exists a matrix $(A| - b)$ such that

$$\widehat{\mathcal{P}} = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : (A| - b) \begin{pmatrix} x \\ \lambda \end{pmatrix} \preceq 0 \right\}.$$

Minkowski–Weyl Theorem: Proof: (ii)

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a finitely generated cone.

By Weyl's theorem, there exists a matrix $(A|-b)$ such that

$$\widehat{\mathcal{P}} = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : (A|-b)\begin{pmatrix} x \\ \lambda \end{pmatrix} \preceq 0 \right\}.$$

Then

$$\mathcal{P} = \{x : Ax \preceq b\},$$

i.e., $\mathcal{P}$ is a polyhedron.

This is the End!

Thank you for your attention!